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Existence and porosity for a class of perturbed optimization problems in Banach spaces ☆

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Abstract

Let X be a Banach space and Z a nonempty closed subset of X. Let $J : Z \to \mathbb{R}$ be an upper semicontinuous function bounded from above. This paper is concerned with the perturbed optimization problem $\sup_{z\in Z}\{J(z) + \|x - z\|\}$, which is denoted by (x, J)-sup. We shall prove in the present paper that if Z is a closed boundedly relatively weakly compact nonempty subset, then the set of all $x \in X$ for which the problem (x, J)-sup has a solution is a dense G_δ-subset of X. In the case when X is uniformly convex and J is bounded, we will show that the set of all points x in X for which there does not exist $z_0 \in Z$ such that $J(z_0) + \|x - z_0\| = \sup_{z\in Z}\{J(z) + \|x - z\|\}$ is a σ-porous subset of X and the set of all points $x \in X \setminus Z^0$ such that there exists a maximizing sequence of the problem (x, J)-sup which has no convergent subsequence is a σ-porous subset of $X \setminus Z^0$, where Z^0 denotes the set of all $z \in Z$ such that z is in the solution set of (z, J)-sup.

Keywords: G_δ-subset; σ-Porous set; Perturbed optimization problem; Upper semicontinuous function

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1. Introduction

Let X be a real Banach space endowed with the norm $\|\cdot\|$. Let Z be a nonempty closed subset of X and let $J : Z \to \mathbb{R}$ be a function defined on Z. Define

$$\Phi(x) = \sup_{z\in Z}\{J(z) + \|x - z\|\} \quad \text{for each } x \in X \tag{1.1}$$

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and

$$\phi(x) = \inf_{z \in Z} \{ J(z) + \|x - z\| \} \quad \text{for each } x \in X. \tag{1.2}$$

In (1.1), the function J is supposed upper semicontinuous and bounded from above, and in (1.2) J is supposed lower semicontinuous and bounded from below. The perturbed optimization problem, which is denoted by (x, J)-sup (respectively (x, J)-inf), is of finding an element $z_0 \in Z$ such that

$$\Phi(x) = J(z_0) + \|x - z_0\| \quad \bigl(\text{respectively } \phi(x) = J(z_0) + \|x - z_0\|\bigr). \tag{1.3}$$

In particular, if $J \equiv 0$, then the perturbed optimization problem (x, J)-sup (respectively (x, J)-inf) reduces to the well-known furthest point problem (respectively the best approximation problem). An element $z_0 \in Z$ satisfying (1.3) is called a solution to the problem (x, J)-sup (respectively (x, J)-inf) and the set of all solutions to the problem (x, J)-sup (respectively (x, J)-inf) is denoted by $F_Z(x)$ (respectively $P_Z(x)$), i.e.,

$$F_Z(x) = \bigl\{ z_0 \in Z\colon\ J(z_0) + \|x - z_0\| = \Phi(x) \bigr\} \tag{1.4}$$

and

$$P_Z(x) = \bigl\{ z_0 \in Z\colon\ J(z_0) + \|x - z_0\| = \phi(x) \bigr\}. \tag{1.5}$$

The problems (x, J)-inf and (x, J)-sup were presented and investigated by Baranger in [3–5], respectively. The existence results have been applied to optimal control problems governed by partial differential equations, see, for example, [3,4,6,8–10,14,24]. Here we are especially interested in the study of the problem (x, J)-sup; while, for the problem (x, J)-inf, the readers are referred to [6,10,11,20,26]. In [5], it was proved that if X is a reflexive and locally uniformly convex Banach space then the set of all $x \in X$ for which the problem (x, J)-sup has a solution is a dense G_δ-subset of X. This result clearly extends Edelstein's [16] and Asplund's [1] results on farthest points. Cobzas studied in [10] the existence problem in an arbitrary Banach space and proved that if Z is a weakly compact subset of X and J is an upper semicontinuous real-valued functional bounded from above, then the set of all $x \in X$ for which the problem (x, J)-sup has a solution is a dense G_δ-subset of X, which extends Lau's result in [23]. For other results on perturbed optimization problems of this kind, one can see, for example, [6,9,10,20].

The purpose of the present paper is to continue to carrying out investigations in this line. Let Z be a closed, boundedly relatively weakly compact and nonempty subset (but unnecessarily bounded) of X. We will show that if J is an upper semicontinuous real-valued functional bounded from above and satisfies $\limsup_{\|z\| \to \infty} \frac{J(z)}{\|z\|} < -1$, then the set of all $x \in X$ for which the problem (x, J)-sup has a solution and any maximizing sequence has a convergent subsequence is a dense G_δ-subset of X. This result clearly extends the corresponding results in [1,5,10,16,23], etc. Furthermore, in the case when X is uniformly convex and J is assumed additionally to be bounded, we will show more, that is, the set of all $x \in X$ such that $F_Z(x) = \emptyset$ is a σ-porous subset of X and the set of all points $x \in X \setminus Z^0$ such that there exists a maximizing sequence of the problem (x, J)-sup which has no convergent subsequence is a σ-porous subset of $X \setminus Z^0$, where Z^0 denotes the set of all $z \in Z$ such that $z \in F_Z(z)$. These results are in the spirit of the idea due to Blasi, Myjak and Papini in [7]. Extensions to convex sets and generalized approximations of this idea of Blasi, Myjak and Papini can be found in [18–22].

We end this section with some standard notations. Let X^* be the dual of X. We use $\langle \cdot, \cdot \rangle$ to denote the inner product connecting X^* and X. The closed (respectively open) ball in X at center x with radius r is denoted by $\mathbf{B}(x, r)$ (respectively $\mathbf{U}(x, r)$) while the corresponding sphere by

$\mathbf{S}(x,r)$. In particular, we write $\mathbf{B} = \mathbf{B}(0,1)$ and $\mathbf{S} = \mathbf{S}(x,r)$. Similarly, the closed ball, open ball and sphere in X^* at center x^* with radius r are denoted by $\mathbf{B}^*(x^*,r)$, $\mathbf{U}^*(x^*,r)$ and $\mathbf{S}^*(x^*,r)$, respectively; while the unit ball and sphere in X^* by $\mathbf{B}^*$ and $\mathbf{S}^*$. For a subset A of X, the linear hull and the closure of A are respectively denoted by $\operatorname{span} A$ and $\bar{A}$. The following notions are well known, see, for example, [13,25].

Definition 1.1. X is said to be

(i) strictly convex if, for any $x_1, x_2 \in \mathbf{S}$, the condition $\|x_1 + x_2\| = 2$ implies that $x_1 = x_2$;
(ii) uniformly convex if, for any sequences $\{x_n\}, \{y_n\} \subseteq \mathbf{S}$, the condition $\lim_{n\to\infty} \|x_n + y_n\| = 2$ implies that $\lim_{n\to\infty} \|x_n - y_n\| = 0$;
(iii) (sequentially) Kadec if, for any sequence $\{x_n\} \subseteq \mathbf{S}$, $x \in \mathbf{S}$, the condition $x_n \to x$ weakly implies that $\lim_{n\to\infty} \|x_n - x\| = 0$.

2. Existence

This section is devoted to establish the generic result of the existence of solutions to the problem (x, J)-sup. The main tool is the Fréchet differentiability of convex functions. We begin with the definition of Fréchet differential.

Definition 2.1. Let A be an open subset of X and f a real-valued function defined on A. Let $x \in A$. Then f is said to be Fréchet differentiable at x if there exists $x^* \in X^*$ such that

$$\lim_{y\to x} \frac{f(y) - f(x) - \langle x^*, y - x\rangle}{\|y - x\|} = 0.$$

x^* is called the Fréchet differential at x which is denoted by $\mathrm{D}f(x)$.

Throughout this section, we always assume the function J is upper semicontinuous and bounded from above, and satisfies

$$\limsup_{\|z\|\to\infty} \frac{J(z)}{\|z\|} < -1. \tag{2.1}$$

Note that (2.1) is understood to hold automatically in the case when Z is bounded. Note also that $\Phi(x) < +\infty$ for each $x \in X$ under the assumption (2.1); hence Φ is a convex function on X. The main result of this section is as follows.

Theorem 2.1. *Let Z be a closed, boundedly relatively weakly compact and nonempty subset of X. Suppose that X is a Kadec Banach space. Then the set of all $x \in X$ such that $F_Z(x) \neq \emptyset$ and every maximizing sequence of the problem (x, J)-sup has a subsequence which converges strongly is a dense G_δ-subset of X.*

Remark 2.1. Recall from [14] that the problem (x, J)-sup is said to be well-posed if it has a unique solution and every maximizing sequence has a subsequence converges strongly to this unique solution. In the special case when $J \equiv 0$, if X is additionally assumed to be strictly convex, then the conclusion of Theorem 2.1 can be improved in such a way that the set of all $x \in X$ such that the problem (x, J)-sup is well-posed is a dense G_δ-subset of X (cf. [22]). However, in the general case, such an improvement as above may be impossible even if X is assumed to be uniformly convex. For example, let $X = \mathbb{R}$ with norm $\|x\| = |x|$. Let $Z = [0, 1]$. Define the function J by $J(z) = |z|$ for each $z \in Z$. Then for each $x > 1$, it is easy to see that $F_Z(x) = [0, 1]$, which is not a singleton.

To prove Theorem 2.1, we first recall the notion of an Asplund space. A Banach space X is said to be Asplund if each continuous convex function f, which is defined on a open convex subset E of X, is Fréchet differentiable on a dense G_δ-subset of E. Also recall that each reflexive Banach space is Asplund, see, for example, [2,17]. Hence the following lemma is clear.

Lemma 2.1. *Let Y be a reflexive Banach space and f a continuous convex function on Y. Then f is Fréchet differentiable on a dense G_δ-subset of Y.*

The following lemma, due to Davis, Figiel, Johnson and Pelczynski [15], see also [12], plays an important role in the proof of Theorem 2.1.

Lemma 2.2. *Let A be a weakly compact subset of a Banach space X and let $Y = \overline{\operatorname{span} A}$. Then there exist a reflexive Banach R and a one-to-one continuous linear mapping $T : R \to Y$ such that $T(\mathbf{B}_R) \supseteq A$, where $\mathbf{B}_R$ denotes the unit ball in R.*

We also need the following simple proposition, the proof of which is direct.

Proposition 2.1. *Let $\Phi : X \mapsto \mathbb{R}$ be defined by* (1.1). *Then*

$$\bigl|\Phi(x) - \Phi(x')\bigr| \leqslant \|x - x'\|, \quad x, x' \in X. \tag{2.2}$$

For $x \in X$ and $\delta > 0$, set

$$Z(x, \delta) = \bigl\{z \in Z\colon\ \|x - z\| + J(z) > \Phi(x) - \delta\bigr\} \tag{2.3}$$

and $H(Z) = \bigcap_n H_n(Z)$, where

$$H_n(Z) = \left\{x \in X\colon\ \begin{array}{l} \text{there are some } \delta > 0 \text{ and } x^* \in \mathbf{S}^* \text{ such that} \\ \inf\limits_{z \in Z(x,\delta)} \langle x^*, x - z\rangle + J(z) > \bigl(1 - 2^{-n}\bigr)\Phi(x) \end{array}\right\}. \tag{2.4}$$

Also set

$$M(Z) = \left\{x \in X\colon\ \begin{array}{l} \text{there is } x^* \in \mathbf{S}^* \text{ such that for each } \varepsilon > 0 \text{ there is } \delta > 0 \\ \text{satisfying } \inf\limits_{z \in Z(x,\delta)} \langle x^*, x - z\rangle + J(z) > (1 - \varepsilon)\Phi(x) \end{array}\right\}. \tag{2.5}$$

Obviously, $M(Z) \subset H(Z)$.

Lemma 2.3. *Let Z be a closed, boundedly relatively weakly compact and nonempty subset of X. Then $H(Z)$ is a dense G_δ-subset of X.*

Proof. To show that $H(Z)$ is a G_δ-subset of X, we only need to prove that $H_n(Z)$ is open for each n. For this end, let $n \in \mathbb{N}$ and $x \in H_n(Z)$. Then there exist $x^* \in X^*$ with $\|x^*\| = 1$ and $\delta > 0$ such that

$$\beta = \inf\bigl\{\langle x^*, x - z\rangle + J(z)\colon\ z \in Z(x, \delta)\bigr\} - \bigl(1 - 2^{-n}\bigr)\Phi(x) > 0. \tag{2.6}$$

Let $\lambda > 0$ be such that $\lambda < \min\{\delta/2, \beta/2\}$. Below we will show that $\mathbf{U}(x,\lambda) \subset H_n(Z)$. Granting this, the openness of $H_n(Z)$ is proved. Let $y \in \mathbf{U}(x,\lambda)$. Let $\delta^* = \delta - 2\lambda$ and $z \in Z(y, \delta^*)$. Since, by Proposition 1.1,

$$\begin{aligned}\|x-z\| + J(z) &\geqslant \|y-z\| + J(z) - \|y-x\| \\ &> \Phi(y) - \delta^* - \lambda \\ &\geqslant \Phi(x) - \delta^* - 2\lambda = \Phi(x) - \delta,\end{aligned}$$

$z \in Z(x,\delta)$ and

$$\langle x^*, x-z\rangle + J(z) \geqslant \beta + \left(1-2^{-n}\right)\Phi(x) \tag{2.7}$$

thanks to (2.6). Now we use Proposition 1.1 to conclude that

$$\begin{aligned}\langle x^*, y-z\rangle + J(z) &= \langle x^*, x-z\rangle + J(z) + \langle x^*, y-x\rangle \\ &\geqslant \beta + \left(1-2^{-n}\right)\Phi(x) - \|x-y\| \\ &\geqslant \beta + \left(1-2^{-n}\right)\Phi(y) - \|x-y\| - \left(1-2^{-n}\right)\|x-y\| \\ &\geqslant \left(1-2^{-n}\right)\Phi(y) + \beta - 2\lambda,\end{aligned}$$

where the first inequality holds because of (2.7). It follows that

$$\inf\bigl\{\langle x^*, y-z\rangle + J(z)\colon\ z \in Z(y,\delta^*)\bigr\} > \left(1-2^{-n}\right)\Phi(y), \tag{2.8}$$

since $z \in Z(y,\delta^*)$ is arbitrary. (2.8) implies that $y \in H_n(Z)$ and the fact that $\mathbf{U}(x,\lambda) \subset H_n(Z)$ is verified.

To prove the density of $H(Z)$ in X, it suffices to prove that $M(Z)$ is dense in X since $M(Z) \subset H(Z)$. To this end, take $x_0 \in X$ and let K denote the weak closure of the set $Z(x_0,1) \cup \{x_0\}$. We claim that $Z(x_0,1)$ is bounded. In fact, if otherwise, there exists a sequence $\{z_n\} \subset Z(x_0,1)$ such that $\|z_n\| \to \infty$. From the assumption (2.1), it follows that

$$\liminf_{n\to\infty}\bigl(\|x_0 - z_n\| + J(z_n)\bigr) \leqslant \|x_0\| + \liminf_{n\to\infty}\bigl(\|z_n\| + J(z_n)\bigr) = -\infty. \tag{2.9}$$

Noting that $\{z_n\} \subset Z(x_0,1)$, we have that

$$\Phi(x_0) \leqslant \liminf_{n\to\infty}\bigl(\|x_0 - z_n\| + J(z_n)\bigr) + 1 = -\infty, \tag{2.10}$$

which leads to a contradiction and the claim holds. Hence K is weakly compact in $Y = \overline{\operatorname{span} K}$. From Lemma 2.2, there exist a reflexive Banach space R and a one-to-one continuous linear mapping $T: R \to Y$ such that $T(\mathbf{B}_R) \supseteq K$. Define a function $f_Z : R \to (-\infty, +\infty)$ by

$$f_Z(u) = \Phi(x_0 + Tu) \quad \text{for each } u \in R. \tag{2.11}$$

Then f_Z is a continuous convex function on R and hence Lemma 2.1 is applicable to concluding that f_Z is Fréchet differentiable on a dense subset of R. Let $1/3 > \varepsilon > 0$. Thus there exists a point of differentiability $v \in R$ with $y = Tv \in \mathbf{U}(0,\varepsilon)$. Let $v^* = \mathrm{D}f_Z(v)$. Then

$$\lim_{h\to 0}\frac{\Phi(x_0 + T(v+h)) - \Phi(x_0 + Tv) - \langle v^*, h\rangle}{\|h\|} = 0, \tag{2.12}$$

and hence

$$\lim_{h\to 0}\frac{\Phi(x_0 + y + Th) - \Phi(x_0 + y) - \langle v^*, h\rangle}{\|h\|} = 0. \tag{2.13}$$

For each $u \in R$, substituting tu for h in the above expression as $t \to 0$, by Proposition 1.1, we have

$$\langle v^*, u\rangle \leqslant \|Tu\|, \quad \forall u \in R. \tag{2.14}$$

Then the Hahn–Banach theorem implies that there is $y^* \in Y^*$ such that $v^* = T^* y^*$. Note that

$$\langle y^*, Tu\rangle = \langle T^* y^*, u\rangle = \langle v^*, u\rangle \leqslant \|Tu\|, \quad \forall u \in R, \tag{2.15}$$

and T has dense range. We have that $\|y^*\| \leqslant 1$ and y^* can be extended to $x^* \in X^*$ with $\|x^*\| \leqslant 1$. Set $x = y + x_0$. Then $\|x - x_0\| < \varepsilon$ and $x \in K + Tv \subset TR$. Observe that, for each $r > 0$, (2.13) implies that

$$\lim_{t\to 0} \frac{\Phi(x + tTh) - \Phi(x) - t\langle x^*, Th\rangle}{t} = 0 \tag{2.16}$$

holds uniformly for all $h \in R$ with $\|h\| \leqslant r$. In particular,

$$\lim_{t\to 0} \frac{\Phi(x + t(z - x)) - \Phi(x) - t\langle x^*, z - x\rangle}{t} = 0 \tag{2.17}$$

holds uniformly for all $z \in K$ as $K \subseteq TR$ is bounded. Below we shall show that $x \in M(Z)$. We claim that for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$\langle x^*, x - z\rangle + J(z) > (1 - \varepsilon/2)\Phi(x) \quad \text{for each } z \in Z(x, \delta). \tag{2.18}$$

Granting this, since $\Phi(x) - \delta \leqslant \|x - z\| + J(z) \leqslant \Phi(x)$ for each $z \in Z(x, \delta)$, we have that

$$1 \geqslant \|x^*\| \geqslant 1 - \frac{\varepsilon}{2}\left(1 + \frac{J(z)}{\Phi(x) - J(z)}\right) \to 1 \quad \text{as } \varepsilon \to 0, \tag{2.19}$$

which implies that $\|x^*\| = 1$; hence $x \in M(Z)$ and the proof is complete as $\|x - x_0\| < \varepsilon$. To verify the claim, suppose on the contrary that there exist $\varepsilon_0 > 0$ and a sequence $\{z_n\}$ in Z such that

$$\lim_{n\to\infty} \big(\|x - z_n\| + J(z_n)\big) = \Phi(x) \tag{2.20}$$

but

$$\big\langle x^*, x - z_n\big\rangle + J(z_n) \leqslant (1 - \varepsilon_0/2)\Phi(x) \quad \text{for each } n \in \mathbb{N}. \tag{2.21}$$

Without loss of generality, we may assume that $\|x - z_n\| + J(z_n) > \Phi(x) - \varepsilon$ holds for each $n \in \mathbb{N}$; hence

$$\|x_0 - z_n\| + J(z_n) > \Phi(x_0) - 1 \quad \text{for each } n \in \mathbb{N} \tag{2.22}$$

by Proposition 1.1. Thus, $\{z_n\} \subseteq K$. Take $t_n \in (-1, 0)$ such that $t_n \to 0$ and $t_n^2 > \Phi(x) - [\|x - z_n\| + J(z_n)]$. Then, by (2.17), one gets that

$$\lim_{n\to\infty} \frac{\Phi(x + t_n(z_n - x)) - \Phi(x)}{t_n} - \langle x^*, z_n - x\rangle = 0. \tag{2.23}$$

Since, for each $t \in (-1, 0)$,

$$\begin{aligned}
\Phi\big(x + t(z_n - x)\big) - \Phi(x) &\geqslant \big\|x + t(z_n - x) - z_n\big\| + J(z_n) - \Phi(x) \\
&= (1 - t)\|x - z_n\| + J(z_n) - \Phi(x) \\
&= -t\|x - z_n\| + \big[\|x - z_n\| + J(z_n) - \Phi(x)\big],
\end{aligned}$$

it follows from (2.23) that

$$\liminf_{n\to\infty}\bigl[-\|x-z_n\|+\langle x^*,x-z_n\rangle\bigr]\geqslant 0. \tag{2.24}$$

This together with (2.20) yields that

$$\lim_{n\to\infty}\bigl(\langle x^*,x-z_n\rangle+J(z_n)\bigr)=\Phi(x), \tag{2.25}$$

which contradicts (2.21). Hence the claim holds. □

Recall that a sequence $\{z_n\}$ in Z is called a maximizing sequence of the problem (x, J)-sup if it satisfies that

$$\lim_{n\to\infty}\bigl(\|x-z_n\|+J(z_n)\bigr)=\Phi(x). \tag{2.26}$$

Remark 2.2. Let $\{z_n\}$ be a maximizing sequence of the problem (x, J)-sup. Then the following assertions hold:

(i) $\{z_n\}$ is bounded because $Z(x,1)$ is bounded as shown in the proof of Lemma 2.3 and $\{z_n\}_{n>N}\subseteq Z(x,1)$ for some $N\in\mathbb{N}$.
(ii) If $\{z_n\}$ has a subsequence which converges to z_0 then $z_0\in F_Z(x)$ because J is upper semicontinuous.

Lemma 2.4. *Let Z be a closed, boundedly relatively weakly compact and nonempty subset of X. Suppose that X is a Kadec Banach space. Then, for each $x\in H(Z)$, any maximizing sequence of the problem (x, J)-sup has a convergent subsequence.*

Proof. Let $x\in H(Z)$. Then, in view of the definition, there exist a positive sequence $\{\delta_n\}$ and a sequence $\{x_n^*\}\subseteq S^*$ such that

$$\inf\bigl\{\bigl\langle x_n^*,x-z\bigr\rangle+J(z)\colon\ z\in Z(x,\delta_n)\bigr\}>\bigl(1-2^{-n}\bigr)\Phi(x)\quad\text{for each } n\in\mathbb{N}. \tag{2.27}$$

Without loss of generality, assume that $\delta_n\leqslant\delta_m$ if $m<n$. Let $\{z_n\}$ be any maximizing sequence of the problem (x, J)-sup, i.e.,

$$\lim_{n\to\infty}\bigl[\|x-z_n\|+J(z_n)\bigr]=\Phi(x). \tag{2.28}$$

Without loss of generality, we may assume that $\{\|x-z_n\|\}$ and $\{J(z_n)\}$ are convergent. Note that $\{z_n\}$ is bounded and Z is boundedly relatively weakly compact. We also assume that, without loss of generality, $z_n\to z_0$ weakly as $n\to\infty$ for some $z_0\in X$. Then we have that

$$\|x-z_0\|+\lim_{n\to\infty}J(z_n)\leqslant\lim_{n\to\infty}\bigl(\|x-z_n\|+J(z_n)\bigr)=\Phi(x) \tag{2.29}$$

since $\|\cdot\|$ is lower semicontinuous. Furthermore, we assume that $z_n\in Z(x,\delta_m)$ for all $n>m$. Thus,

$$\bigl\langle x_m^*,x-z_n\bigr\rangle+J(z_n)>\bigl(1-2^{-m}\bigr)\Phi(x),\quad\forall n>m, \tag{2.30}$$

and so

$$\bigl\langle x_m^*,x-z_0\bigr\rangle+J(z_n)\geqslant\bigl(1-2^{-m}\bigr)\Phi(x),\quad\forall n>m. \tag{2.31}$$

Hence,

$$\|x-z_0\|+\lim_{n\to\infty}J(z_n)\geqslant\Phi(x). \tag{2.32}$$

From (2.28), (2.29) and (2.32), one has that

$$\lim_{n\to\infty} \|x - z_n\| = \|x - z_0\|. \tag{2.33}$$

Note that X is Kadec and $z_n \to z_0$ weakly. It follows from (2.33) that $\lim_{n\to\infty} \|z_0 - z_n\| = 0$, which completes the proof. □

Proof of Theorem 2.1. Let $H(Z) = \bigcap_n H_n(Z)$, where $H_n(Z)$ is defined by (2.4). Then $H(Z)$ is a dense G_δ-subset of X by Lemma 2.3. Let $x \in H(Z)$ and $\{z_n\}$ a maximizing sequence of the problem (x, J)-sup. Then, by Lemma 2.4, $\{z_n\}$ has a subsequence, which converges to, say z_0. By Remark 2.2(ii), one has that $z_0 \in F_Z(x)$. Thus the proof is complete. □

Remark 2.3. Consider a generalization of the problem (x, J)-sup, which is denoted by (x, J_ω)-sup and consists of finding an element $z_0 \in Z$ satisfying

$$\sup_{z\in Z}\bigl\{J(z) + \omega\bigl(\|x - z\|\bigr)\bigr\}, \tag{2.34}$$

where ω is a convex continuous strictly increasing function from $\mathbb{R}^+$ into $\mathbb{R}$. This general perturbed optimization problem was studied in [9], where a similar generic result about the existence of the solution was established in the case when X is a reflexive and Kadec Banach space. We should remark here that the technique used in this section still works and Theorem 2.1 above remains true for the general perturbed optimization problem (x, J_ω)-sup.

3. Porosity

We begin with the notion of the porous set, see, for example, [7,21]. Let (E, d) be a metric space. The closed ball in E with radius r and center x is denoted by $\mathbf{B}_d(x, r)$.

Definition 3.1. A subset G of (E, d) is said to be porous in E if there exist $t \in (0, 1]$ and $r_0 > 0$ such that for every $x \in E$ and $r \in (0, r_0]$ there is a point $y \in E$ such that $\mathbf{B}_d(y, tr) \subseteq \mathbf{B}_d(x, r) \cap (E \setminus G)$. A subset G is said to be σ-porous in E if it is a countable union of sets which are porous in E.

Note that in this definition, the statement "for every $x \in E$" can be replaced by "for every $x \in G$." Clearly, a set which is σ-porous in E is also merger in E, the converse is false in general.

Throughout this section, we shall always assume that Z is a nonempty closed subset of X and the function J is upper semicontinuous and bounded. Without loss of generality, we may assume $J > 0$ on Z. We denote by $\mathcal{V}(Z)$ and $\mathcal{W}(Z)$ respectively the set of all $x \in X$ such that $F_Z(x) \neq \emptyset$ and the set of all $x \in X$ such that any maximizing sequence of the problem (x, J)-sup has a convergent subsequence. Clearly, $\mathcal{W}(Z) \subseteq \mathcal{V}(Z)$. Let Z^0 denote the set of all points $z \in Z$ such that $z \in F_Z(z)$. Then Z^0 is a closed subset of Z. Let $p : \mathcal{V}(Z) \to Z$ be a single-valued selection of the set-valued mapping F_Z satisfying

$$\bigl[x, p(x)\bigr] \cap F_Z(x) = p(x) \quad \text{for each } x \in \mathcal{V}(Z) \setminus Z^0, \tag{3.1}$$

where $[x, y]$ denotes the closed interval with ends x and y. Note that such a single-valued selection $p : \mathcal{V}(Z) \to Z$ satisfying (3.1) exists. In fact, for any $x \in \mathcal{V}(Z) \setminus Z^0$, take $z \in F_Z(x)$ and define

$$t_0 = \inf\bigl\{t \in [0, 1]:\ (1 - t)x + tz \in F_Z(x)\bigr\}. \tag{3.2}$$

Then $t_0 > 0$ as $x \notin Z^0$ and $p(x) := (1 - t_0)x + t_0 z \in F_Z(x)$ is as desired. Letting $\alpha \in [0, \frac{1}{2}]$ and $x \in \mathcal{V}(Z)$, set

$$x_\alpha = (1 + \alpha)x - \alpha p(x). \tag{3.3}$$

Then

$$p(x) \in F_Z(x_\alpha) \quad \text{for each } \alpha \in \left[0, \frac{1}{2}\right]. \tag{3.4}$$

In fact, for any $z \in Z$,

$$\begin{aligned} \|x_\alpha - p(x)\| + J(p(x)) &= (1 + \alpha)\|x - p(x)\| + J(p(x)) \\ &\geqslant \|x - z\| + J(z) + \alpha\|x - p(x)\| \\ &\geqslant \|x_\alpha - z\| + J(z) - \|x - x_\alpha\| + \alpha\|x - p(x)\| \\ &= \|x_\alpha - z\| + J(z). \end{aligned}$$

This shows (3.4).

Let $x \in X$, $\alpha \in [0, \frac{1}{2}]$ and let $\gamma_{x,\alpha}$ be the function defined on [0, 1] by

$$\gamma_{x,\alpha}(\varepsilon) = \varepsilon \min\{\alpha\Phi(x), 1\} \quad \text{for each } \varepsilon \in [0, 1]. \tag{3.5}$$

Define

$$\widetilde{\mathcal{H}} = \bigcap_{k\in\mathbb{N}} \bigcup_{x\in\mathcal{V}(Z)} \bigcup_{\alpha\in[0,1/2]} \mathbf{B}(x_\alpha, \gamma_{x,\alpha}(1/k)).$$

We first state the main theorem of this section.

Theorem 3.1. *Let Z be a nonempty and closed subset of X. Suppose that X is a uniformly convex Banach space. Then the following assertions hold*:

(i) *The set $X \setminus \mathcal{V}(Z)$ is σ-porous in X.*
(ii) *The set $(X \setminus Z^0) \setminus \mathcal{W}(Z)$ is σ-porous in $X \setminus Z^0$.*

To prove Theorem 3.1, we first verify the following lemma.

Lemma 3.1. *Let Z be a nonempty closed subset of X. Suppose that X is a uniformly convex Banach space. Then $\widetilde{\mathcal{H}} \subseteq \mathcal{V}(Z)$ and $\widetilde{\mathcal{H}} \setminus Z^0 \subseteq \mathcal{W}(Z)$.*

Proof. Let $x \in \widetilde{\mathcal{H}}$. By the definition of $\widetilde{\mathcal{H}}$, there exist $\{x^k\} \subseteq \mathcal{V}(Z)$ and $\{\alpha_k\} \subseteq [0, 1/2]$ such that

$$\|x^k_{\alpha_k} - x\| \leqslant \gamma_{x^k,\alpha_k}(1/k). \tag{3.6}$$

For notational convenience, set $z^k = p(x^k)$. Below we shall show that

$$\{z^k\} \text{ has a convergent subsequence.} \tag{3.7}$$

Since

$$\begin{aligned} \|z^k - x\| &\leqslant \|z^k - x^k_{\alpha_k}\| + \gamma_{x^k,\alpha_k}(1/k) \\ &= (1 + \alpha_k)\|x^k - z^k\| + \gamma_{x^k,\alpha_k}(1/k) \quad \text{for each } k = 1, 2, \ldots, \end{aligned}$$

(3.7) is clear in the case when $\liminf_k \|x^k - z^k\| = 0$ because $\lim_k \gamma_{x^k,\alpha_k}(1/k) = 0$. Thus, without loss of generality, we may assume that, for some $\delta > 0$, $\|x^k - z^k\| \geqslant \delta$ for each $k \in \mathbb{N}$. Also, we may assume that $\alpha_k > 0$ for each $k \in \mathbb{N}$, $\lim_{k\to\infty} \alpha_k = \alpha$ and $\{\alpha_k\}$ is monotonic decreasing in the case when $\alpha = 0$.

Note that, by Proposition 1.1,

$$\|x^k_{\alpha_k} - z^k\| + J(z^k) = \Phi(x^k_{\alpha_k}) \to \Phi(x).$$

Hence the sequences $\{J(z^k)\}$ and $\{\|x^k_{\alpha_k} - z^k\|\}$ are bounded, which implies that $\{\|x^k - z^k\|\}$ is bounded because $(1+\alpha_k)\|x^k - z^k\| = \|x^k_{\alpha_k} - z^k\|$ and $0 < \alpha_k \leqslant \frac{1}{2}$. Thus, without loss of generality, we may assume that

$$J(z^k) \to a \quad \text{and} \quad \|x^k - z^k\| \to b \tag{3.8}$$

for some $a, b \in \mathbb{R}$. Since, for each $k \in \mathbb{N}$, $\|x^k - z^k\| \geqslant \delta$, one has that $b \geqslant \delta > 0$. Set $r_k = \frac{\alpha_k}{1+\alpha_k}$. Then

$$\lim_{k\to\infty} r_k = \frac{\alpha}{1+\alpha} \quad \text{and} \quad x^k = (1-r_k)x^k_{\alpha_k} + r_k z^k. \tag{3.9}$$

Let $m > k$. By (3.6), (3.9) and Proposition 1.1, we have

$$\begin{aligned}\left\|\frac{r_k(x^k - z^k) + (1-r_k)(x^k - z^m)}{(1-r_k)}\right\| &= \|x^k_{\alpha_k} - z^m\| \\ &\geqslant \|x^m_{\alpha_m} - z^m\| - \|x^m_{\alpha_m} - x^k_{\alpha_k}\| \\ &= \Phi(x^m_{\alpha_m}) - J(z^m) - \|x^m_{\alpha_m} - x^k_{\alpha_k}\| \\ &\geqslant \Phi(x^k_{\alpha_k}) - J(z^m) - 2\|x^m_{\alpha_m} - x^k_{\alpha_k}\| \\ &\geqslant \Phi(x^k_{\alpha_k}) - J(z^m) - 2\big(\gamma_{x^k,\alpha_k}(1/k) + \gamma_{x^m,\alpha_m}(1/m)\big).\end{aligned}$$

For notational convenience, set

$$\delta_{k,m} = \gamma_{x^k,\alpha_k}(1/k) + \gamma_{x^m,\alpha_m}(1/m). \tag{3.10}$$

The inequality above and the choice of $\delta_{k,m}$ imply that

$$\|r_k(x^k - z^k) + (1-r_k)(x^k - z^m)\| \geqslant (1-r_k)\big(\Phi(x^k_{\alpha_k}) - J(z^m)\big) - 2(1-r_k)\delta_{k,m}. \tag{3.11}$$

Since $z^k \in F_Z(x^k_{\alpha_k})$ by (3.4), it follows that

$$\Phi(x^k_{\alpha_k}) = \|x^k_{\alpha_k} - z^k\| + J(z^k). \tag{3.12}$$

Note that $x^k_{\alpha_k} = (1+\alpha_k)x^k - \alpha_k z^k$ and $r_k = \frac{\alpha_k}{1+\alpha_k}$. This together with (3.12) implies that

$$\begin{aligned}(1-r_k)\big(\Phi(x^k_{\alpha_k}) - J(z^m)\big) &= \|x^k - z^k\| + (1-r_k)\big(J(z^k) - J(z^m)\big) \\ &= r_k\|x^k - z^k\| + (1-r_k)\big(\|x^k - z^k\| \\ &\quad + J(z^k) - J(z^m)\big).\end{aligned} \tag{3.13}$$

Since $\Phi(x^k) = \|x^k - z^k\| + J(z^k)$ as $z^k \in F_Z(x^k)$, one has from (3.13) that

$$\begin{aligned}(1-r_k)\big(\Phi(x^k_{\alpha_k}) - J(z^m)\big) &= r_k\|x^k - z^k\| + (1-r_k)\big(\Phi(x^k) - J(z^m)\big) \\ &\geqslant r_k\|x^k - z^k\| + (1-r_k)\|x^k - z^m\|.\end{aligned}$$

Combining this with (3.11) yields that

$$\begin{aligned}&\left\|r_k\left(x^k-z^k\right)+(1-r_k)\left(x^k-z^m\right)\right\|\\&\quad\geqslant r_k\left\|x^k-z^k\right\|+(1-r_k)\left\|x^k-z^m\right\|-2(1-r_k)\delta_{k,m}.\end{aligned}\tag{3.14}$$

Take $x^*\in X^*$ with $\|x^*\|=1$ such that

$$\left\langle x^*,r_k\left(x^k-z^k\right)+(1-r_k)\left(x^k-z^m\right)\right\rangle=\left\|r_k\left(x^k-z^k\right)+(1-r_k)\left(x^k-z^m\right)\right\|.$$

Then

$$\begin{aligned}&\left\langle x^*,r_k\left(x^k-z^k\right)+(1-r_k)\left(x^k-z^m\right)\right\rangle\\&\quad\geqslant r_k\left\|x^k-z^k\right\|+(1-r_k)\left\|x^k-z^m\right\|-2(1-r_k)\delta_{k,m}\end{aligned}\tag{3.15}$$

thanks to (3.14). Consequently,

$$\begin{aligned}\left\langle x^*,r_k\left(x^k-z^k\right)\right\rangle&\geqslant r_k\left\|x^k-z^k\right\|+(1-r_k)\left\|x^k-z^m\right\|-2(1-r_k)\delta_{k,m}\\&\qquad-\left\langle x^*,(1-r_k)\left(x^k-z^m\right)\right\rangle\\&\geqslant r_k\left\|x^k-z^k\right\|-2(1-r_k)\delta_{k,m}\end{aligned}$$

because $\|x^k-z^m\|\geqslant\langle x^*,x^k-z^m\rangle$. Multiplying the both sides of the inequality above by $(r_k\|x^k-z^k\|)^{-1}$ and noting that $r_k=\frac{\alpha_k}{1+\alpha_k}$, we have

$$\left\langle x^*,\frac{x^k-z^k}{\|x^k-z^k\|}\right\rangle\geqslant1-\frac{(1-r_k)2\delta_{k,m}}{r_k\|x^k-z^k\|}=1-\frac{2\delta_{k,m}}{\alpha_k\|x^k-z^k\|}.\tag{3.16}$$

Similarly, we have that

$$\left\langle x^*,\frac{x^k-z^m}{\|x^k-z^m\|}\right\rangle\geqslant1-\frac{2\delta_{k,m}}{\|x^k-z^m\|}\geqslant1-\frac{2\delta_{k,m}}{\alpha_k\|x^k-z^m\|}.\tag{3.17}$$

Combining two inequalities above shows that

$$\left\|\frac{x^k-z^k}{\|x^k-z^k\|}+\frac{x^k-z^m}{\|x^k-z^m\|}\right\|\geqslant2-\left(\frac{2\delta_{k,m}}{\alpha_k\|x^k-z^k\|}+\frac{2\delta_{k,m}}{\alpha_k\|x^k-z^m\|}\right).\tag{3.18}$$

We claim that

$$\lim_{k,m\to\infty}\left\|x^k-z^m\right\|=b\tag{3.19}$$

and

$$\lim_{k,m\to\infty}\left(\frac{2\delta_{k,m}}{\alpha_k\|x^k-z^k\|}+\frac{2\delta_{k,m}}{\alpha_k\|x^k-z^m\|}\right)=0.\tag{3.20}$$

In fact, by (3.6), we have

$$\lim_{k,m\to\infty}\left\|x^k_{\alpha_k}-x^m_{\alpha_m}\right\|=0.\tag{3.21}$$

Hence, one has

$$\begin{aligned}\lim_{k,m\to\infty}\left\|x^k-z^m\right\| &\geqslant \lim_{k,m\to\infty}\left(\left\|x^k_{\alpha_k}-z^m\right\|-\left\|x^k_{\alpha_k}-x^k\right\|\right)\\ &\geqslant \lim_{k,m\to\infty}\left(\left\|x^m_{\alpha_m}-z^m\right\|-\left\|x^k_{\alpha_k}-x^m_{\alpha_m}\right\|-\left\|x^k_{\alpha_k}-x^k\right\|\right)\\ &= \lim_{k,m\to\infty}\left(\left\|x^k_{\alpha_k}-z^k\right\|-\left\|x^k_{\alpha_k}-x^k\right\|\right)\\ &= \lim_{k\to\infty}\left\|x^k-z^k\right\|,\end{aligned}$$

where the first equality holds by (3.21) while the last equality is because

$$\left\|x^k_{\alpha_k}-z^k\right\|=(1+\alpha_k)\left\|x^k-z^k\right\| \quad \text{and} \quad \left\|x^k_{\alpha_k}-x^k\right\|=\alpha_k\left\|x^k-z^k\right\|.$$

Thus by (3.8) it is seen that

$$\lim_{k,m\to\infty}\left\|x^k-z^m\right\|\geqslant b. \tag{3.22}$$

On the other hand, since

$$\left\|x^k-z^m\right\|+J\left(z^m\right)\leqslant\left\|x^k-z^k\right\|+J\left(z^k\right)$$

and $\lim_{k\to\infty}J(z^k)=a$ by (3.8), one has

$$\lim_{k,m\to\infty}\left\|x^k-z^m\right\|\leqslant\lim_{k,m\to\infty}\left(\left\|x^k-z^k\right\|+J\left(z^k\right)-J\left(z^m\right)\right)=b,$$

which together with (3.22) implies (3.19). Recalling that $b>0$ and (3.19), to verify (3.20), it suffices to show that

$$\lim_{k\to\infty}\frac{\delta_{k,m}}{\alpha_k}=0 \tag{3.23}$$

clearly, (3.23) is trivial in the case when $\alpha\neq 0$. In the case when $\alpha=0$, we have that $0<\alpha_m\leqslant\alpha_k$ as assumed before and $x^k\to x$ by (3.6). Hence, by (3.5) and (3.10), we have

$$\lim_{k\to\infty}\frac{\delta_{k,m}}{\alpha_k}=\lim_{k,m\to\infty}\frac{\gamma_{x^k,\alpha_k}(1/k)+\gamma_{x^m,\alpha_m}(1/m)}{\alpha_k}\leqslant\lim_{k,m\to\infty}\left(\frac{\Phi(x^k)}{k}+\frac{\Phi(x^m)}{m}\right)=0.$$

Hence (3.23) holds and the claim is proved. It follows from (3.18) and (3.20) that

$$\liminf_{k,m}\left\|\frac{x^k-z^k}{\|x^k-z^k\|}+\frac{x^k-z^m}{\|x^k-z^m\|}\right\|\geqslant 2. \tag{3.24}$$

This together with (3.19) implies that

$$\liminf_{k,m}\left\|\frac{x^k-z^k}{b}+\frac{x^k-z^m}{b}\right\|\geqslant 2. \tag{3.25}$$

Since X is a uniformly convex Banach space, (3.25) implies that the sequence $\{z^k\}$ is a Cauchy sequence; hence $\{z^k\}$ is convergent and (3.7) holds. Consequently, by Remark 2.2, $x\in\mathcal{V}(Z)$ and the first conclusion of the lemma is proved.

To verify the second conclusion, suppose additional that $x\notin Z^0$ and let $\{y^k\}$ be any maximizing sequence of the problem (x,J)-sup. We have to show that $\{y^k\}$ has a convergent subsequence. Since $\{z^k\}$ has a convergent subsequence, we may assume that $\{z^k\}$ itself converges to, say z_0. Then $z_0\in F_Z(x)$ by Remark 2.2 and $z_0\neq x$ as $x\notin Z^0$. Recall that $r_k=\frac{\alpha_k}{1+\alpha_k}$. Then

$$\lim_{k\to\infty}x^k=\frac{x}{1+\alpha}+\frac{\alpha z_0}{1+\alpha} \quad \text{and} \quad \lim_{k\to\infty}\left(x^k-z^k\right)=\frac{x-z_0}{1+\alpha} \tag{3.26}$$

thanks to (3.9). This implies that $\{x^k\}$ converges and $\lim_k \|x^k - z^k\| > 0$. Thus, without loss of generality, we may assume $\lim_k \|x^k - y^k\| = a' > 0$. Set $c_k = \min\{\|x^k - z^k\|, \|x^k - y^k\|\}$. Then $\liminf_k c_k > 0$. Furthermore, passing to subsequence of $\{y^k\}$ if necessary, we may assume

$$\Phi(x) - \gamma_{x^k,\alpha_k}(1/k) \leqslant \|x - y^k\| + J(y^k) \leqslant \Phi(x) \quad \text{for each } k \in \mathbb{N}. \tag{3.27}$$

Then by (3.6), (3.27) and Proposition 1.1, we have

$$\begin{aligned}\left\|\frac{r_k(x^k - z^k) + (1-r_k)(x^k - y^k)}{(1-r_k)}\right\| &= \|x^k_{\alpha_k} - y^k\| \\ &\geqslant \|x - y^k\| - \|x - x^k_{\alpha_k}\| \\ &\geqslant \Phi(x) - J(y^k) - 2\gamma_{x^k,\alpha_k}(1/k) \\ &\geqslant \Phi(x^k_{\alpha_k}) - J(y^k) - 3\gamma_{x^k,\alpha_k}(1/k).\end{aligned}$$

The inequality above implies that

$$\begin{aligned}&\|r_k(x^k - z^k) + (1-r_k)(x^k - y^k)\| \\ &\quad \geqslant (1-r_k)\big(\Phi(x^k_{\alpha_k}) - J(y^k)\big) - 3(1-r_k)\gamma_{x^k,\alpha_k}(1/k).\end{aligned} \tag{3.28}$$

Since

$$\begin{aligned}(1-r_k)\big(\Phi(x^k_{\alpha_k}) - J(y^k)\big) &= \|x^k - z^k\| + (1-r_k)\big(J(z^k) - J(y^k)\big) \\ &= r_k\|x^k - z^k\| + (1-r_k)\big(\|x^k - z^k\| + J(z^k) - J(y^k)\big) \\ &= r_k\|x^k - z^k\| + (1-r_k)\big(\Phi(x^k) - J(y^k)\big) \\ &\geqslant r_k\|x^k - z^k\| + (1-r_k)\|x^k - y^k\|,\end{aligned}$$

it follows that

$$\begin{aligned}&\|r_k(x^k - z^k) + (1-r_k)(x^k - y^k)\| \\ &\quad \geqslant r_k\|x^k - z^k\| + (1-r_k)\|x^k - y^k\| - 3(1-r_k)\gamma_{x^k,\alpha_k}(1/k).\end{aligned} \tag{3.29}$$

Using the similar arguments as in the proof of (3.18), we have that

$$\left\|\frac{x^k - z^k}{\|x^k - z^k\|} + \frac{x^k - y^k}{\|x^k - y^k\|}\right\| \geqslant 2 - \frac{6\gamma_{x^k,\alpha_k}(1/k)}{\alpha_k c_k}, \tag{3.30}$$

which implies that

$$\liminf_k \left\|\frac{x^k - z^k}{\|x^k - z^k\|} + \frac{x^k - y^k}{\|x^k - y^k\|}\right\| \geqslant 2. \tag{3.31}$$

As X is uniformly convex, (3.31) implies that

$$\lim_k \left\|\frac{x^k - z^k}{\|x^k - z^k\|} - \frac{x^k - y^k}{\|x^k - y^k\|}\right\| = 0. \tag{3.32}$$

Then the sequence $\{y^k\}$ converges because $\{x^k - z^k\}$, $\{\|x^k - z^k\|\}$, $\{\|x^k - y^k\|\}$ and $\{x^k\}$ are convergent. Therefore, $x \in \mathcal{W}(Z)$ and the second conclusion is proved. □

Proof of Theorem 3.1. We shall only verify the assertion (ii) because (i) can be done similarly. For $k, m \in \mathbb{N}$, define

$$\mathcal{H}_k = (X \setminus Z^0) \setminus \bigcup_{x \in \mathcal{V}(Z)\setminus Z^0} \bigcup_{\alpha \in [0,1/2]} \big(\mathbf{B}(x_\alpha, \gamma_{x,\alpha}(1/k)) \setminus Z^0\big)$$

and

$$\mathcal{H}_k^m = \big\{x \in \mathcal{H}_k\colon\ 1/m < \Phi(x) < m\big\}.$$

By Lemma 3.1, we have

$$(X \setminus Z^0) \setminus \mathcal{W}(Z) \subseteq (X \setminus Z^0) \setminus (\widetilde{\mathcal{H}} \setminus Z^0) = \bigcup_{k\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \mathcal{H}_k^m.$$

It suffices to show that $\mathcal{H}_k^m$ is porous in $X \setminus Z^0$ for every $k, m \in \mathbb{N}$. To do this, let $k, m \in \mathbb{N}$ and define

$$r_0 = \frac{1}{2m}, \qquad s = \frac{1}{4k}.$$

Let $x \in \mathcal{H}_k^m$ and $0 < r \leqslant r_0$ be arbitrary. Let η be such that

$$0 < \eta < \frac{r}{4} \quad \text{and} \quad \frac{1}{m} - \eta < \Phi(x) < m + \eta. \tag{3.33}$$

Since Z^0 is closed and $x \notin Z^0$, by Theorem 2.1, there exists $\hat{x} \in \mathcal{V}(Z) \setminus Z^0$ such that

$$\|x - \hat{x}\| < \eta. \tag{3.34}$$

Hence, by Proposition 1.1 and (3.33), we have

$$\big|\Phi(x) - \Phi(\hat{x})\big| < \eta \tag{3.35}$$

and

$$\frac{1}{m} < \Phi(\hat{x}) < m. \tag{3.36}$$

Recall that p is a single-valued selection and that $\hat{x}_\alpha = (1+\alpha)\hat{x} - \alpha p(\hat{x})$ for each $\alpha \in [0, 1/2]$. It follows from (3.33)–(3.36) that

$$\begin{aligned} \|\hat{x}_{1/2} - x\| &\geqslant \|\hat{x}_{1/2} - \hat{x}\| - \|\hat{x} - x\| \\ &\geqslant \|\hat{x} - p(\hat{x})\|/2 - r/4 \\ &\geqslant 3r/4 - J\big(p(\hat{x})\big)/2. \end{aligned}$$

Hence

$$\|\hat{x}_{1/2} - x\| + J\big(p(\hat{x})\big)/2 \geqslant 3r/4. \tag{3.37}$$

Let f be the function defined on $[0, 1]$ by $f(t) := \|\hat{x}_t - x\| + tJ(p(\hat{x}))$ for each $t \in [0, 1]$. Then f is continuous on $[0, 1]$. Moreover, by (3.37), $f(1/2) \geqslant 3r/4$ and, by (3.34), $f(0) \leqslant \eta < r/4$ thanks to (3.33). Hence, there exists $\alpha \in (0, 1/2]$ such that $\|\hat{x}_\alpha - x\| + \alpha J(p(\hat{x})) = 3r/4$. We claim that $\hat{x}_\alpha \notin Z^0$. In fact, otherwise,

$$J(\hat{x}_\alpha) = \|\hat{x}_\alpha - p(\hat{x})\| + J\big(p(\hat{x})\big). \tag{3.38}$$

Noting that $\hat{x}_\alpha = (1+\alpha)\hat{x} - \alpha p(\hat{x})$, one has

$$\|\hat{x}_\alpha - p(\hat{x})\| = \|\hat{x} - p(\hat{x})\| + \|\hat{x} - \hat{x}_\alpha\|.$$

This together with (3.38) implies that

$$\begin{aligned} J(\hat{x}_\alpha) + \|\hat{x} - \hat{x}_\alpha\| &= \|\hat{x}_\alpha - p(\hat{x})\| + J(p(\hat{x})) + \|\hat{x} - \hat{x}_\alpha\| \\ &= \|\hat{x} - p(\hat{x})\| + J(p(\hat{x})) + 2\|\hat{x} - \hat{x}_\alpha\| \\ &= \Phi(\hat{x}) + 2\|\hat{x} - \hat{x}_\alpha\| \\ &> \Phi(\hat{x}), \end{aligned}$$

which contradicts that $p(\hat{x}) \in F_Z(\hat{x})$ as $\hat{x} \neq \hat{x}_\alpha$. Hence the claim stands. Since for each $z \in \mathbf{B}(\hat{x}_\alpha, sr) \setminus Z^0$,

$$\|z - x\| \leqslant \|z - \hat{x}_\alpha\| + \|\hat{x}_\alpha - x\| \leqslant sr + 3r/4 \leqslant r,$$

we have

$$\mathbf{B}(\hat{x}_\alpha, sr) \setminus Z^0 \subseteq \mathbf{B}(x, r) \setminus Z^0.$$

Thus to complete the proof, it suffices to show that

$$\mathbf{B}(\hat{x}_\alpha, sr) \setminus Z^0 \subseteq (X \setminus Z^0) \setminus \mathcal{H}_k^m. \tag{3.39}$$

Recall that $\|\hat{x}_\alpha - x\| + \alpha J(p(\hat{x})) = 3r/4$. It follows that

$$\|\hat{x}_\alpha - \hat{x}\| \geqslant \|\hat{x}_\alpha - x\| - \|x - \hat{x}\| \geqslant 3r/4 - \alpha J(p(\hat{x})) - \eta \geqslant r/2 - \alpha J(p(\hat{x})),$$

thanks to (3.33) and (3.34). Hence we have that

$$\alpha\Phi(\hat{x})/k = \left[\|\hat{x}_\alpha - \hat{x}\| + \alpha J(p(\hat{x}))\right]/k \geqslant \frac{r}{2k} \geqslant sr. \tag{3.40}$$

Since $sr < \frac{1}{4k} \cdot \frac{1}{2m} \leqslant \frac{1}{k}$, it follows that $sr \leqslant \gamma_{\hat{x},\alpha}(1/k)$ thanks to (3.5). This shows that

$$\mathbf{B}(\hat{x}_\alpha, sr) \setminus Z^0 \subseteq \mathbf{B}(\hat{x}_\alpha, \gamma_{\hat{x},\alpha}(1/k)) \setminus Z^0. \tag{3.41}$$

As $\mathcal{H}_k^m \subseteq \mathcal{H}_k$ and $\mathcal{H}_k \cap (\mathbf{B}(\hat{x}_\alpha, \gamma_{\hat{x},\alpha}(1/k)) \setminus Z^0) = \emptyset$, (3.39) follows and the proof is complete. □

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