Strong uniqueness of best approximations in spaces of bounded linear operators

LI Chong[1] & HE Jinsu[2]

1. Department of Mathematics, Zhejiang University, Hangzhou 310027, China;
2. Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China
Correspondence should be addressed to Li Chong (email: cli@zju.edu.cn)

Received February 10, 2003

Abstract The present paper is concerned with problems of the strong uniqueness of the best approximation and the characterization of a uniqueness element in operator spaces. Some results on the strong uniqueness of the best approximation operator from RS-sets are proved and the uniqueness element of a sun in the compact operator space from c_0 to c_0 is characterized by the strict Kolmogorov's condition. Some recent results due to Lewicki and others are extended and improved.

Keywords: best approximation operator, RS-set, sun, strong uniqueness, uniqueness element.

DOI: 10.1360/02ys0336

Let X and Y be two real Banach spaces and let $\mathcal{L}(X,Y)$ and $\mathcal{K}(X,Y)$ denote the space of all bounded and compact linear operators from X to Y, respectively. Let $\mathcal{G}$ be a closed subset in $\mathcal{L}(X,Y)$. For an operator $L \in \mathcal{L}(X,Y)$, an element $V^* \in \mathcal{G}$ is called a best approximation operator to L from $\mathcal{G}$ if it satisfies

$$\|L - V^*\| \leqslant \|L - V\|, \quad \forall V \in \mathcal{G}.$$

The set of all best approximation operators to L from $\mathcal{G}$ is denoted by $P_{\mathcal{G}}(L)$, that is,

$$P_{\mathcal{G}}(L) = \{V^* \in \mathcal{G} : \ \|L - V^*\| = d(L,\mathcal{G})\},$$

where

$$d(L,\mathcal{G}) = \inf_{V\in\mathcal{G}} \|L - V\|.$$

The problem of best approximation in the space of bounded linear operators has a long history (see, for example, refs. [1—6]). Some applications to minimal projections in Banach spaces can be found in ref. [7] while some recent results on the uniqueness of minimal projections are given in refs. [8—10]. Recently, Lewicki obtained characterization theorems of a best approximation operator from a finite dimensional subspace of $\mathcal{K}(X,Y)$ and $\mathcal{L}(X,Y)$[1,3]. At the same time, Li[4] considered the nonlinear best approximation problem in $\mathcal{L}(X,Y)$ and characterized suns of $\mathcal{K}(X,Y)$ and $\mathcal{L}(X,Y)$. The results generalize those results in $\mathcal{K}(c_0,c_0)$ and $\mathcal{K}(C,C)$ due to Malbrock [5,6]. In addition, Lewicki also obtained some special results on strong uniqueness of the best approximation operator from finite dimensional subspaces of $\mathcal{L}(X,Y)$ and $\mathcal{K}(c_0,c_0)$. However, up to now, the strong uniqueness problem for the nonlinear cases in $\mathcal{K}(X,Y)$ and $\mathcal{L}(X,Y)$ has not been explored yet. The purpose of the present paper is to investigate some problems in this direction. The study is mainly concerned with two aspects. One is with the strong uniqueness of the best approximation from RS-sets in $\mathcal{L}(X,Y)$ and the other with the characterization of a uniqueness element of a sun by the strict Kolmogorov condition in the space $\mathcal{K}(c_0,c_0)$.

1 Strong uniqueness in Banach spaces

In order to establish the strong uniqueness of the best approximation operator, we first give a generalized strong uniqueness theorem for a polyhedron in an arbitrary Banach space X. The theorems and their proofs are similar to those in ref. [11], but we still write them here for completeness.

Definition 1.1. A closed convex subset $\mathcal{G}$ is called a polyhedron if it is the intersection of a finite number of closed half-spaces, that is,

$$\mathcal{G}=\bigcap_{i=1}^{k}\{x\in X:\ \ \langle x_i^*,x\rangle\leqslant d_i\},$$

for some $x_i^*\in X^*\setminus\{0\}$ and real scalars d_i. $\mathcal{G}$ is called a polyhedron of finite dimensions if it is the intersection of a finite dimensional subspace with a polyhedron.

Now let $\mathcal{G}$ be a nonempty closed subset in the Banach space X and let M be a subspace containing $\mathcal{G}$. Let $\mathbb{B}^*$ denote the closed unit ball of the dual X^*. As in refs. [12, 13], we endow $\mathbb{B}^*$ with the $\sigma(X^*,M)$-topology: for any net $\{x_\alpha^*\}\subseteq\mathbb{B}^*$, $x_0^*\in\mathbb{B}^*$, $x_\alpha^*\to x_0^*$ in the $\sigma(X^*,M)$-topology if and only if $\langle x_\alpha^*,x\rangle\to\langle x_0^*,x\rangle$ for any $x\in M$. For a subset K of $\mathbb{B}^*$, we use $\overline{K^\sigma}$ to denote the closure of K in the $\sigma(X^*,M)$-topology while $\mathrm{ext}K$ stands for the set of all extreme points from K when K is a closed convex subset. In the remainder of this section, let x be a fixed element of X. Take a convex set $K\subseteq\mathbb{B}^*$ such that

(i) K is $\sigma(X^*,M)$-compact;

(ii) for any $g\in\mathcal{G}$, $\sup_{x^*\in K}\langle x^*,x-g\rangle=\|x-g\|$.

Define

$$P_{\mathcal{G}}(x)=\left\{g_0\in\mathcal{G}:\ \|x-g_0\|=\inf_{g\in\mathcal{G}}\|x-g\|\right\}$$

and

$$E_{x-g_0}=\{x^*\in\mathrm{ext}K:\ \langle x^*,x-g_0\rangle=\|x-g_0\|\}.$$

Then we have the following theorem.

Theorem 1.1. Let $\mathcal{G}$ be a polyhedron of finite dimensions of X. Suppose that $g_0\in P_{\mathcal{G}}(x)$ and

$$\max_{a^*\in E_{x-g_0}}\langle a^*,g_0-g\rangle>0,\quad\forall g\in\mathcal{G}\setminus\{g_0\}.\tag{1.1}$$

Then g_0 is strongly unique; that is, there exists a constant $r > 0$ such that

$$\|x - g\| \geqslant \|x - g_0\| + r\|g - g_0\|, \quad \forall g \in \mathcal{G}.$$

Proof. Assume that

$$\mathcal{G} = \bigcap_{i=1}^{k} \{x \in X : \ \langle x_i^*, x\rangle \leqslant d_i\}$$

for some $x_i^* \in X^* \setminus \{0\}$ and real scalars d_i, $i = 1, 2, \cdots, k$. Set $I = \{1, 2, \cdots, k\}$,

$$I_0 = \{i \in I : \ \langle x_i^*, g\rangle = d_i \text{ for all } g \in \mathcal{G}\}$$

and

$$H_i = \{g \in \mathcal{G} : \ \langle x_i^*, g\rangle = d_i\}, \quad \forall i \in I.$$

For $g \in \mathcal{G}$, define

$$J(g) = \{i \in I : \ g \in H_i\}.$$

In addition, if $J(g_0) \neq I$, define

$$\mathcal{G}_0 = \bigcup_{i \notin J(g_0)} H_i.$$

Then $\mathcal{G}_0$ is a closed nonempty subset of $\mathcal{G}$ so that $t^* = d(g_0, \mathcal{G}_0) > 0$. Let

$$\lambda^* = \begin{cases} t^*, & \text{if } J(g_0) \neq I, \\ 1, & \text{if } J(g_0) = I \end{cases}$$

and

$$g_\lambda = \left(1 - \frac{\lambda}{\|g_0 - g\|}\right) g_0 + \frac{\lambda}{\|g_0 - g\|} g,$$

where $g \in \mathcal{G} \setminus \{g_0\}$. We will show that $g_{\lambda^*} \in \mathcal{G}$. In fact, it is clear, if $J(g_0) = I$. Then we may assume that $J(g_0) \neq I$. For convenience, define $l_i(\lambda) = \langle x_i^*, g_\lambda\rangle$ for any $i \in I$ and $I^+ = \{i \in I : \ l_i(\lambda^*) > d_i\}$. Then if $g_{\lambda^*} \notin \mathcal{G}$, we have

$$I^+ \neq \varnothing, \quad I^+ \bigcap J(g_0) = \varnothing,$$

since $l_i(\lambda^*) \leqslant d_i$ for all $i \in J(g_0)$. For any $i \in I^+$, take $0 < \lambda_i < t^*$ satisfying $l_i(\lambda_i) = d_i$ and let $\lambda = \min_{i \in I^+} \lambda_i$. From $l_i(\lambda) = d_i$ for any $i \in I_0$, it follows that $g_\lambda \in \partial\mathcal{G}$, the relative boundary of $\mathcal{G}$, and $J(g_\lambda) \cap I^+ \neq \varnothing$. Take $i_0 \in J(g_\lambda) \cap I^+$. Then $g_\lambda \in H_{i_0}$ but $g_0 \notin H_{i_0}$. This implies that $g_\lambda \in \partial\mathcal{G}$ and hence

$$t^* = d(g_0, \mathcal{G}_0) \leqslant \|g_0 - g_\lambda\| = \lambda < t^*.$$

This is a contradiction, so $g_{\lambda^*} \in \mathcal{G}$.

Now define

$$r(g) = \max_{a^* \in E_{x - g_0}} \frac{\lambda^* \langle a^*, g_0 - g\rangle}{\|g_0 - g\|}.$$

Then $r = \inf_{g \in \mathcal{G} \setminus \{g_0\}} r(g) > 0$. In fact, suppose on the contrary that there exists a sequence $\{g_n\} \subset \mathcal{G} \setminus \{g_0\}$ such that $r(g_n) \to 0$ as $n \to \infty$. Due to compactness, we may assume that $\dfrac{\lambda^*(g_0 - g_n)}{\|g_0 - g_n\|} \to \bar{g} \neq 0$. Observe that

$$g_0 - \frac{\lambda^*(g_0 - g_n)}{\|g_0 - g_n\|} = \left(1 - \frac{\lambda^*}{\|g_0 - g_n\|}\right) g_0 + \frac{\lambda^*}{\|g_0 - g_n\|} g_n = (g_n)_{\lambda^*}.$$

It follows from $(g_n)_{\lambda^*} \in \mathcal{G}$ that $g_0 - \bar{g} \in \mathcal{G} \setminus \{g_0\}$; hence

$$\max_{a^* \in E_{x-g_0}} \langle a^*, g_0 - (g_0 - \bar{g})\rangle = \lim_{n\to\infty} r(g_n) = 0,$$

which contradicts (1.1). Hence $r > 0$. This implies that

$$\max_{a^* \in E_{x-g_0}} \langle a^*, g_0 - g\rangle \geqslant \frac{r}{\lambda^*}\|g_0 - g\|, \quad \forall g \in \mathcal{G}.$$

Let $a_0^* \in E_{x-g_0}$ be such that

$$\langle a_0^*, g_0 - g\rangle = \max_{a^* \in E_{x-g_0}} \langle a^*, g_0 - g\rangle.$$

Then, we have

$$\begin{aligned} \|x - g\| &\geqslant \langle a_0^*, x - g\rangle \\ &= \langle a_0^*, x - g_0\rangle + \langle a_0^*, g_0 - g\rangle \\ &\geqslant \|x - g_0\| + \frac{r}{\lambda^*}\|g_0 - g\| \end{aligned}$$

for all $g \in \mathcal{G}$; that is, g_0 is strongly unique. The proof of the theorem is complete.

Remark 1.1. In general, condition (1.1) is called the strict Kolmogorov condition.

Next we will give the definition of an RS-set on K and consider the strong uniqueness of best approximation from an RS-set on K. The concept of an RS-set was introduced first by Amir[14] to investigate the uniqueness of restricted Chebyshev centers in Banach spaces. The following definition of an RS-set on K, which is convenient to apply, is a generalization of an RS-set. We further assume that $\mathrm{ext}K$ has at least $n+1$ linearly independent points.

Definition 1.2. An n-dimensional subspace $\mathcal{G}$ of the Banach space X is called an interpolating subspace on K if no nontrivial linear combination of n linearly independent points from the set $\mathrm{ext}K$ annihilates $\mathcal{G}$. In particular, $\mathcal{G}$ is called an interpolating subspace if $\mathcal{G}$ is an interpolating subspace on $K = B^*$.

Definition 1.3. Let $\{y_1, y_2, \cdots, y_n\}$ be n linearly independent elements of X. We call the set

$$\mathcal{G} = \left\{ g = \sum_{i=1}^{n} c_i y_i : \ \ c_i \in J_i \right\} \tag{1.2}$$

an RS-set on K if each J_i is a subset of the reals $\mathbb{R}$ of one of the following types:

(I) the whole of $\mathbb{R}$,

(II) a nontrivial proper closed (bounded or unbounded) interval of $\mathbb{R}$,

(III) a singleton, and, in addition, every subset of $\{y_1, y_2, \cdots, y_n\}$ consisting of all y_i with J_i of type (I) and some y_i with J_i of type (II) spans an interpolating subspace on K. In particular, $\mathcal{G}$ is called an RS-set when $K = B^*$.

Clearly, an RS-set on K is a finite dimensional polyhedron. Thus, by Theorem 1.1, in order to prove the strong uniqueness of the best approximation from an RS-set, it suffices to show that condition (1.1) holds. For this purpose, define

$$\alpha_i = \inf J_i, \quad \beta_i = \sup J_i, \quad i = 1, \cdots, n.$$

Then

$$\begin{aligned} &\alpha_i = -\infty, \quad \beta_i = +\infty, && \text{if } J_i \text{ is of type (I);} \\ &-\infty \leqslant \alpha_i < \beta_i \leqslant +\infty, && \text{if } J_i \text{ is of type (II);} \\ &\alpha_i = \beta_i \neq +\infty, && \text{if } J_i \text{ is of type (III).} \end{aligned}$$

Define linear functionals c_i, $i = 1, 2, \cdots, n$, on the space Y spanned by $\{y_1, y_2, \cdots, y_n\}$ as follows:

$$c_i(y) = c_i, \quad \forall y = \sum_{i=1}^{n} c_i y_i, \quad i = 1, 2, \cdots, n.$$

Set

$$\begin{aligned} &I_0 = \{i : \ \alpha_i = \beta_i\}, \\ &I_+(g_0) = \{i : \ c_i(g_0) = \alpha_i\} \setminus I_0, \\ &I_-(g_0) = \{i : \ c_i(g_0) = \beta_i\} \setminus I_0, \\ &I(g_0) = I_+(g_0) \cup I_-(g_0) \end{aligned}$$

and

$$\sigma_i(g_0) = 1, \text{ if } i \in I_+(g_0); \quad \sigma_i(g_0) = -1, \text{ if } i \in I_-(g_0).$$

Finally, let

$$P = \{g \in Y : \ c_i(g) = 0, \ \forall i \in I_0\}.$$

Theorem 1.2. Suppose that $\mathcal{G}$ is an RS-set on K in X. Then the following three statements are equivalent:

(i) $g_0 \in P_{\mathcal{G}}(x)$;

(ii) For any $g \in P$,

$$\max\left\{\max_{a^* \in E_{x-g_0}} \langle a^*, g\rangle, \ \max_{i \in I(g_0)} \sigma_i(g_0) c_i(g)\right\} \geqslant 0; \tag{1.3}$$

(iii) There exist $A(x - g_0) = \{a_1^*, \cdots, a_l^*\} \subseteq E_{x-g_0}$, $B(g_0) = \{i_1, \cdots, i_m\} \subseteq I(g_0)$ and positive numbers $\lambda_1, \cdots, \lambda_l, \lambda_1', \cdots, \lambda_m'$ $(1 + m \leqslant l + m \leqslant \dim P + 1)$ such that

$$\sum_{i=1}^{l} \lambda_i \langle a_i^*, g\rangle + \sum_{j=1}^{m} \lambda_j' \sigma_{i_j}(g_0) c_{i_j}(g) = 0, \quad \forall g \in P. \tag{1.4}$$

Proof. (i)$\Longrightarrow$(ii). Suppose that condition (1.3) does not hold for some $g \in P$. Then

$$\langle a^*, g\rangle < 0, \quad \forall a^* \in E_{x-g_0} \tag{1.5}$$

and

$$c_i(g)\sigma_i(g_0) < 0, \quad \forall i \in I(g_0). \tag{1.6}$$

Write $g^t = g_0 - tg$. It follows from (1.6) that there is $t_0 > 0$ such that $g^t \in \mathcal{G}$ for all $0 < t \leqslant t_0$. Observe that (1.5) implies $g_0 \notin P_{\mathcal{G}^0}(x)$, where $\mathcal{G}^0 = \{g^t : \ 0 \leqslant t \leqslant t_0\}$, so that there exists $0 < t \leqslant t_0$ such that $\|x - g^t\| < \|x - g_0\|$, which contradicts (i) and proves the implication (i)$\Longrightarrow$(ii).

(ii)$\Longrightarrow$(iii). Set

$$\mathcal{U} = \{\mathbf{b}(a^*) : \ \ a^* \in E_{x-g_0}\} \bigcup \{\mathbf{c}(i) : \ \ i \in I(g_0)\},$$

where

$$\mathbf{b}(a^*) = (\langle a^*, y_1\rangle, \langle a^*, y_2\rangle, \cdots, \langle a^*, y_n\rangle),$$

$$\mathbf{c}(i) = (c_i(y_1), c_i(y_2), \cdots, c_i(y_n))\sigma_i(g_0).$$

By (ii) and the linear inequalities theorem in ref. [15], the origin of the space $\mathbb{R}^n$ belongs to the convex hull of the set $\mathcal{U}$. In view of Caratheodory's theorem one can find $A(x - g_0) = \{a_1^*, \cdots, a_l^*\} \subseteq E_{x-g_0}$, $B(g_0) = \{i_1, \cdots, i_m\} \subseteq I(g_0)$ and positive numbers $\lambda_1, \cdots, \lambda_l$, $\lambda_1', \cdots, \lambda_m'$ ($1 + m \leqslant l + m \leqslant \dim P + 1$) such that (1.4) holds. The proof of (ii)$\Longrightarrow$(iii) is complete.

(iii)$\Longrightarrow$ (i). Suppose that (iii) holds but (i) does not hold. Then there exists an element $g_1 \in \mathcal{G}$ such that $c_i(g_1) \in (\alpha_i, \beta_i)$ for all $i \notin I_0$ and $\|x - g_1\| < \|x - g_0\|$. Consequently,

$$\langle a^*, x - g_1\rangle < \langle a^*, x - g_0\rangle, \quad \forall a^* \in E_{x-g_0}$$

and hence,

$$\langle a^*, g_0 - g_1\rangle < 0, \quad \forall a^* \in E_{x-g_0}. \tag{1.7}$$

In addition, it follows that

$$c_i(g_0 - g_1)\sigma_i(g_0) < 0, \quad \forall i \in I(g_0). \tag{1.8}$$

Clearly, $\overline{g} = g_0 - g_1 \in P$. However, (1.7) and (1.8) imply that (1.4) does not hold for $\overline{g}$ which contradicts (iii). The proof of the theorem is complete.

Lemma 1.1. Suppose that $\mathcal{G}$ is an RS-set on K. Suppose that $x \in X \setminus \mathcal{G}$, $g_0 \in P_{\mathcal{G}}(x)$. Let $A(x - g_0) = \{a_1^*, \cdots, a_l^*\} \subseteq E_{x-g_0}$ and $B(g_0) = \{i_1, \cdots, i_m\} \subseteq I(g_0)$ with positive numbers $\lambda_1, \cdots, \lambda_l, \lambda_1', \cdots, \lambda_m'$ satisfy (1.4). Then there are at least $\dim P - m$ linearly independent elements in $A(x - g_0)$.

Proof. Set

$$Q = \{g \in P : \ c_{i_j}(g) = 0, \ j = 1, \cdots, m\}.$$

Then Q is an interpolating subspace of dimension $N = \dim P - m$. With no loss of generality, we may assume that $a_1^*, \cdots, a_{l'}^*$ are linearly independent and that (1.4) can be rewritten as

$$\sum_{i=1}^{l'} \tilde{\lambda}_i \langle a_i^*, g\rangle + \sum_{j=1}^{m} \lambda_j' \sigma_{i_j}(g_0) c_{i_j}(g) = 0, \quad \forall g \in P. \tag{1.9}$$

To complete the proof, it suffices to show that $l' \geqslant N$. Suppose on the contrary that $l' < N$. Since Q is an interpolating subspace of dimension $N = \dim P - m$, there exists an element $g_0 \in Q \setminus \{0\}$ such that

$$\langle a_i^*, g_0\rangle = \tilde{\lambda}_i, \ i = 1, \cdots, l',$$

which, together with (1.9), implies that $\tilde{\lambda}_i = 0,\ i = 1, 2, \cdots, l'$. Consequently,

$$\sum_{i=1}^{l} \lambda_i \langle a_i^*, g \rangle = 0, \quad \forall g \in X.$$

This yields $g_0 \in P_X(x)$, which contradicts $x \in X \setminus \mathcal{G}$. The proof is complete.

Theorem 1.3. Suppose $\mathcal{G}$ is an RS-set on K. Then, there exists a strongly unique best approximation to x from $\mathcal{G}$.

Proof. Clearly, $P_{\mathcal{G}}(x) \neq \varnothing$. Assume that $x \in X \setminus \mathcal{G}$ since it is trivial when $x \in \mathcal{G}$. By Theorem 1.1 we only need to show that (1.1) holds. Let $g_0 \in P_{\mathcal{G}}(x)$. Suppose that there exists $g_1 \in \mathcal{G} \setminus \{g_0\}$ such that

$$\max_{a^* \in E_{x-g_0}} \langle a^*, g_0 - g_1 \rangle \leqslant 0.$$

From Theorem 1.2, we have

$$\sum_{i=1}^{l} \lambda_i \langle a_i^*, g_0 - g_1 \rangle + \sum_{j=1}^{m} \lambda'_j \sigma_{i_j}(g_0) c_{i_j}(g_0 - g_1) = 0.$$

Since $\sigma_{i_j}(g_0) c_{i_j}(g_0 - g_1) \leqslant 0,\ j = 1, \cdots, m$, it follows that

$$0 \geqslant \sum_{i=1}^{l} \lambda_i \langle a_i^*, g_0 - g_1 \rangle = -\sum_{j=1}^{m} \lambda'_j \sigma_{i_j}(g_0) c_{i_j}(g_0 - g_1) \geqslant 0.$$

This implies that

$$\langle a_i^*, g_0 - g_1 \rangle = 0,\ i = 1, \cdots, l,$$

$$c_{i_j}(g_0 - g_1) = 0,\ j = 1, \cdots, m;$$

hence $g_0 - g_1 \in Q$. Then Lemma 1.1 implies that $g_0 = g_1$, which contradicts the fact that $g_1 \in \mathcal{G} \setminus \{g_0\}$, so (1.1) is true. The proof is complete.

2 Strong uniqueness of the best approximation operator

In this section, we will use Theorem 1.3 to deduce strong uniqueness results of best approximation operators. Let $\mathbf{B}_X$ denote the closed unit ball of X. First take $K = \mathbf{B}_{Y^*} \times \mathbf{B}_{X^{**}}$ and $M = \mathcal{K}(X, Y)$. Then the $\sigma(\mathcal{K}(X, Y)^*, M)$-topology is just the weak*-topology in $\mathcal{K}(X, Y)^*$ since[16]

$$\mathbf{B}_{\mathcal{K}(X,Y)^*} = \mathbf{B}_{Y^*} \times \mathbf{B}_{X^{**}}.$$

Then we have the following theorem directly from Theorem 1.3.

Theorem 2.1. Let $\mathcal{G}$ be an RS-set in $\mathcal{K}(X, Y)$. Then, for any $L \in \mathcal{K}(X, Y)$, the best approximation operator to L from $\mathcal{G}$ is strongly unique.

To give more strong uniqueness results in $\mathcal{L}(X, Y)$, we introduce the concept of a strict RS-set.

Definition 2.1. An n-dimensional subspace $\mathcal{G} \subseteq \mathcal{K}(X, Y)$ is called a strictly interpolating subspace of $\mathcal{L}(X, Y)$ if no nontrivial linear combination of n linearly independent points from the set $\overline{\{\text{ext}(\mathbf{B}_{Y^*}\times\mathbf{B}_{X^{**}})\}}^*$ annihilates $\mathcal{G}$, where $\overline{\{\text{ext}(\mathbf{B}_{Y^*}\times\mathbf{B}_{X^{**}})\}}^*$ denotes the weak*-closure of $\text{ext}(\mathbf{B}_{Y^*}\times\mathbf{B}_{X^{**}})$ in $\mathcal{L}(X,Y)^*$.

Definition 2.2. Let $\{T_1,T_2,\cdots,T_n\}$ be n linearly independent elements of $\mathcal{K}(X,Y)$. We call the set

$$\mathcal{G}=\left\{T=\sum_{i=1}^{n}c_iT_i:\ \ c_i\in J_i\right\} \tag{2.1}$$

a strictly RS-set of $\mathcal{L}(X,Y)$ if each J_i is a subset of $\mathbb{R}$ of one of the three types (I), (II) and (III), and, in addition, every subset of $\{T_1,T_2,\cdots,T_n\}$ consisting of all T_i with J_i of type (I) and some T_i with J_i of type (II) spans a strictly interpolating subspace of $\mathcal{L}(X,Y)$.

Theorem 2.2. Let $\mathcal{G}\subseteq\mathcal{K}(X,Y)$ be a strict RS-set in $\mathcal{L}(X,Y)$. Then, for any $L\in\mathcal{L}(X,Y)$, the best approximation operator to L from $\mathcal{G}$ is strongly unique.

Proof. Take $K=\overline{\{\mathbf{B}_{Y^*}\times\mathbf{B}_{X^{**}}\}}^*$ and $M=\mathcal{L}(X,Y)$. Since $\text{ext}K\subseteq\overline{\{\text{ext}(\mathbf{B}_{Y^*}\times\mathbf{B}_{X^{**}})\}}^*$, $\mathcal{G}$ is an RS-set on K in $\mathcal{L}(X,Y)$. It follows from Theorem 1.3 that the best approximation operator to L from $\mathcal{G}$ is strongly unique and the proof is complete.

Recall that a set $F\subseteq\mathbf{B}_{Y^*}\times\mathbf{B}_{X^{**}}$ is a norm attaining set for $S\subseteq\mathcal{L}(X,Y)$ if for any $L\in S$, $\|L\|=\sup_{a^*\in F}|\langle a^*,L\rangle|$ (see, for example, ref. [3]).

Definition 2.3. Let $F\subseteq\mathbf{B}_{Y^*}\times\mathbf{B}_{X^{**}}$. An n-dimensional subspace $\mathcal{G}$ spanned by $\{T_1,\cdots,T_n\}$ in $\mathcal{L}(X,Y)$ is called a δ_F-interpolating subspace on F of $\mathcal{L}(X,Y)$ if there exists $\delta_F>0$ such that, for any n linearly independent points $a_1^*,\cdots,a_n^*$ from the set F,

$$|\det(\langle a_i^*,T_j\rangle)_{i,j=1}^n|\geqslant\delta_F.$$

Definition 2.4. Let $\{T_1,T_2,\cdots,T_n\}$ be n linearly independent elements of $\mathcal{L}(X,Y)$. We call the set $\mathcal{G}$ defined by (2.1) a δ_F-RS-set on F of $\mathcal{L}(X,Y)$ if each J_i is a subset of $\mathbb{R}$ of one of the three types (I), (II) and (III), and, in addition, every subset of $\{T_1,T_2,\cdots,T_n\}$ consisting of all T_i with J_i of type (I) and some T_i with J_i of type (II) spans a δ_F-interpolating subspace on F of $\mathcal{L}(X,Y)$.

Theorem 2.3. Let $L\in\mathcal{L}(X,Y)$ and F a norm attaining set for $\{L\}\cup\mathcal{G}$. Suppose that $\mathcal{G}$ is a δ_F-RS-set in $\mathcal{L}(X,Y)$ on F. Then, the best approximation operator to L from $\mathcal{G}$ is strongly unique.

Proof. Take $K=\overline{(\text{co}F\cup(-F))}^*$ and $M=\mathcal{L}(X,Y)$, where $-F=\{-a^*:a^*\in F\}$. It is not difficult to verify that $\mathcal{G}$ is an RS-set on K in $\mathcal{L}(X,Y)$ under the assumption of Theorem 2.3, so the result follows from Theorem 1.3. The proof is complete.

Remark 2.1. In the case where $\mathcal{G}$ is subspace, Theorems 2.1 and 2.3 were given in ref. [3] but Theorem 2.2 seems new even in the case where $\mathcal{G}$ is a subspace. Clearly, the above results extend and improve some results due to Lewicki in ref. [3].

3 Uniqueness element of a sun in $\mathcal{K}(c_0,c_0)$

Definition 3.1. Let $\mathcal{G}$ be a subset of $\mathcal{K}(c_0, c_0)$ and $V_0 \in \mathcal{G}$. V_0 is called a uniqueness element of $\mathcal{G}$ if V_0 is a unique best approximation operator to L from $\mathcal{G}$ whenever both $L \in \mathcal{K}(c_0, c_0)$ and $V_0 \in P_{\mathcal{G}}(L)$.

Remark 3.1. The notation of a uniqueness element was first introduced in ref. [17] to characterize the uniqueness of a best uniform approximation in terms of the strict Kolmogorov condition in the continuous function space. Recently, it also has been applied to the problems of strong uniqueness of restricted rational approximation, generalized weight approximations and simultaneous approximation[18−20].

Definition 3.2. Let $\mathcal{G}$ be a subset of $\mathcal{K}(c_0, c_0)$. $\mathcal{G}$ is called a sun, if, for any $L \in \mathcal{K}(c_0, c_0)$ and $V_0 \in \mathcal{G}$, $V_0 \in P_{\mathcal{G}}(L)$ implies that $V_0 \in P_{\mathcal{G}}(L_\alpha)$ for all $\alpha \geqslant 0$, where $L_\alpha = V_0 + \alpha(L - V_0)$.

Write

$$E_{L-V_0} = \{u^* \in \mathrm{ext}\mathbf{B}_{\mathcal{K}(c_0,c_0)^*} : \langle u^*, L - V_0\rangle = \|L - V_0\|\}.$$

Then we are ready to give the main theorem of this section.

Theorem 3.1. Let $\mathcal{G}$ be a sun of $\mathcal{K}(c_0, c_0)$ and $V_0 \in \mathcal{G}$. Then the following statements are equivalent.

(i) V_0 is a uniqueness element of $\mathcal{G}$.

(ii) For any $L \in \mathcal{K}(c_0, c_0) \setminus \mathcal{G}$, $V_0 \in P_{\mathcal{G}}(L)$ if and only if (L, V_0) satisfies the strict Kolmogorov condition, that is,

$$\max\{\langle u^*, V_0 - V\rangle : \ u^* \in E_{L-V_0}\} > 0, \quad \forall V \in \mathcal{G} \setminus \{V_0\}.$$

Proof. It is clear that (ii)⇒ (i). In order to show (i)⇒(ii), by the well-known characterization theorem of Kolmogorov's type of the best approximation from a sun (see for example ref. [21]), it suffices to show that, for any $L \in \mathcal{K}(c_0, c_0)$, (L, V_0) satisfies the strict Kolmogorov condition if $V_0 \in P_{\mathcal{G}}(L)$. Suppose that there exist $L_0 \in \mathcal{K}(c_0, c_0)$, $V_0 \in P_{\mathcal{G}}(L_0)$ such that

$$\max\{\langle u^*, V_0 - V_1\rangle : \ u^* \in E_{L_0-V_0}\} \leqslant 0 \tag{3.1}$$

for some $V_1 \in \mathcal{G} \setminus \{V_0\}$. Let

$$e_i = (0, \cdots, 0, 1, 0, \cdots), \quad i = 1, 2, \cdots.$$

For $L \in \mathcal{K}(c_0, c_0)$, define the functional $e_i \circ L \in c_0^* = l^1$ as follows:

$$e_i \circ L(x) = \langle e_i, L(x)\rangle, \quad \forall x \in c_0.$$

Since

$$\mathrm{ext}\mathbf{B}_{\mathcal{K}(c_0,c_0)^*} = \mathrm{ext}\mathbf{B}_{l^1} \times \mathrm{ext}\mathbf{B}_{l^\infty}$$

from ref. [16], we have

$$E_{L_0-V_0} = \{(e_i, x) \in \mathrm{ext}\mathbf{B}_{l^1} \times \mathrm{ext}\mathbf{B}_{l^\infty} : \ \langle (e_i, x), (L_0 - V_0)\rangle = \|L_0 - V_0\|\}, \tag{3.2}$$

where $\langle (e_i, x), L\rangle$ is defined by

$$\langle (e_i, x), L\rangle = \langle x, e_i \circ L\rangle.$$

Set

$$I=\{i\in\mathbb{N}:\ \|e_i\circ(L_0-V_0)\|=\|L_0-V_0\|\}$$

and

$$Z_i=\{x\in \mathrm{ext}\mathbf{B}_{l^\infty}:\ \langle(e_i,x),L_0-V_0\rangle=\|L_0-V_0\|\},\quad \forall i\in I.$$

Then $I\neq\varnothing$ and $Z_i\neq\varnothing$ for all $i\in I$. Obviously, we also have

$$E_{L_0-V_0}=\{(e_i,x):\ i\in I, x\in Z_i\}. \tag{3.3}$$

For $L\in\mathcal{K}(c_0,c_0)$, let $[L(i,k)]_{i,k=1}^{+\infty}$ denote the matrix corresponding to the operator L, that is,

$$L(i,k)=e_i\circ L(e_k),\quad \forall i,k=1,2,\cdots.$$

Set

$$L(i)=(L(i,1),L(i,2),\cdots),\quad i\in\mathbb{N}.$$

Then

$$\|L_0-V_0\|=\sup_{1\leqslant i<+\infty}\sum_{k=1}^{+\infty}|L_0(i,k)-V_0(i,k)|=\sup_{1\leqslant i<+\infty}\|L_0(i)-V_0(i)\|_1$$

and, for any $i\in I$, $x=(x_k)\in Z_i$, we have

$$x_k=\mathrm{sgn}(L_0(i,k)-V_0(i,k)),\quad \text{if } L_0(i,k)-V_0(i,k)\neq 0. \tag{3.4}$$

We will construct an operator $L\in\mathcal{K}(c_0,c_0)$ such that

$$\|L(i)-V_1(i)\|_1\leqslant\|L(i)-V_0(i)\|_1,\quad i\in I \tag{3.5}$$

and

$$\langle(e_i,x),L-V_0\rangle=\|L-V_0\|,\quad \forall i\in I,\ x\in Z_i. \tag{3.6}$$

To this end, we always assume that $\beta_i\geqslant 1$ in the following. The L is defined as follows.

If $i\in I$ and $\|V_1(i)-V_0(i)\|_1=0$, define

$$L(i)=\beta_i(L_0(i)-V_0(i))+V_0(i). \tag{3.7}$$

If $i\in I$ and $\|V_1(i)-V_0(i)\|_1\neq 0$, define

$$U_i=\{k\in\mathbb{N}:\ L_0(i,k)=V_0(i,k)\}.$$

Then $U_i\neq\mathbb{N}$ due to the fact that $\|e_i\circ(L_0-V_0)\|=\|L_0-V_0\|>0$. Set

$$F_i=\{k\in\mathbb{N}\setminus U_i:\ x_k=\mathrm{sgn}(V_1(i,k)-V_0(i,k))\},$$

where $x=(x_k)\in Z_i$. By (3.4) F_i is well-defined. In addition, $F_i\neq\varnothing$. In fact, let $E_i=\mathbb{N}\setminus(U_i\cup F_i)$ and define

$$y_k=\begin{cases} x_k, & k\in F_i\cup E_i,\\ -\mathrm{sgn}(V_1(i,k)-V_0(i,k)), & k\in U_i.\end{cases}$$

Then $y=(y_k)\in\mathrm{ext}\mathbf{B}_{l^\infty}$ and $\langle(e_i,y),L_0-V_0\rangle=\|L_0-V_0\|$. From (3.1), we have

$$\langle(e_i,y),V_0-V_1\rangle\leqslant 0,$$

so that

$$\sum_{k\in F_i}|V_1(i,k)-V_0(i,k)|-\sum_{k\in U_i\cup E_i}|V_1(i,k)-V_0(i,k)|=\langle(e_i,y),V_1-V_0\rangle\geqslant 0.$$

This implies that $F_i \neq \varnothing$ since $\|V_1(i) - V_0(i)\|_1 \neq 0$. Now define

$$L(i) = (L(i,1), \cdots, L(i,k), \cdots), \tag{3.8}$$

where

$$L(i,k) = \begin{cases} \beta_i(V_1(i,k) - V_0(i,k)) + V_0(i,k), & k \in F_i, \\ V_0(i,k), & k \in \mathbb{N} \setminus F_i. \end{cases} \tag{3.9}$$

Finally, if $i \in \mathbb{N} \setminus I$, define

$$L(i) = L_0(i). \tag{3.10}$$

Clearly, when $L(i)$ is defined by (3.7),

$$\langle (e_i, x), L - V_0 \rangle = \beta_i \langle (e_i, x), L_0 - V_0 \rangle = \beta_i \|L_0 - V_0\| = \|L(i) - V_0(i)\|_1 \tag{3.11}$$

and, when $L(i)$ is defined by (3.8),

$$\langle (e_i, x), L - V_0 \rangle = \beta_i \sum_{k \in F_i} |V_1(i,k) - V_0(i,k)| = \|L(i) - V_0(i)\|_1 \tag{3.12}$$

for any $i \in I$ and $x \in Z_i$.

From the Schur Theorem, it follows that there exists i_0 such that

$$M = \|e_i \circ (L_0 - V_0)\| < \|L_0 - V_0\|/2, \quad \forall i \geqslant i_0.$$

so I is finite and

$$\sup_{i \in \mathbb{N} \setminus I} \|e_i \circ (L_0 - V_0)\| < \|L_0 - V_0\|. \tag{3.13}$$

By (3.11) and (3.12), for any $i \in I$, we can take suitable $\beta_i \geqslant 1$ such that

$$\|L(i) - V_0(i)\|_1 = a > \|L_0 - V_0\|, \tag{3.14}$$

where a is a fixed constant. It follows that

$$\|L(i) - V_0(i)\|_1 = a > \sup_{i \in \mathbb{N} \setminus I} \|e_i \circ (L - V_0)\|, \quad i \in I.$$

Hence, (3.6) holds. To show (3.5), note that (3.5) is trivial if $L(i)$ is defined by (3.7). If $L(i)$ is defined by (3.8), then, using (3.9), we have

$$\begin{aligned} \|L(i) - V_1(i)\|_1 &= \sum_{k \in F_i} (\beta_i - 1)|V_1(i,k) - V_0(i,k)| + \sum_{k \notin F_i} |V_1(i,k) - V_0(i,k)| \\ &= \sum_{k \in F_i} \beta_i |V_1(i,k) - V_0(i,k)| - \sum_{k \in F_i} |V_1(i,k) - V_0(i,k)| \\ &\quad + \sum_{k \notin F_i} |V_1(i,k) - V_0(i,k)| \\ &\leqslant \sum_{k \in F_i} \beta_i |V_1(i,k) - V_0(i,k)| \\ &= \|L(i) - V_0(i)\|_1. \end{aligned}$$

So (3.5) holds.

Since $\mathcal{G}$ is a sun and $V_0 \in P_{\mathcal{G}}(L_0)$, we have

$$\max\{\langle u^*, V_0 - V \rangle : \ u^* \in E_{L_0 - V_0}\} \geqslant 0, \quad \forall V \in \mathcal{G}.$$

This, together with (3.3) and (3.6), implies that $V_0 \in P_{\mathcal{G}}(L)$. Let $L^t = V_0 + t(L - V_0)$. Then $V_0 \in P_{\mathcal{G}}(L^t)$ for all $t > 0$. Take $\alpha_0 > 0$ satisfying $M + \alpha_0 < \|L_0 - V_0\|$. Then, from (3.10) and (3.13) , we get

$$\sup_{i \notin I} \|L(i) - V_0(i)\|_1 + \alpha_0 < \|L - V_0\|. \tag{3.15}$$

Now selecting $t > \max\{\frac{\|V_1 - V_0\|}{\alpha_0}, 1\}$, we obtain

$$\sup_{i \notin I} \|L^t(i) - V_1(i)\|_1 \leqslant t\|L - V_0\| - t\alpha_0 + \|V_1 - V_0\| < \|L^t - V_0\|. \tag{3.16}$$

For any $i \in I$,

$$\begin{aligned} \|L^t(i) - V_1(i)\|_1 \leqslant& \|L^t(i) - L(i)\|_1 + \|L(i) - V_1(i)\|_1 \\ \leqslant& \|L^t(i) - L(i)\|_1 + \|L(i) - V_0(i)\|_1 \\ =& \|L^t(i) - V_0(i)\|_1. \end{aligned}$$

It follows from (3.16) that $\|L^t - V_1\| \leqslant \|L^t - V_0\|$; hence $V_1 \in P_{\mathcal{G}}(L^t)$, which contradicts the fact that V_0 is a uniqueness element of $\mathcal{G}$. The proof is complete.

From Theorems 1.1 and 3.1, we immediately have the following corollaries.

Corollary 3.1. Let $\mathcal{G}$ be a polyhedron in $\mathcal{K}(c_0, c_0)$ with finite dimension. Let $L \in \mathcal{K}(c_0, c_0),\ V_0 \in P_{\mathcal{G}}(L)$. If V_0 is a uniqueness element of $\mathcal{G}$, V_0 is strongly unique.

Corollary 3.2. Let $\mathcal{G}$ be a polyhedron in $\mathcal{K}(c_0, c_0)$ with finite dimension. If $\mathcal{G}$ is a Chebyshev subset, then any $L \in \mathcal{K}(c_0, c_0)$ has a strong unique best approximation operator from $\mathcal{G}$.

Remark 3.1. Corollary 3.2 extends the same result in ref. [3] under the assumption that $\mathcal{G}$ is a finite dimensional subspace in $\mathcal{K}(c_0, c_0)$.

Acknowledgements This work was supported in part by the National Natural Science Foundations of China (Grant No 10271025).

References

1. Lewicki, G., Kolmogorov's type criteria for spaces of compact operators, J. Approx. Theory, 1991, 64(2): 181—202.
2. Lewicki, G., Best approximation in spaces of bounded linear operators, Dissertatione Math., 1994, 330, 130.
3. Lewicki, G., Best approximation in finite dimensional subspaces of $L(W, V)$, J. Approx. Theory, 1995, 81(2): 151—165.
4. Li, C., Sun sets in spaces of bounded linear operators, Chinese Ann. Math. (in Chinese), 1995, 16A(1): 60—69.
5. Malbrock, J., Chebyshev subspaces in the space of bounded linear operators from c_0 to c_0, J. Approx. Theory, 1973, 9(2): 149—-164.
6. Malbrock, J., Best approximation in the space of bounded linear operators from $C(X)$ to $C(Y)$, J. Approx. Theory, 1975, 15(2): 132—-142.
7. Odyniec, W., Lewicki, G., Minimal projections in Banach spaces, Lecture Notes in Mathematics, 1449, Berlin Heidelberg: Springer-Verlag, 1990.
8. Lewicki, G., On minimal projections onto two dimensional subspaces of $l_\infty^{(4)}$, J. Approx. Theory, 1997, 88(1): 92—108.
9. Lewicki, G., The uniqueness of norm-one projections in spaces of James-type spaces, J. Approx. Theory, 1999, 100(1): 73—93.
10. Skrzypek, L., Uniqueness of minimal projections in smooth matrix spaces, J. Approx. Theory, 2000, 107(2): 315—336.

11. He, J. S., Li, C., Strong uniqueness of best approximations from RS-sets in Banach spaces, J. Southeast Univ., 2001, 17(1): 70—72.
12. Freilich, J. H., Mclaughlin, H. W., Approximation of bounded sets, J. Approx. Theory, 1982, 34(2): 146—158.
13. Xu, S. Y., Li, C., Characterization of the best simultaneous approximation, Acta Math. Sinica (in Chinese), 1987, 30(3): 528—534.
14. Amir, D., Uniqueness of best simultaneous approximations and strictly interpolating subspaces, J. Approx. Theory, 1984, 40(3): 196—201.
15. Cheney, E. W., Introduction to Approximation Theory, New York: McGraw-Hill, 1966.
16. Ruess, W. M., Stegall, C., Extreme points in duals of operator spaces, Math. Ann., 1982, 261: 535—546.
17. Yang, W. S., Li, C., Watson G. A., Characterization and uniqueness of nonlinear uniform approximation, Proc. Edinburgh Math. Soc., 1997, 40(3): 473—482.
18. Li, C., Watson, G. A., Strong uniqueness in restricted rational approximation, J Approx. Theory, 1997, 89(1): 96—113.
19. Li, C., Wang, X. H., Generalized weight approximation by maximal families with applications, Acta Math. Sinica (New Ser.), 1998,14(sub): 635—646.
20. Li, C., Watson G. A., On best simultaneous approximation, J Approx. Theory, 1997, 91(2): 332—348.
21. Braess, D., Nonlinear Approximation Theory, Berlin Heidelberg: Springer-Verlag, 1986.