Acta Mathematica Sinica, English Series
Feb., 2005, Vol.21, No.1, pp. 31–38
Published online Dec. 1, 2004
DOI: 10.1007/s10114-004-0350-2
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series
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On Best Approximations from RS-sets in Complex Banach Spaces

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Abstract The concept of an RS-set in a complex Banach space is introduced and the problem of best approximation from an RS-set in a complex space is investigated. Results consisting of characterizations, uniqueness and strong uniqueness are established.

Keywords Complex RS-set, Best approximation, Uniqueness, Strong uniqueness
MR(2000) Subject Classification 41A29, 41A50

1 Introduction

Let X be a complex Banach space and G a closed subset in X. For a point $x \in X$, an element $g^* \in G$ is called a best approximation to x from G if it satisfies that

$$\|x - g^*\| \le \|x - g\|, \quad \forall g \in G.$$

The set of all best approximations to x from G is denoted by $P_G(x)$, that is,

$$P_G(x) = \{g^* \in G : \ \|x - g^*\| = d(x, G)\},$$

where $d(x, G) = \inf_{g \in G} \|x - g\|$.

Motivated by the work of Rozema and Smith in [1], Amir [2] introduced the concept of RS-sets in a real Banach space and gave the uniqueness results for the restricted Chebyshev center of any compact set with respect to an RS-set. Recently, there are several papers concerned with the uniqueness of the best approximation from an RS-set [3, 4, 5].

In the spirit of Amir's idea of RS-sets in a real Banach space, one natural problem is, whether an RS-set in a complex Banach space can be defined and similar uniqueness results hold? The purpose of the present paper is to introduce the concept of RS-sets in a complex Banach space and investigate the problems of characterization, uniqueness and strong uniqueness of best approximations from an RS-set in a complex Banach space. Some results that are similar to the real case are given. It should be noted that this problem has never been considered before. The related works about approximations from generalized polynomials having restricted ranges in complex-valued continuous function spaces are referred to ones in Smirnov and Smirnov [6] and Li [7].

Received August 13, 2001, Accepted April 19, 2002
Supported in part by the National Natural Science Foundation of China (Grant No. 10271025)

2 Preliminaries

Let B^* denote the closed unit ball of the dual X^* and $\text{ext}B^*$ the set of all extreme points of B^*. For $x \in X$, let $E(x)$ denote the set of all extremal supporting functional at x, i.e.,

$$E(x) = \{a^* \in \text{ext}B^* : \langle a^*, x\rangle = \|x\|\}.$$

Now let us give the definition of an RS-set in a complex Banach space. For the end, recall first the notion of an RS-set in a real Banach space.

Definition 2.1 *An n-dimensional subspace G of a Banach space X, real or complex, is called an interpolating subspace if no non-trivial linear combination of n linearly independent extreme points of the ball B^* annihilates G.*

Definition 2.2 *Let X be a real Banach space and $\{y_1, y_2, \ldots, y_n\}$ n linearly independent elements of X. We call the set*

$$G = \left\{ g = \sum_{i=1}^{n} c_i y_i : \quad c_i \in J_i \right\} \tag{2.1}$$

a real RS-set if each J_i is a subset of the real $\mathbb{R}$ of one of the following types:

(I) *The whole of $\mathbb{R}$;*

(II) *A non-trivial proper closed (bounded or unbounded) interval of $\mathbb{R}$;*

(III) *A singleton;*

and in addition every subset of $\{y_1, y_2, \ldots, y_n\}$ consisting of all y_i with J_i of type (I) *and some y_i with J_i of type* (II) *spans an interpolating subspace.*

Note that the subset of type (II) is, indeed, a non-trivial proper closed convex subset of R with non-empty interior. This motivates us to define an RS-set in a complex Banach space.

Definition 2.3 *Let X be a complex Banach space and $\{y_1, y_2, \ldots, y_n\}$ n linearly independent elements of X. We call the set G defined by* (2.1) *an RS-set if each J_i is a subset of the complex plane $\mathbb{C}$ of one of the following types:*

(I) *The whole of $\mathbb{C}$;*

(II) *A non-trivial proper closed convex (bounded or unbounded) subset with non-empty interior of $\mathbb{C}$;*

(III) *A singleton;*

and in addition every subset of $\{y_1, y_2, \ldots, y_n\}$ consisting of all y_i with J_i of type (I) *and some y_i with J_i of type* (II) *spans an interpolating subspace.*

In the next sections, we assume that X is a complex Banach space and G a complex RS-set of X.

3 Characterizations of Best Approximations

Set $I_0 = \{i : \text{ if } J_i \text{ is of type III}\}$, $I_1 = \{i : \text{ if } J_i \text{ is of type II}\}$. For $i \in I_1$, let $F_i(\cdot)$ be a convex function defined on the complex plane $\mathbb{C}$ such that

$$\partial J_i = \{z \in \mathbb{C} : \quad F_i(z) = 0\}, \quad \text{int}J_i = \{z \in \mathbb{C} : \quad F_i(z) < 0\},$$

where ∂J_i and $\text{int}J_i$ denote the boundary and the interior of J_i, respectively. Note that such a convex function exists since the convex set J_i has a non-empty interior.

To give characterizations of best approximations from an RS-set, we need to introduce the concepts of the subdifferential and directional derivative of a real function.

Definition 3.1 *Let F be a real function defined on $\mathbb{C}$ and $z,\ u \in \mathbb{C}$. The subdifferential of F at z, denoted by $\partial F(z)$, is defined by*

$$\partial F(z) = \{u \in \mathbb{C} :\ F(v) \geq F(z) + \text{Re}(v - z)\overline{u},\ \forall v \in \mathbb{C}\},$$

while the directional derivative of F at z with respect to u, denoted by $F'(z)(u)$, is defined by

$$F'(z)(u) = \lim_{t \to +0} \frac{F(z + tu) - F(z)}{t}.$$

As is well known [8], if F is convex then $\partial F(z)$ is a non-empty closed bounded convex set in $\mathbb{C}$ and

$$F'(z)(u) = \max \text{Re}\overline{\partial F(z)}u. \tag{3.1}$$

The following proposition, of which the proof is direct, is useful in the rest of the paper.

Proposition 3.1 *Let $z^* \in \mathbb{C}$ satisfy that $F(z^*) = 0$ and $z \in \mathbb{C}$. If $F(z) \leq 0 (< 0)$, then*

$$\max \text{Re}\,\overline{\partial F(z^*)}(z - z^*) \leq 0 (< 0). \tag{3.2}$$

Now, for $g = \sum_{i=1}^n c_i y_i$, define

$$c_i(g) = c_i, \quad I(g) = \{i \in I_1 :\ c_i(g) \in \partial J_i\}, \quad \sigma_i(g) = -\partial F_i(c_i(g)), \quad \forall i \in I_1.$$

Let

$$P = \left\{g \in \text{span}\{y_1, \ldots, y_n\} :\ \ c_i(g) = 0,\ \forall i \in I_0\right\}.$$

Then we are ready to state the main theorem of this section.

Theorem 3.1 *Let $x \in X,\ g^* \in G$. Then the following statements are equivalent:*

i) $g^* \in P_G(x)$;

ii) *For any $g \in P$,*

$$\max\Big\{\max_{a^* \in E(x-g^*)} \text{Re}\langle a^*, g\rangle,\ \max_{i \in I(g^*)} \max \text{Re} c_i(g)\overline{\sigma_i(g^*)}\Big\} \geq 0, \tag{3.3}$$

where $c_i(g)\overline{\sigma_i(g^*)}$ means $\{c_i(g)\overline{\sigma} :\ \ \sigma \in \sigma_i(g^*)\}$;

iii) *There exist $A(x-g^*) = \{a_1^*, a_2^*, \ldots, a_k^*\} \subset E(x-g^*)$, $B(g^*) = \{i_1, i_2, \ldots, i_m\} \subset I(g^*)$, $\sigma_{i_j} \in \sigma_{i_j}(g^*)$, $j = 1, \ldots, m$ $(k \geq 1,\ k + m \leq 2\text{dim}P + 1)$ and positive scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$; $\lambda_1', \lambda_2', \ldots, \lambda_m'$, such that*

$$\sum_{i=1}^k \lambda_i \langle a_i^*, g\rangle + \sum_{j=1}^m \lambda_j' c_{i_j}(g)\overline{\sigma}_{i_j} = 0, \quad \forall g \in P. \tag{3.4}$$

Proof i)$\Longrightarrow$ii) Since it is trivial when $x \in G$, we assume that $x \in X \setminus G$. Suppose that the condition (3.3) does not hold for some $g \in P$. Then,

$$\text{Re}\,\langle a^*, g\rangle < 0, \quad \forall a^* \in E(x - g^*) \tag{3.5}$$

and

$$\max \text{Re}\, c_i(g)\overline{\sigma_i(g^*)} < 0, \quad \forall i \in I(g^*). \tag{3.6}$$

Write $g_t = g^* - tg$. It follows from (3.1) and (3.6) that

$$\lim_{t\to 0+} \frac{F_i(c_i(g_t)) - F_i(c_i(g^*))}{t} < 0,$$

for all $i \in I(g^*)$ so that, for each $i \in I(g^*)$, there is $t_i > 0$ such that

$$F_i(c_i(g_t)) < 0, \quad \forall\, 0 < t \le t_i. \tag{3.7}$$

Taking into account that $g^*(t) \in \mathrm{int}J_i$ for all $i \notin I(g^*) \cup I_0$, we obtain that, for each $i \notin I_0$, there is $t_i > 0$ such that (3.7) holds. Set $t_0 = \min_{i\notin I_0} t_i$. Then $g_t \in G$ for all $0 < t \le t_0$. Observe that (3.5) implies that g^* is not a best approximation to x from the convex set $G_0 = \{g_t :\ 0 \le t \le t_0\}$. One has that there exists $0 < t \le t_0$ such that $\|x - g_t\| < \|x - g^*\|$, which contradicts i) and proves the implication i)$\Longrightarrow$ii).

ii)$\Longrightarrow$iii) Set

$$\mathcal{U} = \{\mathbf{b}(a^*) = (\langle a^*, y_1\rangle, \langle a^*, y_2\rangle, \ldots, \langle a^*, y_n\rangle) :\ \ a^* \in \overline{E(x-g^*)}\} \bigcup \mathcal{C},$$

where

$$\mathcal{C} = \bigcup_{i\in I(g^*)} \{\mathbf{c}(i) = (c_i(y_1), c_i(y_2), \ldots, c_i(y_n))\sigma_i(g^*)\}.$$

Note that

$$\max \mathrm{Re}\, \overline{\partial F_i(z)} u \le F_i(z+u) - F_i(z), \quad \forall u \in \mathbb{C}.$$

It follows that $\sigma_i(g^*) = -\partial F_i(c_i(g^*))$ is uniformly bounded on $I(g^*)$. This implies that $\mathcal{C}$ is compact and so is $\mathcal{U}$. Thus from ii) and the linear inequality theorem in [9], we get that the origin of the space $\mathbb{C}^n$ belongs to the convex hull of the set $\mathcal{U}$. In view of Caratheodory's theorem in [9], one can apply Krein–Milman Theorem to find $\{a_1^*, \ldots, a_k^*\} \subset E(x-g^*)$, $\{i_1, \ldots, i_m\} \subset I(g^*)$, $\mathbf{c}_s(i_j) \in \mathbf{c}(i_j)$, $s = 1, \ldots, m_j, j = 1, \ldots, m$ and positive scalars $\lambda_1, \ldots, \lambda_k$, λ'_{js}, $s = 1, \ldots, m_j$, $j=1, \ldots, m$ such that

$$\sum_{l=1}^{k} \lambda_l + \sum_{j=1}^{m}\sum_{s=1}^{m_j} \lambda'_{js} = 1,\ \sum_{l=1}^{k} \lambda_l \mathbf{b}(a_l^*) + \sum_{j=1}^{m}\sum_{s=1}^{m_j} \lambda'_{js}\mathbf{c}_s(i_j) = 0,\ \ k + \sum_{j=1}^{m} m_j \le 2\mathrm{dim}P + 1. \tag{3.8}$$

Assume $\mathbf{c}_s(i_j) = (c_{i_j}(y_1), c_{i_j}(y_2), \ldots, c_{i_j}(y_n))\sigma_{js}$ for some $\sigma_{js} \in \sigma_{i_j}(g^*)$, $s = 1, \ldots, m_j$, $j = 1, \ldots, m$. Set

$$\lambda'_j = \sum_{s=1}^{m_j} \lambda'_{js}, \quad \sigma_{i_j} = \frac{\sum_{s=1}^{m_j} \lambda'_{js}\sigma_{js}}{\lambda'_j}, \quad j = 1, \ldots, m.$$

By the convexity of $\sigma_{i_j}(g^*)$, it follows that $\sigma_{i_j} \in \sigma_{i_j}(g^*)$. Then we obtain (3.4) from (3.8). Let $(c_1^0, c_2^0, \ldots, c_n^0)$ satisfy $c_i^0 = 0$, $\forall i \in I_0$, $c_i^0 \in \mathrm{int}J_i$, $\forall i \in I_1$ and $g_0 = \sum_{i=1}^{n} c_i^0 y_i$. It follows that

$$\mathrm{Re}\, c_{i_j}(g_0 - g^*)\overline{\sigma}_{i_j} > 0, \quad j = 1, \ldots, m.$$

This implies that $k \ge 1$. The proof of ii)$\Longrightarrow$iii) is complete.

iii)$\Longrightarrow$i) Suppose that i) does not hold. Then there exists an element $g_1 \in G$ such that $c_i(g_1) \in \mathrm{int}J_i$ for all $i \in I_1$ and $\|x - g_1\| < \|x - g^*\|$. It follows from Proposition 3.1 that

$$\max \mathrm{Re}\, c_i(g^* - g_1)\overline{\sigma_i(g^*)} < 0, \quad \forall\, i \in I(g^*). \tag{3.9}$$

From

$$\mathrm{Re}\,\langle a^*, x - g_1\rangle < \mathrm{Re}\,\langle a^*, x - g^*\rangle, \quad \forall\, a^* \in E(x - g^*),$$

we have that

$$\mathrm{Re}\,\langle a^*, g^* - g_1\rangle < 0, \quad \forall\, a^* \in E(x - g^*). \tag{3.10}$$

Clearly, $g_0 = g^* - g_1 \in P$. However, (3.9) and (3.10) imply that (3.4) does not hold for g_0, that is, iii) does not hold. The proof of the theorem is complete.

4 Uniqueness and Strong Uniqueness of Best Approximations

Lemma 4.1 *Suppose G is a complex RS-set in X, $x \in X \backslash G$ and $g^* \in P_G(x)$. Let $A(x-g^*) = \{a_1^*, \dots, a_k^*\} \subset E(x - g^*)$ and $B(g^*) = \{i_1, \dots, i_m\} \subset I(g^*)$ such that* (3.4) *holds. Then there are at least* $\mathrm{dim}P - m$ *linearly independent elements in $A(x - g^*)$.*

Proof Let positive numbers $\lambda_1, \dots, \lambda_k,\ \lambda_1', \dots, \lambda_m',\ A(x - g^*)$ and $B(g^*)$ be such that (3.4) holds. Set

$$Q = \Big\{g = \sum_{i=1}^{n} c_i y_i \in P:\ c_{i_j} = 0,\ j = 1, \dots, m\Big\}.$$

Then Q is an interpolating subspace of dimension $N = \mathrm{dim}P - m$. With no loss of generality, we may assume that $a_1^*, \dots, a_{l'}^*$ are linearly independent and (3.4) can be rewritten as

$$\sum_{i=1}^{l'} \tilde{\lambda}_i \langle a_i^*, g\rangle + \sum_{j=1}^{m} \lambda_j' \sigma_{i_j} c_{i_j}(g) = 0, \quad \forall\, g \in P. \tag{4.1}$$

To complete the proof, it suffices to show that $l' \geq N$. Suppose on the contrary that $l' < N$. Since Q is an interpolating subspace of dimension $N = \mathrm{dim}P - m$, there exists an element $g_0 \in Q \setminus \{0\}$ such that $\tilde{\lambda}_i \langle a_i^*, g_0\rangle = |\tilde{\lambda}_i|^2,\ i = 1, \dots, l'$. This with (4.1) implies that $\tilde{\lambda}_i = 0,\ i = 1, \dots, l'$. Hence, $m \geq 1$ since $x \in X \backslash G$. Thus, by (4.1),

$$\sum_{j=1}^{m} \lambda_j' \sigma_{i_j} c_{i_j}(g) = 0, \quad \forall\, g \in P. \tag{4.2}$$

Note that there exists $g = \sum_{i=1}^{n} c_i g_i \in P$ such that $c_i \in \mathrm{int}J_i$ for each $i \notin I_0$. Then (4.2) does not hold and we have a contradiction. The proof is complete.

Recall that a convex subset J of C is strictly convex if, for any two distinct elements $z_1, z_2 \in J$, $\frac{1}{2}(z_1 + z_2) \in \mathrm{int}J$.

Theorem 4.1 *Let G be a complex RS-set in X. Suppose that J_i is strictly convex for each $i \notin I_0$. Then, for each $x \in X$, x has a unique best approximation to x from G.*

Proof The case when $x \in G$ is trivial. Now let $x \in X \setminus G$. Suppose that x has two best approximations $g_1^*,\ g_2^*$. Write $g^* = (g_1^* + g_2^*)/2$. Then, using standard techniques, we have that

$$E(x - g^*) \subset E(x - g_1^*) \cap E(x - g_2^*) \subset \{a^* \in \mathrm{ext}B^*:\ \ \langle a^*, g_1^* - g_2^*\rangle = 0\}$$

and $I(g^*) \subset I(g_1^*) \cap I(g_2^*)$. This implies that $c_i(g_1^*) = c_i(g_2^*)$ by the strict convexity of J_i for each $i \in I(g^*)$. Let

$$P_0 = \{g \in P:\ \ c_i(g) = 0,\ \ i \in I(g^*)\}.$$

In view of the definition of a complex *RS*-set, P_0 is an interpolating subspace of dimension $\mathrm{dim}P - |I(g^*)|$, where $|I(g^*)|$ denotes the cardinality of the set $I(g^*)$. Clearly, $g_1^* - g_2^* \in P_0$. It follows from Lemma 4.1 that $g_1^* - g_2^* = 0$ and so the proof is complete.

Remark 4.1 Because of the convexity of G, the conclusion of Theorem 4.1 is clearly true if X is a strictly convex Banach space whenever J_i is strictly convex or not for each $i \notin I_0$. However, if X is not strictly convex, the conclusion of Theorem 4.1 may not be true without the strict convexity assumption of J_i, as shown in the following example:

Example 4.1 Let $Q = \{-1, 0, 1\}$ and $X = C(Q)$, the complex continuous function space defined on Q with the uniform norm. Define $g_1 = 1$, $g_2 = t - \frac{1}{2}$, $\forall\, t \in Q$, $J_1 = \{z \in C : \operatorname{Re} z \geq 1\}$, $J_2 = \{z \in C : \operatorname{Re} z \leq 1\}$ and $f(t) = \begin{cases} -\frac{1}{2}, & t=-1, \\ 0, & t=0, \\ \frac{3}{2}, & t=1. \end{cases}$ Then, $G = \{g = c_1 + c_2(t - \frac{1}{2}) : \operatorname{Re} c_1 \geq 1,\ \operatorname{Re} c_1 \leq 1\}$. Take

$$g_1^* = 1 + \Big(t - \frac{1}{2}\Big), \quad g_2^* = 1 + \frac{i}{8} + \Big(1 + \frac{i}{4}\Big)\Big(t - \frac{1}{2}\Big).$$

Obviously, $\|f - g_1^*\| = |(f - g_1^*)(0)| = \frac{1}{2}$. Since, for any $g = c_1 + c_2(t - \frac{1}{2}) \in G$,

$$\|f - g\| \geq |(f - g)(0)| = \Big|c_1 - \frac{1}{2}c_2\Big| \geq \frac{1}{2},$$

we have that $g_1^* \in P_G(f)$. On the other hand, it is easy to check that

$$\|f - g_2^*\| \leq |(f - g_2^*)(0)| = \frac{1}{2},$$

so that $g_2^* \in P_G(f)$.

Now let's consider the strong uniqueness of the best approximation to f from G. The following definition is well known.

Definition 4.1 *Let $x \in X$ and $g^* \in P_G(x)$. g^* is called strongly unique of order $\alpha > 0$ if there exists a constant $c_\alpha > 0$ such that $\|x - g\|^\alpha \geq \|x - g^*\|^\alpha + c_\alpha \|g - g^*\|^\alpha$, $\forall\, g \in G$.*

Theorem 4.2 *Let G be a complex RS-set. Suppose that ∂J_i has a positive curvature at z^* for any $i \in I_1$, $z^* \in \partial J_i$. Then each $x \in X$ has a strongly unique best approximation of order 2 to x from G.*

Proof The proof for the case when $x \in G$ is trivial, so that we assume that $x \notin G$. Let g^* be the unique best approximation to x from G. Then it follows from Theorem 3.1 that there exist sets $A(x - g^*) = \{a_1^*, a_2^*, \ldots, a_k^*\} \subset E(x - g^*)$, $B(g^*) = \{i_1, i_2, \ldots, i_m\} \subset I(g^*), \sigma_{i_j} \in \sigma_{i_j}(g^*)$, $j = 1, \ldots, m$ ($k \geq 1$, $k + m \leq 2\dim P + 1$) and positive scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$, $\lambda_1', \lambda_2', \ldots, \lambda_m'$ such that (3.4) holds. Without loss of generality, we may take $\lambda_1, \lambda_2, \ldots, \lambda_k$ to satisfy $\sum_{i=1}^k \lambda_i = 1$. For $j = 1, 2, \ldots, m$, let $\kappa_{i_j} > 0$ and u_{i_j} denote the curvature and the center of curvature of J_{i_j} at $c_{i_j}(g^*)$, respectively. Define $\bar{c}_{i_j} = 2u_{i_j} - c_{i_j}(g^*)$, $r_{i_j} = 2|u_{i_j} - c_{i_j}(g^*)| = 2/\kappa_{i_j}$ for $j = 1, 2, \ldots, m$. Then there exists a neighborhood U_{i_j} of $c_{i_j}(g^*)$ such that

$$|z - \bar{c}_{i_j}| \leq r_{i_j}, \quad \text{for all } z \in J_{i_j} \cap U_{i_j},\ j = 1, 2, \ldots, m. \tag{4.3}$$

Observe that, for any $i_j \in B(g^*)$ and $\sigma \in \sigma_{i_j}(g^*)$, $\sigma = d(\bar{c}_{i_j} - c_{i_j}(g^*))$ for some $d > 0$. Without loss of generality, assume that $d = 1$. Also we may assume that $\|x - g^*\| = 1$. It follows from (3.4) that

$$\sum_{l=1}^k \lambda_l \langle a_l^*, g\rangle + \sum_{j=1}^m \lambda_j' c_{i_j}(g) \overline{(\bar{c}_{i_j} - c_{i_j}(g^*))} = 0, \quad \forall\, g \in P. \tag{4.4}$$

For any $g \in P$, set

$$\|g\|_2 = \left(\sum_{l=1}^{k} \lambda_l |\langle a_l^*, g\rangle|^2 + \sum_{j=1}^{m} \lambda_j' |c_{i_j}(g)|^2 \right)^{1/2}.$$

It is easy to see that $\|\cdot\|_2$ is a norm on P so that it is equivalent to the original norm. Consequently, there exists a constant $\gamma > 0$ such that $\|g\|_2 \geq \gamma\|g\|$, $\forall g \in P$. Set

$$\gamma(g) = \frac{\|x-g\|^2 - \|x-g^*\|^2}{\|g-g^*\|^2}, \quad \forall\, g \in G,\ g \neq g^*.$$

We will show that $\gamma(g)$ has positive lower bounds on $G \setminus \{g^*\}$. Suppose on the contrary that there exists a sequence $\{g_n\} \subset G$ such that $\gamma(g_n) \to 0$. Then $\|x-g_n\| \to \|x-g^*\|$. With no loss of generality, we may assume that $g_n \to g^*$ due to the uniqueness of the best approximation. It follows from (4.3) that $|c_{i_j}(g_n) - \bar{c}_{i_j}| \leq r_{i_j}$, $\forall\, i_j \in B(g^*)$, for all n large enough. This with (4.4) implies that

$$\begin{aligned} \|x-g_n\|^2 &\geq \sum_{l=1}^{k} \lambda_l |\langle a_l^*, x-g_n\rangle|^2 + \sum_{j=1}^{m} \lambda_j' |\bar{c}_{i_j} - c_{i_j}(g_n)|^2 - \sum_{j=1}^{m} \lambda_j' r_{i_j}^2 \\ &= \sum_{l=1}^{k} \lambda_l |\langle a_l^*, x-g^*\rangle|^2 + \sum_{l=1}^{k} \lambda_l |\langle a_l^*, g^*-g_n\rangle|^2 + \sum_{j=1}^{m} \lambda_j' |c_{i_j}(g^*) - c_{i_j}(g_n)|^2 \\ &= \|x-g^*\|^2 + \|g^*-g_n\|_2^2 \geq \|x-g^*\|^2 + \gamma^2\|g^*-g_n\|^2. \end{aligned}$$

This means that $\gamma(g_n) \geq \gamma^2$, which contradicts that $\gamma(g_n) \to 0$ and completes the proof.

In order to give the more general strong uniqueness theorems, we introduce the notion of a uniformly convex function and some useful properties, see, for example, [10].

Definition 4.2 *A function $F : \mathbb{C} \to \mathbb{R}$ is uniformly convex at $z^* \in \mathbb{C}$ if there exists $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\delta(x) > 0$ for $x > 0$ such that*

$$F(\lambda z^* + (1-\lambda)z) \leq \lambda F(z^*) + (1-\lambda)F(z) - \lambda(1-\lambda)\delta(|z^*-z|), \quad \forall z \in \mathbb{C},\ \ 0 < \lambda < 1.$$

Define the modulus of convexity of F at z^*

$$\mu_{z^*}(x) = \inf \frac{\lambda F(z^*) + (1-\lambda)F(z) - F(\lambda z^* + (1-\lambda)z)}{\lambda(1-\lambda)},$$

where the infimum is taken over all $z \in \mathbb{C}$ and λ satisfying $|z^*-z| = x,\ 0 < \lambda < 1$. Clearly, F is uniformly convex at z^* if and only if $\mu_{z^*}(x) > 0$ for $x > 0$.

Definition 4.3 *A function $F : \mathbb{C} \to \mathbb{R}$ has the modulus of convexity of order $\alpha > 0$ at $z^* \in \mathbb{C}$ if there exists $d_\alpha > 0$ such that $\mu_{z^*}(x) > d_\alpha x^\alpha$ for $x > 0$.*

Proposition 4.1 *A function $F : \mathbb{C} \to \mathbb{R}$ has the modulus of convexity of order $\alpha > 0$ at $z^* \in \mathbb{C}$ if and only if there exists $d_\alpha > 0$ such that $F(z) \geq F(z^*) + \mathrm{Re}(z-z^*)\overline{u} + d_\alpha|z-z^*|^\alpha$, $\forall z \in \mathbb{C},\ u \in \partial F(z^*)$.*

Theorem 4.3 *Let G be a complex RS-set. Suppose that, for any $i \in I_1,\ z^* \in \partial J_i,\ F_i(\cdot)$ has the modulus of convexity of order $\alpha > 0$ at z^*. Then each $x \in X$ has a strongly unique best approximation of order $r = \max\{\alpha, 2\}$ to x from G.*

Proof As in the proof of Theorem 4.1, we assume that $x \notin G$, $g^* \in P_G(x)$ and $\|x-g^*\| = 1$. Let $A(x-g^*) = \{a_1^*, a_2^*, \ldots, a_k^*\} \subset E(x-g^*)$, $B(g^*) = \{i_1, i_2, \ldots, i_m\} \subset I(g^*), \sigma_{i_j} \in \sigma_{i_j}(g^*),\ j =$

$1, \ldots, m$ ($k \geq 1$, $k+m \leq 2\dim P + 1$) and positive scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$; $\lambda'_1, \lambda'_2, \ldots, \lambda'_m$ be such that (3.4) holds and $\sum_{i=1}^{k} \lambda_i = 1$. For any $g \in P$, set

$$\|g\|_r = \left(\sum_{l=1}^{k} \lambda_l |\langle a_l^*, g\rangle|^r + \sum_{j=1}^{m} \lambda'_j |c_{i_j}(g)|^r \right)^{1/r}.$$

Again $\|\cdot\|_r$ is a norm on P equivalent to the original norm so that $\|g\|_r \geq \gamma\|g\|$, $\forall g \in P$ for some constant $\gamma > 0$. Set $\gamma_r(g) = \frac{\|x-g\|^r - \|x-g^*\|^r}{\|g-g^*\|^r}$, $\forall g \in G,\ g \neq g^*$. Then $\gamma_r(g)$ has positive lower bounds on $G \setminus \{g^*\}$. In fact, if otherwise, there exists a sequence $\{g_n\} \subset G$ such that $\gamma_r(g_n) \to 0$. Then $\|x - g_n\| \to \|x - g^*\|$. With no loss of generality, we may assume that $g_n \to g^*$ since $P_G(x)$ is a singleton. From Proposition 3.1 and (3.4), we have that

$$\begin{aligned}
\|x - g_n\|^2 \geq & \sum_{l=1}^{k} \lambda_l |\langle a_l^*, x - g_n\rangle|^2 + 2\sum_{j=1}^{m} \lambda'_j \mathrm{Re}\,(c_{i_j}(g^*) - c_{i_j}(g_n))\overline{\sigma}_{i_j} + 2d_\alpha \sum_{j=1}^{m} \lambda'_j |c_{i_j}(g^*) - c_{i_j}(g_n)|^\alpha \\
= & \|x - g^*\|^2 + \sum_{l=1}^{k} \lambda_l |\langle a_l^*, g_n - g^*\rangle|^2 + 2d_\alpha \sum_{i=1}^{m} \lambda'_i |c_{i_j}(g^*) - c_{i_j}(g_n)|^\alpha \\
\geq & \|x - g^*\|^2 + \sum_{l=1}^{k} \lambda_l |\langle a_l^*, g_n - g^*\rangle|^r + 2d_\alpha \sum_{i=1}^{m} \lambda'_i |c_{i_j}(g^*) - c_{i_j}(g_n)|^r \\
\geq & \|x - g^*\|^2 + \min\{1, 2\,d_\alpha\}\|g_n - g^*\|_r^r \geq \|x - g^*\|^2 + \min\{1, 2\,d_\alpha\}\gamma^r \|g_n - g^*\|^r,
\end{aligned}$$

for all n large enough. Since

$$\|x - g_n\|^r - \|x - g^*\|^r \geq (r/2)\|x - g^*\|^{r-2} \left(\|x - g_n\|^2 - \|x - g^*\|^2\right),$$

it follows that $\gamma_r(g_n) \geq \min\{1, 2d_\alpha\}(r/2)\|x - g^*\|^{r-2}\gamma^r > 0$, which contradicts that $\gamma_r(g_n) \to 0$ and completes the proof.

Remark 4.2 In the case when F_i has a continuous twice derivative, we can show that ∂J_i has a positive curvature at z^* that implies that $F_i(\cdot)$ has the modulus of convexity of order 2 at z^* for any $i \in I_1$, $z^* \in \partial J_i$. Hence, in this case, Theorem 4.2 is a direct consequence of Theorem 4.3.

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