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On best restricted range approximation in continuous complex-valued function spaces

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Abstract

To provide a Kolmogorov-type condition for characterizing a best approximation in a continuous complex-valued function space, it is usually assumed that the family of closed convex sets in the complex plane used to restrict the range satisfies a strong interior-point condition, and this excludes the interesting case when some Ω_t is a line-segment or a singleton. The main aim of the present paper is to remove this restriction by virtue of a study of the notion of the strong CHIP for an infinite system of closed convex sets in a continuous complex-valued function space.

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1. Introduction

Throughout this paper $C(Q)$ will denote the Banach space of all complex-valued continuous functions on a compact metric space Q endowed with the uniform norm (the “Sup-norm”). Let \mathcal{P} denote a finite-dimensional subspace of $C(Q)$, and let $\{\Omega_t : t \in Q\}$ be a family of nonempty closed convex sets in the complex plane \mathbb{C} . Set

$$\mathcal{P}_\Omega = \{p \in \mathcal{P} : p(t) \in \Omega_t \text{ for each } t \in Q\}. \quad (1.1)$$

The captioned problem is that of finding an element $p^* \in \mathcal{P}_\Omega$ for a function $f \in C(Q)$

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such that

$$\|f - p^*\| = \inf_{p \in \mathcal{P}_\Omega} \|f - p\| \quad (1.2)$$

(such a p^* is called a best restricted range approximation to f from \mathcal{P} with respect to $\{\Omega_t\}$). This problem was first presented and formulated by Smirnov and Smirnov in [24,25]; their approach followed the standard path for the corresponding issue in the real-valued continuous function space theory. In [24], while it was pointed out that this problem for the general class of restrictions was quite difficult, they took up the special case when each Ω_t is a disc in \mathbb{C} . Later, in a series of papers by them and by others [26–28,11,14], a more general class of $\{\Omega_t\}$ has been considered but each of them is still under a general strong interior-point condition assumption that there exists an element \bar{p} of \mathcal{P}_Ω such that $\text{int} \cap_{t \in Q} (\Omega_t - \bar{p}(t)) \neq \emptyset$ (hence $\text{int} \Omega_t \neq \emptyset$ for each $t \in Q$). This unfortunately excludes the interesting case when some Ω_t is a line-segment or a singleton in \mathbb{C} . Our results in Section 3 further relax the restriction by allowing the interesting case just mentioned. Letting

$$C_t = \{u \in C(Q) : u(t) \in \Omega_t\} \quad \text{for each } t \in Q, \quad (1.3)$$

we note that $\{\mathcal{P}, C_t : t \in Q\}$ is a family of closed convex sets in $C(Q)$ with the intersection $\mathcal{P} \cap (\bigcap_{t \in Q} C_t) = \mathcal{P}_\Omega$. The main aim of this paper is to provide some characterizations for p^* to satisfy (1.2) in a reasonable case (under appropriate continuity assumption of the set-valued mapping $t \mapsto \Omega_t$, and a suitably relaxed interior-point condition). One such characterization is given, as in the corresponding real case, by a condition of the Kolmogorov type. Our results are obtained here by virtue of a study of the strong CHIP (the strong conical hull intersection property) for an infinite family of closed convex sets in a Banach space. The notion of the strong CHIP was first introduced by Deutsch et al. [7,8] for a finite family of closed convex sets in a Euclidean space (or a Hilbert space) and was recently extended by Li and Ng in [14] to an arbitrary family of closed convex sets in a Banach space. In [16], this notion was studied extensively and some useful sufficient conditions for the strong CHIP were established.

We end this introduction with a short remark that having obtained the characterization results as presented in Section 3, the issue of the uniqueness of solutions for the corresponding problems can be addressed along a well-established path (cf. [11]) and we need not further elaborate here.

2. Notations and preliminary results

We begin with the notations used in this paper, most of which are standard (cf. [5,10]). In particular, we assume that X is a complex (or real at times) Banach space. For a set Z in X (or in \mathbb{R}^n), the interior (*resp.* relative interior, closure, convex hull, convex cone hull, linear hull, affine hull, boundary, relative boundary) of Z is denoted by $\text{int } Z$ (*resp.* $\text{ri } Z$, \bar{Z} , $\text{conv } Z$, $\text{cone } Z$, $\text{span } Z$, $\text{aff } Z$, $\text{bd } Z$, $\text{rb } Z$); the normal cone of Z at z_0 is denoted by $N_Z(z_0)$ and defined by

$$N_Z(z_0) = \{x^* \in X^* : \text{Re} \langle x^*, z - z_0 \rangle \leq 0 \quad \text{for each } z \in Z\}. \quad (2.1)$$

The distance from z_0 to Z is denoted by $dZ(z_0)$.

Our main tools are the following Theorems 2.1 and 2.2 taken from [16, Corollaries 4.2 and 5.1]. It would be convenient for us to repeat some of the definitions introduced in [16] as well as some other more standard notions in this regard. Let I denote an index-set which is assumed to be a compact metric space. A family $\{C, C_i : i \in I\}$ is called a closed convex set system with base-set C (CCS-system with base-set C) if C and C_i are nonempty closed convex subsets of X for each $i \in I$.

Definition 2.1. A CCS-system $\{C, C_i : i \in I\}$ (with base-set C) is said to satisfy:

(i) the interior-point condition if

$$C \cap \left(\bigcap_{i \in I} \text{int } C_i \right) \neq \emptyset; \tag{2.2}$$

(ii) the strong interior-point condition if

$$C \cap \left(\text{int } \bigcap_{i \in I} C_i \right) \neq \emptyset; \tag{2.3}$$

(iii) the weak–strong interior-point condition with the pair (I_1, I_2) if there exist two disjoint finite subsets I_1 and I_2 of I such that each C_i ($i \in I_2$) is a polyhedron and

$$\text{ri } C \cap \left(\text{int } \bigcap_{i \in I \setminus (I_1 \cup I_2)} C_i \right) \cap \left(\bigcap_{i \in I_1} \text{ri } C_i \right) \cap \bigcap_{i \in I_2} C_i \neq \emptyset. \tag{2.4}$$

Any point \bar{x} belonging to the set on the left-hand side of (2.2) (resp. (2.3), (2.4)) is called an interior point (resp. a strong interior point, a weak–strong interior point with the pair (I_1, I_2)) of the CCS-system $\{C, C_i : i \in I\}$.

It is trivial that (2.2) \implies (2.3). The converse also holds in some cases, one of which will be described in terms of the continuity of some set-valued functions (cf. [16]). For set-valued functions there are many different notions of continuity. In Definitions 2.2 and 2.3 below, we recall two frequently used ones. We assume that Q is a compact metric space.

Definition 2.2. Let $F : Q \rightarrow 2^X$ be a set-valued function defined on Q and let $t_0 \in Q$. Then F is said to be

(i) lower semicontinuous at t_0 , if, for any $y_0 \in F(t_0)$ and any $\varepsilon > 0$, there exists an open neighbourhood $U(t_0)$ of t_0 such that, for each $t \in U(t_0)$, $\mathbf{B}(y_0, \varepsilon) \cap F(t) \neq \emptyset$.

(ii) upper semicontinuous at t_0 if, for any open neighbourhood V of $F(t_0)$, there exists an open neighbourhood $U(t_0)$ of t_0 such that $F(t) \subseteq V$ for each $t \in U(t_0)$.

(iii) lower (resp. upper) semicontinuous on Q if it is lower (resp. upper) semicontinuous at each $t \in Q$.

Definition 2.3 (cf. Singer [23, p. 55]). Let $F : Q \rightarrow 2^X$ be a set-valued function defined on Q and let $t_0 \in Q$. Then F is said to be

- (i) upper Kuratowski semicontinuous at t_0 if, for any sequence $\{t_k\} \subseteq Q$, the relations $\lim_{k \rightarrow \infty} t_k = t_0, \lim_{k \rightarrow \infty} x_{t_k} = x_{t_0}, x_{t_k} \in F(t_k), k = 1, 2, \dots$ imply $x_{t_0} \in F(t_0)$.
- (ii) lower Kuratowski semicontinuous at t_0 if, for any sequence $\{t_k\} \subseteq Q$, the relations $\lim_{k \rightarrow \infty} t_k = t_0, y_0 \in F(t_0)$ imply $\lim_{k \rightarrow \infty} d_F(t_k)(y_0) = 0$;
- (iii) Kuratowski continuous at t_0 if F is both upper Kuratowski semicontinuous and lower Kuratowski semicontinuous at t_0 .
- (iv) Kuratowski continuous on Q if it is Kuratowski continuous at each point of Q .

Remark 2.1. Clearly,

- (i) F is upper semicontinuous $\implies F$ is upper Kuratowski semicontinuous.
- (ii) F is lower semicontinuous $\iff F$ is lower Kuratowski semicontinuous.

Moreover, the converse of (i) holds provided that the union $\cup_{t \in Q} F(t)$ is compact.

Let $\{A_i : i \in J\}$ be a family of subsets of X . The set $\sum_{i \in J} A_i$ is defined by

$$\sum_{i \in J} A_i = \begin{cases} \left\{ \sum_{i \in J_0} a_i : a_i \in A_i, J_0 \subseteq J \text{ being finite} \right\} & \text{if } J \neq \emptyset, \\ \{0\} & \text{if } J = \emptyset. \end{cases} \tag{2.5}$$

Definition 2.4. Let $\{C_i : i \in I\}$ be a collection of convex subsets of X and $x \in \bigcap_{i \in I} C_i$. The collection is said to have

- (a) the strong CHIP at x if

$$N_{\bigcap_{i \in I} C_i}(x) = \sum_{i \in I} N_{C_i}(x). \tag{2.6}$$

- (b) the strong CHIP if it has the strong CHIP at each point of $\bigcap_{i \in I} C_i$.

Theorem 2.1. Let $x_0 \in C \cap (\bigcap_{i \in I} C_i)$. The system $\{C, C_i : i \in I\}$ has the strong CHIP at x_0 if the following conditions are satisfied:

- (a) The system $\{C, C_i : i \in I\}$ satisfies the weak-strong interior-point condition with (I_1, I_2) .
- (b) The set-valued mapping $i \mapsto C_i$ is lower semicontinuous on I .
- (c) At least one of the sets in the family $\{C, C_i : i \in I_1\}$ is finite-dimensional.

Theorem 2.2. Suppose that the CCS-system $\{C, C_i : i \in I\}$ satisfies the interior-point condition, $\dim C < +\infty$ and that the set-valued function $i \mapsto (aff C) \cap C_i$ is Kuratowski continuous. Then the system $\{C, C_i : i \in I\}$ has the strong CHIP.

We end this section with two results on characterizations of the strong CHIP of a (possibly infinite) system $\{C, C_i : i \in I\}$ of closed convex sets. The first result, which is valid in a general Banach space and will be used in the next section, is given in terms of the optimality conditions of a constrained best approximation while the second result in the Hilbert space setting is given as a dual formulation of a constrained best approximation (see for example, [3,4,7–9,12–15,17,18]). To this end, we need a well-known result on the characterization of the best approximation by a convex set in X , which was established independently by

Deutsch [6] and Rubenstein [20] (see also [1]). For a closed convex subset W of X , let P_W denote the projection operator defined by

$$P_W(x) = \{y \in W : \|x - y\| = d_W(x)\}.$$

Where $d_W(x)$ denotes the distance from x to W . Recall that the duality map J from X to 2^{X^*} is defined by

$$J(x) := \{x^* \in X^* : \langle x^*, x \rangle = \|x\|^2, \|x^*\| = \|x\|\}. \tag{2.7}$$

Proposition 2.1. *Let W be a closed convex set in X . Then for any $x \in X$, $z_0 \in P_W(x)$ if and only if $z_0 \in W$ and there exists $x^* \in J(x - z_0)$ such that $\text{Re} \langle x^*, z - z_0 \rangle \leq 0$ for any $z \in W$, that is, $J(x - z_0) \cap N_W(z_0) \neq \emptyset$. In particular, when X is smooth, $z_0 \in P_W(x)$ if and only if $z_0 \in W$ and $J(x - z_0) \in N_W(z_0)$.*

Theorem 2.3. *Let $K = C \cap (\bigcap_{i \in I} C_i)$ and $x_0 \in K$. Consider the following statements.*

- (i) *The system $\{C, C_i : i \in I\}$ has the strong CHIP at x_0 .*
- (ii) *For each $x \in X$, $x_0 \in P_K(x)$ if and only if*

$$J(x - x_0) \cap \left(N_C(x_0) + \sum_{i \in I} N_{C_i}(x_0) \right) \neq \emptyset. \tag{2.8}$$

- (iii) *For each $x \in X$, $x_0 \in P_K(x)$ if and only if*

$$J(x - x_0)|_{C-x_0} \cap \left(N_C(x_0)|_{C-x_0} + \sum_{i \in I} N_{C_i}(x_0)|_{C-x_0} \right) \neq \emptyset. \tag{2.9}$$

Then the following implications hold.

- (1) (i) \implies (ii) \iff (iii).
- (2) (i) \iff (ii) \iff (iii) if X is both reflexive and smooth.

Proof. Note the following equivalence:

$$\begin{aligned} & J(x - x_0) \cap \left(N_C(x_0) + \sum_{i \in I} N_{C_i}(x_0) \right) \neq \emptyset \\ \iff & J(x - x_0)|_{C-x_0} \cap \left(N_C(x_0)|_{C-x_0} + \sum_{i \in I} N_{C_i}(x_0)|_{C-x_0} \right) \neq \emptyset. \end{aligned} \tag{2.10}$$

Indeed, implication \implies in (2.10) is trivial; hence it suffices to show the converse implication. Thus, let $x^* \in J(x - x_0)$ be such that $x^*|_{C-x_0} \in J(x - x_0)|_{C-x_0} \cap \left(N_C(x_0)|_{C-x_0} + \sum_{i \in I} N_{C_i}(x_0)|_{C-x_0} \right)$. Then there exist $x_0^* \in N_C(x_0)$, a finite subset J of I and $x_i^* \in N_{C_i}(x_0)$ for each $i \in J$ such that $x^*|_{C-x_0} = \sum_{i=0}^m x_i^*|_{C-x_0}$. Write $y^* = x^* - \sum_{i=0}^m x_i^*$. Then $y^* \in N_C(x_0)$ and so $x^* = y^* + \sum_{i=0}^m x_i^* \in N_C(x_0) + \sum_{i \in I} N_{C_i}(x_0)$. Hence, $x^* \in J(x - x_0) \cap \left(N_C(x_0) + \sum_{i \in I} N_{C_i}(x_0) \right)$. Therefore (2.10) is true.

Now, using (2.10), one can complete the proof in the same way as that given for [15, Theorem 3.1]. \square

For the remainder of this section, let X denote a Hilbert space (over \mathbb{R} or \mathbb{C}), and we consider $X^* = X$ as usual. In particular, the normal cone of a nonempty set Z at z_0 can be redefined as $N_Z(z_0) = \{y \in X : \operatorname{Re} \langle y, z - z_0 \rangle \leq 0 \text{ for all } z \in Z\}$. Let $I(x_0) = \{i \in I : x_0 \in \operatorname{bd} C_i\}$. Then, similar to the proof of [14, Theorem 4.1], we obtain the following theorem.

Theorem 2.4. *Let X be a Hilbert space, $K = C \cap (\bigcap_{i \in I} C_i)$ and let $x_0 \in K$. Then the following statements are equivalent.*

- (i) *The system $\{C, C_i : i \in I\}$ has the strong CHIP at x_0 .*
- (ii) *For any $x \in X$, $P_K(x) = x_0$ if and only if there exists a finite (possibly empty) set $I_0 \subseteq I$ such that $P_C(x - \sum_{i \in I_0} h_i) = x_0$ for some $h_i \in N_{C_i}(x_0)$ for each $i \in I_0$.*

Now, let C be a closed convex set in X , $\{h_i : i \in I\} \subset X \setminus \{0\}$ and let $\{\Omega_i : i \in I\}$ be a family of nonempty closed convex subsets of the scalar field. Define

$$\widehat{C}_i = \{x \in X : \langle x, h_i \rangle \in \Omega_i\}, \quad i \in I, \tag{2.11}$$

and

$$\widehat{K} = C \cap \left(\bigcap_{i \in I} \widehat{C}_i \right). \tag{2.12}$$

Let $x_0 \in \widehat{K}$. For convenience, we shall write $\widetilde{h}_i(\cdot)$ for the function $\langle h_i, \cdot \rangle$ on X , and h_i^0 for the scalar $\langle h_i, x_0 \rangle$. Then we have the following assertion:

$$N_{\widehat{C}_i}(x_0) = \{\bar{\alpha}h_i : \alpha \in N_{\Omega_i}(h_i^0)\} \text{ for each } i \in I. \tag{2.13}$$

This assertion was proved in the proof of [14, Theorem 4.2]. Here we give a direct and much simpler proof. In fact, it is direct that the set on the left-hand side contains the one on the right-hand side of (2.13). To show the converse inclusion, let h_i^\perp denote the orthogonal complement of h_i and let $x^* \in N_{\widehat{C}_i}(x_0)$. Then, for each $x \in h_i^\perp$ and $\gamma \in \mathbb{C}$, $\operatorname{Re} \langle x^*, \gamma x \rangle \leq 0$ since $\gamma x + x_0 \in \widehat{C}_i$; hence $x^* \perp h_i^\perp$ and $x^* = \bar{\alpha}h_i$ for some scalar $\alpha \in \mathbb{C}$. Since, for each $\beta \in \Omega_i$, there exists $x \in \widehat{C}_i$ such that $\langle h_i, x \rangle = \beta$, we have that

$$\operatorname{Re} \bar{\alpha}(\beta - h_i^0) = \langle x^*, x - x_0 \rangle \leq 0.$$

This means that $\alpha \in N_{\Omega_i}(h_i^0)$. Therefore x^* belongs to the set on the right-hand side of (2.13) and (2.13) is proved. Thus, by (2.13) and Theorem 2.4, we immediately obtain the following perturbation theorem, which was given in [14]. Note that the proof here is much simpler than that in [14].

Corollary 2.1. *Let X be a Hilbert space and let $x_0 \in \widehat{K}$, where \widehat{K} is defined by (2.12). Then the following statements are equivalent.*

- (i) *The collection of closed convex sets $\{C, \widehat{C}_i : i \in I\}$ has the strong CHIP at x_0 , and*
- (ii) *For any $x \in X$, $P_{\widehat{K}}(x) = x_0$ if and only if there exists a finite (possibly empty) set $I_0 \subseteq I$ such that $P_C(x - \sum_{i \in I_0} \bar{\alpha}_i h_i) = x_0$ for some $\alpha_i \in N_{\Omega_i}(h_i^0)$ for each $i \in I_0$.*

3. Characterization for constrained approximation in complex-valued function spaces

Let $C(Q)$ denote the Banach space of all complex-valued continuous functions on a compact metric space Q endowed with the uniform norm:

$$\|f\| = \max_{t \in Q} |f(t)| \quad \text{for each } f \in C(Q). \tag{3.1}$$

Let \mathcal{P} be an n -dimensional subspace of $C(Q)$ and $\{\Omega_t : t \in Q\}$ a family of nonempty closed convex sets in the complex plane \mathbb{C} . For brevity, we write $\{\Omega_t\}$ for $\{\Omega_t : t \in Q\}$. Note that, for each $t \in Q$, Ω_t is either a point or a linear-segment, or a “planar” convex set (of real dimension 2) in the complex plane \mathbb{C} . Set

$$\mathcal{P}_\Omega = \{p \in \mathcal{P} : p(t) \in \Omega_t \text{ for each } t \in Q\}. \tag{3.2}$$

The problem considered here is that of finding an element $p^* \in \mathcal{P}_\Omega$ for a function $f \in C(Q)$ such that

$$\|f - p^*\| = \inf_{p \in \mathcal{P}_\Omega} \|f - p\|, \tag{3.3}$$

(such a p^* is called a best-restricted range approximation to f from \mathcal{P} with respect to $\{\Omega_t\}$; see [24,28,11,14]).

We assume that

$$Q = Q_S \cup Q_E \cup Q_N, \tag{3.4}$$

where

$$Q_S = \{t \in Q : \Omega_t \text{ is a singleton}\},$$

$$Q_E = \{t \in Q \setminus Q_S : \text{int } \Omega_t = \emptyset\},$$

$$Q_N = \{t \in Q : \text{int } \Omega_t \neq \emptyset\}.$$

We also assume for the whole section that

$$Q_S \cup Q_E \text{ is finite.} \tag{3.5}$$

We introduce some short notation of conditions for easy reference.

- IC₀: \mathcal{P} contains the constant functions and there exists an element $\bar{p} \in \mathcal{P}_\Omega$ such that $\bar{p}(t) \in \text{int } \Omega_t$ for each $t \in Q$, that is,

$$0 \in \bigcap_{t \in Q} \text{int } (\Omega_t - \bar{p}(t)). \tag{3.6}$$

- IC: There exists an element $\bar{p} \in \mathcal{P}_\Omega$ such that

$$0 \in \text{int} \left(\bigcap_{t \in Q_N} (\Omega_t - \bar{p}(t)) \right) \cap \left(\bigcap_{t \in Q_E} \text{ri}(\Omega_t - \bar{p}(t)) \right). \tag{3.7}$$

- UKC: The set-valued function $t \mapsto \Omega_t$ is upper Kuratowski semicontinuous on Q .
- LKC: The set-valued function $t \mapsto \Omega_t$ is lower Kuratowski semicontinuous on Q .
- KC: The set-valued function $t \mapsto \Omega_t$ is Kuratowski continuous on Q .

We will see later that these conditions closely relate to some corresponding properties of the CCS-system $\{\mathcal{P}, C_t : t \in Q\}$ in $C(Q)$, where C_t is defined by (1.3). Let $f \in C(Q)$ and $p^* \in \mathcal{P}_Q$. We fix this pair of functions f, p^* in what follows. Define

$$\sigma(t) = f(t) - p^*(t) \quad \text{for each } t \in Q. \tag{3.8}$$

Set

$$M(\sigma) = \{t \in Q : |\sigma(t)| = \|\sigma\|\}$$

and

$$B(p^*) = \{t \in Q : p^*(t) \in \text{bd}\Omega_t\}, \quad B^{rb}(p^*) = \{t \in Q \setminus Q_S : p^*(t) \in \text{rb}\Omega_t\}.$$

(Here we adopt the convention that $\text{bd}\Omega_t = \Omega_t$ if Ω_t is a singleton.) Note that

$$B^{rb}(p^*) = (B(p^*) \cap Q_N) \cup \{t \in Q_E : p^*(t) \in \text{rb}\Omega_t\} \tag{3.9}$$

and in particular that $B^{rb}(p^*) \subseteq B(p^*)$. Moreover, $B^{rb}(p^*) = B(p^*)$ in the case when Q_S and Q_E are empty (e.g., when IC_0 holds).

Let $\text{span}_R(\Omega_t - p^*(t))$ denote the real subspace spanned by $\Omega_t - p^*(t)$ in \mathbb{C} . Then $\text{span}_R(\Omega_t - p^*(t))$ is the whole complex plane \mathbb{C} if $t \in Q_N$, a line in \mathbb{C} if $t \in Q_E$ and a singleton $\{0\}$ if $t \in Q_S$. Set

$$\mathcal{P}_R = \{p \in \mathcal{P} : p(t) \in \text{span}_R(\Omega_t - p^*(t)) \text{ for each } t \in Q_E \cup Q_S\}. \tag{3.10}$$

Note that \mathcal{P}_R is a real subspace of \mathcal{P} and that $\mathcal{P}_R = \mathcal{P}$ if $Q = Q_N$. Let m denote the real dimension of \mathcal{P}_R : $\dim_R \mathcal{P}_R = m$, and let ψ_1, \dots, ψ_m be a real basis of \mathcal{P}_R , that is, each element of \mathcal{P}_R can be uniquely expressed as a real linear combination of ψ_1, \dots, ψ_m . Moreover, let $\{\phi_1, \dots, \phi_n\}$ be a (complex) basis of \mathcal{P} , that is, each element of \mathcal{P} can be uniquely expressed as a complex linear combination of ϕ_1, \dots, ϕ_n .

We define

$$\tau(t) = \{\tau \in -N_{\Omega_t}(p^*(t)) : |\tau| = 1\} \quad \text{for each } t \in Q. \tag{3.11}$$

Note that if $t \in Q_N \cap B(p^*)$ and $\tau \in \tau(t)$ then

$$\text{Re } \bar{\tau}(z - p^*(t)) > 0 \tag{3.12}$$

for all $z \in \text{int } \Omega_t$. Since $\text{int } \Omega_t = \emptyset$ if $t \in Q \setminus Q_N$, we have to define two more set-valued functions from Q to the unit sphere of \mathbb{C} :

$$\tau_r(t) = \begin{cases} \tau(t) & \text{for each } t \in Q \setminus Q_E, \\ \{\tau \in \mathbb{C} : |\tau| = 1, \text{Re } \bar{\tau}(z - p^*(t)) > 0 \\ \quad \forall z \in \text{ri } \Omega_t\} & \text{for each } t \in Q_E \end{cases} \tag{3.13}$$

and

$$\tau_r^+(t) = \begin{cases} \tau(t) & \text{for each } t \in Q \setminus Q_E, \\ \emptyset & \text{for each } t \in Q_E \text{ with } p^*(t) \in \text{ri } \Omega_t, \\ \frac{z-p^*(t)}{|z-p^*(t)|} & \text{for each } t \in Q_E \text{ with } p^*(t) \in \text{rb } \Omega_t, z \in \Omega_t \setminus p^*(t). \end{cases} \tag{3.14}$$

(Note that $\frac{z-p^*(t)}{|z-p^*(t)|}$ does not depend on the particular choice of z as Ω_t is a line-segment for $t \in Q_E$.)

Remark 3.1. (i) For any $t \in Q$, $\tau(t) \neq \emptyset \iff t \in B(p^*)$.

(ii) For any $t \in Q_E$,

$$\tau_r(t) \neq \emptyset \iff t \in B^{rb}(p^*) \iff \tau_r^+(t) \text{ is a singleton.} \tag{3.15}$$

(iii) If $t \in B^{rb}(p^*) \cap Q_E$ and $\tau \in -N_{\Omega_t}(p^*(t))$ with $|\tau| = 1$, then

$$\tau \notin \tau_r(t) \iff \text{Re } \bar{\tau}(z - p^*(t)) = 0 \text{ for each } z \in \Omega_t \iff \text{Re } \bar{\tau}(z - p^*(t)) = 0 \text{ for some } z \in \Omega_t. \tag{3.16}$$

(iv) For any $t \in Q$, $\tau_r^+(t)$ is compact

$$\tau_r^+(t) \subseteq \tau_r(t) \subseteq \tau(t). \tag{3.17}$$

Let $t \in B^{rb}(p^*) \cap Q_E$, $\tau \in \tau_r(t)$ and let $\text{Pr}_t(\tau)$ denote the projection of τ on the subspace $\text{span}_R(\Omega_t - p^*(t))$. Then $\text{Pr}_t(\tau) \neq 0$,

$$\frac{\text{Pr}_t(\tau)}{|\text{Pr}_t(\tau)|} \in \tau_r^+(t) \text{ and } \text{Re } z\bar{\tau} = \text{Re } z\overline{\text{Pr}_t(\tau)} \text{ for each } z \in \text{span}_R(\Omega_t - p^*(t)). \tag{3.18}$$

For each $t \in Q$, let $\mathbf{c}(t) \subset \mathbb{C}^n$, $\mathbf{c}_r(t) \subset \mathbb{R}^m$ and $\mathbf{c}_r^+(t)$ be defined, respectively, by

$$\mathbf{c}(t) := \{(\phi_1(t)\bar{\tau}, \dots, \phi_n(t)\bar{\tau}) : \tau \in \tau(t)\}, \tag{3.19}$$

$$\mathbf{c}_r(t) := \{(\text{Re } \psi_1(t)\bar{\tau}, \dots, \text{Re } \psi_m(t)\bar{\tau}) : \tau \in \tau_r(t)\} \tag{3.20}$$

and

$$\mathbf{c}_r^+(t) := \{(\text{Re } \psi_1(t)\bar{\tau}, \dots, \text{Re } \psi_m(t)\bar{\tau}) : \tau \in \tau_r^+(t)\}. \tag{3.21}$$

Set

$$\mathcal{U} = \bigcup_{t \in B(p^*)} \mathbf{c}(t), \quad \mathcal{U}_r = \bigcup_{t \in B^{rb}(p^*)} \mathbf{c}_r(t), \quad \mathcal{U}_r^+ = \bigcup_{t \in B^{rb}(p^*)} \mathbf{c}_r^+(t). \tag{3.22}$$

Note that these sets are bounded and that, by (3.17) and (3.18),

$$\mathcal{U}_r^+ \subseteq \mathcal{U}_r \subseteq \bigcup_{0 < \eta \leq 1} (\eta \mathcal{U}_r^+). \tag{3.23}$$

Recalling (3.8), we define $\mathbf{b}(t) \in \mathbb{C}^n$ and $\mathbf{b}_r(t) \in \mathbb{R}^m$, respectively, by

$$\mathbf{b}(t) = (\phi_1(t), \dots, \phi_n(t))\overline{\sigma(t)} = (\phi_1(t)\overline{\sigma(t)}, \dots, \phi_n(t)\overline{\sigma(t)})$$

for each $t \in Q$ (3.24)

and

$$\mathbf{b}_r(t) = \text{Re}(\psi_1(t), \dots, \psi_m(t))\overline{\sigma(t)} \quad \text{for each } t \in Q. \tag{3.25}$$

We define

$$\mathcal{V} = \{\mathbf{b}(t) : t \in M(\sigma)\}, \quad \mathcal{V}_r = \{\mathbf{b}_r(t) : t \in M(\sigma)\}. \tag{3.26}$$

Clearly they are compact sets. Set

$$\mathcal{W} = \mathcal{V} \bigcup \mathcal{U}, \quad \mathcal{W}_r = \mathcal{V}_r \bigcup \mathcal{U}_r, \quad \mathcal{W}_r^+ = \mathcal{V}_r \bigcup \mathcal{U}_r^+. \tag{3.27}$$

Note that they are bounded sets. Also, by (3.23),

$$\mathcal{W}_r^+ \subseteq \mathcal{W}_r \subseteq \bigcup_{0 < t \leq 1} (t \mathcal{W}_r^+). \tag{3.28}$$

This implies that

$$\text{co } \mathcal{W}_r^+ \subseteq \text{co } \mathcal{W}_r \subseteq \text{co } \bigcup_{0 < t \leq 1} (t \mathcal{W}_r^+) \subseteq \bigcup_{0 < t \leq 1} (t \text{co } \mathcal{W}_r^+), \tag{3.29}$$

where the last inclusion can be verified by a routine verification.

Let

$$C_t = \{u \in C(Q) : u(t) \in \Omega_t\} \quad \text{for each } t \in Q. \tag{3.30}$$

Then

$$\mathcal{P}_\Omega = \mathcal{P} \cap \left(\bigcap_{t \in Q} C_t \right). \tag{3.31}$$

Clearly $\{\mathcal{P}, C_t : t \in Q\}$ is a CCS-system with a base-set \mathcal{P} . To prepare for our main result, we first give a few lemmas. These lemmas will show in particular that the conditions introduced at the beginning of this section for the system $\{\Omega_t\}$ are naturally linked to some desirable properties of the system $\{\mathcal{P}, C_t : t \in Q\}$ so that the results in Section 2 are applicable. The first of the lemmas describes the connections of the conditions IC_0 , IC for the system $\{\Omega_t\}$ and the interior-point conditions for the system $\{\mathcal{P}, C_t : t \in Q\}$ while the second describes the connection of the normal cones of Ω_t and that of the corresponding C_t .

Lemma 3.1. (i) *The system $\{\Omega_t\}$ satisfies IC_0 if and only if the CCS-system $\{\mathcal{P}, C_t : t \in Q\}$ satisfies the interior-point condition. Furthermore, $0 \notin \text{conv } \mathcal{U}$ if the system $\{\Omega_t\}$ satisfies IC_0 .*

(ii) The system $\{\Omega_t\}$ satisfies IC if and only if the CCS-system $\{\mathcal{P}, C_t : t \in Q\}$ satisfies the weak-strong interior-point condition with the pair (Q_E, Q_S) . Furthermore, $0 \notin \text{conv } \mathcal{U}_r$ if the system $\{\Omega_t\}$ satisfies IC.

Proof. Let $\alpha > 0$ and $f_0 \in C_t$. We claim that

$$\mathbf{B}(f_0, \alpha) \subseteq C_t \iff \mathbf{B}(f_0(t), \alpha) \subseteq \Omega_t \quad \text{for each } t \in Q_N, \tag{3.32}$$

$$\mathbf{B}(f_0, \alpha) \bigcap \text{aff } C_t \subseteq C_t \iff \mathbf{B}(f_0(t), \alpha) \bigcap \text{aff } \Omega_t \subseteq \Omega_t \quad \text{for each } t \in Q_E. \tag{3.33}$$

We shall only prove (3.33) (the proof of (3.32) is similar). To do this, we need only establish the necessity part. Note first the following obvious fact:

$$\text{aff } C_t = \{u \in C(Q) : u(t) \in \text{aff } \Omega_t\} \quad \text{for each } t \in Q. \tag{3.34}$$

Let $t \in Q_E$ and assume that

$$\mathbf{B}(f_0, \alpha) \bigcap \text{aff } C_t \subseteq C_t. \tag{3.35}$$

Let $z \in \mathbf{B}(f_0(t), \alpha) \bigcap \text{aff } \Omega_t$. We have to show that $z \in \Omega_t$. By the Tietze Extension Theorem, there exists $s \in C(Q)$ such that $\|s\| = s(t) = 1$. Define

$$f(w) = f_0(w) + s(w)(z - f_0(t)) \quad \forall w \in Q.$$

Then $\|f - f_0\| \leq \|z - f_0(t)\| \leq \alpha$. Since $f(t) = z \in \text{aff } \Omega_t$, $f \in \text{aff } C_t$ by (3.34). Consequently, $f \in C_t$ and hence $z = f(t) \in \Omega_t$, as required. Therefore, our claim stands.

By (3.32), we have that

$$\text{int } C_t = \{u \in C(Q) : u(t) \in \text{int } \Omega_t\} \quad \text{for each } t \in Q. \tag{3.36}$$

Thus the first part of (i) is clear. Again by (3.32),

$$\text{int } \bigcap_{t \in Q_N} C_t = \left\{ u \in C(Q) : u(t) \in \text{int } \bigcap_{t \in Q_N} \Omega_t \right\}, \tag{3.37}$$

while, by (3.33),

$$\text{ri } C_t = \{u \in C(Q) : u(t) \in \text{ri } \Omega_t\} \quad \text{for each } t \in Q_E. \tag{3.38}$$

Combining (3.37) and (3.38), the first part of (ii) is also clear.

Thus, to complete the proof, it remains to show that (a): $0 \notin \text{conv } \mathcal{U}_r$ if IC is satisfied and that (b): $0 \notin \text{conv } \mathcal{U}$ if IC_0 is satisfied. We shall only prove (a) as the proof for (b) is similar. Suppose that there exist $\lambda_1, \dots, \lambda_s \in [0, 1]$ with $\sum_{j=1}^s \lambda_j = 1$ and $t'_1, \dots, t'_s \in B^{rb}(p^*)$, $\tau'_j \in \tau_r(t'_j)$, $j = 1, \dots, s$ such that

$$\text{Re } \sum_{j=1}^s p(t'_j) \lambda_j \overline{\tau'_j} = 0 \tag{3.39}$$

holds for each $p \in \{\psi_1, \dots, \psi_m\}$ and hence for each $p \in \mathcal{P}_R$. Assuming IC with some $\bar{p} \in \mathcal{P}_\Omega$ satisfying (3.7), let $p := \bar{p} - p^*$. Since, each $t'_j \in B^{rb}(p^*)$ and each $\tau'_j \in \tau_r(t'_j)$, we obtain, by (3.12), (3.7) and (3.13) that

$$\operatorname{Re} p(t'_j)\bar{\tau}'_j = \operatorname{Re} (\bar{p}(t'_j) - p^*(t'_j))\bar{\tau}'_j > 0 \quad \text{for each } j = 1, \dots, s. \tag{3.40}$$

This contradicts (3.39) and hence $0 \notin \operatorname{conv} \mathcal{U}_r$. \square

Lemma 3.2. *Let $t \in Q$ and assume that $p^* \in C_t$. Then*

$$N_{C_t}(p^*) = \{\bar{\alpha}e_t : \alpha \in N_{\Omega_t}(p^*(t))\}, \tag{3.41}$$

where e_t denotes the point-functional on $C(Q)$ defined by

$$\langle e_t, u \rangle = u(t) \quad \text{for each } u \in C(Q). \tag{3.42}$$

Proof. Let $u \in C(Q)$. Let $z \in \Omega_t$ be such that $d_{\Omega_t}(u(t)) = |z - u(t)|$. By the Tietze Extension Theorem, there exists a function $w \in C(Q)$ such that $\|w\| = |u(t) - z|$ and $w(t) = u(t) - z$ (so $u - w \in C_t$). Then $d_{C_t}(u) \leq \|u - (u - w)\| = |z - u(t)| = d_{\Omega_t}(u(t))$. Consequently,

$$d_{C_t}(u) = d_{\Omega_t}(u(t)) \quad \text{for each } u \in C(Q) \tag{3.43}$$

as it is straightforward to verify that $d_{C_t}(u) \geq d_{\Omega_t}(u(t))$. Since $p^* \in C_t$ (and so $p^*(t) \in \Omega_t$), (3.43) and the proof of [14, Lemma 5.2 (iii)] imply that

$$\partial d_{C_t}(p^*) = \{\bar{\alpha}e_t \in C(Q)^* : \alpha \in \partial d_{\Omega_t}(p^*(t))\}. \tag{3.44}$$

Recalling from [5] that

$$\begin{aligned} \partial d_{C_t}(p^*) &= \{x^* \in N_{C_t}(p^*) : \|x^*\| \leq 1\} \quad \text{and} \\ \partial d_{\Omega_t}(p^*(t)) &= \{\alpha \in N_{\Omega_t}(p^*(t)) : |\alpha| \leq 1\}, \end{aligned} \tag{3.45}$$

it follows that (3.41) holds. \square

Lemma 3.3. (i) *If UKC is satisfied, then the set-valued function $t \mapsto C_t$ is upper Kuratowski semicontinuous on Q .*

(ii) *If LKC is satisfied, then the set-valued function $t \mapsto C_t$ is lower Kuratowski semicontinuous on Q (and so is the set-valued function $t \mapsto \mathcal{P} \cap C_t$ if $1 \in \mathcal{P}$).*

Proof. Let $t_0 \in Q$ and $\{t_k\} \subseteq Q$ be a sequence convergent to t_0 .

(i) Let $f_k \in C_{t_k}$ for each k be such that $\|f_k - \bar{f}\| \rightarrow 0$. Then, $f_k(t_k) \in \Omega_{t_k}$ for each k and $f_k(t_k) \rightarrow \bar{f}(t_0)$ as $k \rightarrow \infty$. By the condition UKC, it follows that $\bar{f}(t_0) \in \Omega_{t_0}$ and so $\bar{f} \in C_{t_0}$. This proves (i).

(ii) Let $f_0 \in C_{t_0}$ (or $f_0 \in \mathcal{P} \cap C_{t_0}$ if $1 \in \mathcal{P}$). Then $f_0(t_0) \in \Omega_{t_0}$ and, by the condition LKC, there exists $z_k \in \Omega_{t_k}$ for each k such that $|z_k - f_0(t_0)| \rightarrow 0$. Define $f_k \in C(Q)$ by

$$f_k(t) = f_0(t) + z_k - f_0(t_k) \quad \text{for each } t \in Q.$$

Thus $f_k(t_k) = z_k \in \Omega_{t_k}$ and hence $f_k \in C_{t_k}$ (and $f_k \in \mathcal{P} \cap C_{t_k}$ if $1 \in \mathcal{P}$). Moreover, we also have that

$$\|f_k - f_0\| = |z_k - f_0(t_k)| \leq |z_k - f_0(t_0)| + |f_0(t_0) - f_0(t_k)| \rightarrow 0.$$

Thus (ii) is proved. \square

Lemma 3.4. *Suppose that the condition LKC is satisfied. Then $B(p^*)$ is closed and \mathcal{W} is compact in \mathbb{C}^n .*

Proof. Let $\{t_k\} \subseteq B(p^*)$ and $\{\tau_k\} \subseteq \cup_{t \in B(p^*)} \tau(t)$ be such that $\tau_k \in \tau(t_k)$, $t_k \rightarrow t_0 \in Q$ and $\tau_k \rightarrow \tau \in \mathbb{C}$. Then $|\tau_k| = |\tau| = 1$. Moreover, since $Q \setminus Q_N$ is finite, we assume, without loss of generality, that each $t_k \in Q_N$. Then, for each k ,

$$\operatorname{Re} \overline{-\tau_k}(z - p^*(t_k)) \leq 0 \quad \text{for each } z \in \Omega_{t_k}. \tag{3.46}$$

By the condition LKC, for each $z \in \Omega_{t_0}$, there exists $z_k \in \Omega_{t_k}$ such that $z_k \rightarrow z$. Noting that $p^*(t_k) \rightarrow p^*(t_0)$, it follows from (3.46) that

$$\operatorname{Re} \overline{-\tau}(z - p^*(t_0)) \leq 0 \quad \text{for all } z \in \Omega_{t_0}. \tag{3.47}$$

Since $p^*(t_0) \in \Omega_{t_0}$ as $p^* \in \mathcal{P}_\Omega$, this means that $-\tau \in N_{\Omega_{t_0}}(p^*(t_0))$. Since $\tau \neq 0$, this implies that $p^*(t_0) \in \operatorname{bd}\Omega_{t_0}$ and so $t_0 \in B(p^*(t_0))$. Hence, $B(p^*)$ is closed and hence $\tau \in \cup_{t \in B(p^*)} \tau(t)$. This shows that $\cup_{t \in B(p^*)} \tau(t)$ is closed and hence compact since it is bounded. By definition, it is now easily verified that \mathcal{U} is compact. Since \mathcal{V} is compact, it follows that \mathcal{W} is compact. \square

Lemma 3.5. *Suppose that the conditions LKC and IC hold. Then $B^{rb}(p^*)$ is closed and the closure of \mathcal{W}_r^+ is contained in \mathcal{W}_r .*

Proof. As in the proof of Lemma 3.4, let $\{t_k\} \subseteq B^{rb}(p^*)$ and $\tau_k \in \tau_r^+(t_k)$ for each k such that $t_k \rightarrow t_0 \in Q$ and $\tau_k \rightarrow \tau \in \mathbb{C}$. Thus, by (3.9) and (3.17), one has $\{t_k\} \subseteq B(p^*)$ and $\tau_k \in \tau(t_k)$ for each k . By Lemma 3.4, it follows that $t_0 \in B(p^*)$ and $-\tau \in N_{\Omega_{t_0}}(p^*(t_0))$ thanks to LKC. It suffices to show that $t_0 \in B^{rb}(p^*)$ and $\tau \in \tau_r(t_0)$. If $t_0 \in Q_N$, they are done by the proof of Lemma 3.4 because one then has $t_0 \in B(p^*) \cap Q_N \subseteq B^{rb}(p^*)$ and $\tau \in \tau(t_0) = \tau_r(t_0)$. Thus, we may assume henceforth that $t_0 \notin Q_N$. Note that if $t_k \in Q_E$ for infinitely many k , then, since Q_E is finite, one has $t_k = t_0$ for these k (say for all k by considering a subsequence if necessary). Hence $t_0 \in B^{rb}(p^*)$ and $\tau_k \in \tau_r(t_0)$. However, in view of (3.15), $\tau_r(t_0)$ must be a singleton in the present case, so $\tau \in \tau_r(t_0)$. Therefore, without loss of generality, we may assume that $t_k \in Q_N$ for each k . In view of (3.27), to complete the proof, it is sufficient to show that $t_0 \in Q_E$, $p^*(t_0) \in \operatorname{rb}\Omega_{t_0}$ and

$$\operatorname{Re} \bar{\tau}(z - p^*(t_0)) > 0 \quad \text{for some } z \in \operatorname{ri}\Omega_{t_0}. \tag{3.48}$$

By IC, there exists $\bar{p} \in \mathcal{P}_\Omega$ satisfying (3.7). Let $\delta > 0$ be such that

$$\mathbf{B}(0, \delta) \subset \bigcap_{t \in Q_N} (\Omega_t - \bar{p}(t)). \tag{3.49}$$

We will show that there exists an integer $N > 0$ such that

$$\mathbf{B}\left(\bar{p}(t_0) - p^*(t_0), \frac{\delta}{2}\right) \subset \bigcap_{k \geq N} (\Omega_{t_k} - p^*(t_k)). \tag{3.50}$$

Indeed, take $N > 0$ such that $|(\bar{p}(t_k) - p^*(t_k)) - (\bar{p}(t_0) - p^*(t_0))| < \frac{\delta}{2}$ for each $k \geq N$. Then

$$\mathbf{B}\left(\bar{p}(t_0) - p^*(t_0), \frac{\delta}{2}\right) \subset \mathbf{B}(\bar{p}(t_k) - p^*(t_k), \delta) \quad \text{for each } k \geq N. \tag{3.51}$$

On the other hand, by (3.49),

$$\mathbf{B}(\bar{p}(t_k) - p^*(t_k), \delta) \subset \Omega_{t_k} - p^*(t_k) \quad \text{for each } k. \tag{3.52}$$

Consequently, (3.50) follows from (3.51) and (3.52). Set $\Omega^* := \bigcap_{k \geq N} (\Omega_{t_k} - p^*(t_k))$. Then $0 \in \text{bd } \Omega^*$ and $\bar{p}(t_0) - p^*(t_0) \in \text{int } \Omega^*$ by (3.50). In particular,

$$\text{Re } \bar{\alpha}(\bar{p}(t_0) - p^*(t_0)) < 0 \quad \text{for each } \alpha \in N_{\Omega^*}(0) \setminus \{0\}.$$

Hence, there exists a positive number b such that, for each $\alpha \in N_{\Omega^*}(0)$ with $|\alpha| = 1$,

$$\text{Re } \bar{\alpha}(\bar{p}(t_0) - p^*(t_0)) \leq -b < 0. \tag{3.53}$$

Since $-\tau_k \in N_{\Omega_{t_k}}(p^*(t_k))$, $|\tau_k| = 1$ and $N_{\Omega_{t_k}}(p^*(t_k)) \subseteq N_{\Omega^*}(0)$ for each $n \geq N$, we have that

$$\text{Re } \overline{-\tau_k}(\bar{p}(t_0) - p^*(t_0)) \leq -b < 0 \quad \text{for each } k \geq N. \tag{3.54}$$

Noting that $\tau_k \rightarrow \tau$, it follows that

$$\text{Re } \overline{-\tau}(\bar{p}(t_0) - p^*(t_0)) \leq -b < 0. \tag{3.55}$$

Thus Ω_{t_0} contains more than one point ($\bar{p}(t_0)$, $p^*(t_0)$ being distinct members of Ω_{t_0}). It follows that Ω_{t_0} is a line-segment (recalling that $t_0 \notin Q_N$, i.e., $t_0 \in Q_E$. Consequently, by (3.7), $\bar{p}(t_0) \in \text{ri } \Omega_{t_0}$. Therefore (3.48) holds by (3.55). Since $0 \neq -\tau \in N_{\Omega_{t_0}}(p^*(t_0))$ (noting $\bar{p}(t_0) \in \Omega_{t_0}$), it follows from (3.55) that $p^*(t_0)$ must be an end point of Ω_{t_0} , i.e., $p^*(t_0) \in \text{rb } \Omega_{t_0}$. The proof of Lemma 3.5 is complete. \square

Lemma 3.6. *Let Φ be a complex linear functional on \mathcal{P} such that*

$$\text{Re } \Phi(p) = 0 \quad \text{for each } p \in \mathcal{P}_R. \tag{3.56}$$

Then there exist a nonnegative integer s with $s \leq 2n - m$, $\{t''_j\}_{j=1}^s \subseteq Q_E \cup Q_S$ and $\{\tau''_j\}_{j=1}^s \subset \mathbb{C} \setminus \{0\}$ with each $\tau''_j \in -N_{\Omega_{t''_j}}(p^(t''_j))$ such that*

$$\Phi(p) + \sum_{j=1}^s p(t''_j) \overline{\tau''_j} = 0 \quad \text{for each } p \in \mathcal{P}. \tag{3.57}$$

Proof. We assume that $Q_E \cup Q_S \neq \emptyset$ (the result is trivial otherwise). For each $t \in Q_E$, $\text{span}_R(\Omega_t - p^*(t))$ is a line passing through the origin. Hence there exists $\tau_t \in \mathbb{C}$ with $|\tau_t| = 1$ which is “perpendicular” to $\text{span}_R(\Omega_t - p^*(t))$ in the sense that

$$\text{Re } \overline{\tau_t} \alpha = 0 \iff \alpha \in \text{span}_R(\Omega_t - p^*(t)). \tag{3.58}$$

Thus, defining the real linear functional ξ_t on \mathcal{P} by

$$\xi_t(p) = \text{Re } \overline{\tau_t} p(t) \quad \text{for each } p \in \mathcal{P}, \tag{3.59}$$

we can characterize the kernel of ξ_t for $t \in Q_E$:

$$p \in \text{Ker } \xi_t \iff p(t) \in \text{span}_R(\Omega_t - p^*(t)). \tag{3.60}$$

For each $t \in Q_S$, two linear functionals on \mathcal{P} (respectively, denoted by ξ_t and ξ'_t) will be useful for us. They are defined by

$$\xi_t(p) = \text{Re } p(t) \quad \text{for each } p \in \mathcal{P},$$

$$\xi'_t(p) = \text{Re } ip(t) \quad \text{for each } p \in \mathcal{P},$$

where $i = \sqrt{-1}$. Thus, for $t \in Q_S$,

$$p(t) = 0 \iff p \in \text{Ker } \xi_t \cap \text{Ker } \xi'_t.$$

By (3.10), we have that

$$\mathcal{P}_R = \left(\bigcap_{t \in Q_E \cup Q_S} \text{Ker } \xi_t \right) \cap \left(\bigcap_{t \in Q_S} \text{Ker } \xi'_t \right). \tag{3.61}$$

It will be convenient to list the functionals

$$\{\xi_t : t \in Q_E \cup Q_S\} \cup \{\xi'_t : t \in Q_S\} := \{\xi^1, \xi^2, \dots, \xi^r\}, \tag{3.62}$$

where $r = |Q_E| + 2|Q_S|$, and for example $|Q_E|$ stands for the number of elements in Q_E . Letting $q := 2n - m$, the difference of real dimensions of \mathcal{P} and \mathcal{P}_R , one has $q \leq r$. Recalling that $\{\psi_1, \dots, \psi_m\}$ is a basis of \mathcal{P}_R , there exist $\psi_{m+1}, \dots, \psi_{2n} \in \mathcal{P}$ such that $\{\psi_1, \dots, \psi_{2n}\}$ is a real basis of \mathcal{P} . Since $\mathcal{P}_R \cap \text{span}_R\{\psi_{m+1}, \dots, \psi_{m+q}\} = \{0\}$, it is easy to verify that the vectors $\{\vec{a}_i : i = m + 1, \dots, m + q\} \subset \mathbb{R}^r$ are (real) linearly independent, where each \vec{a}_i is defined by

$$\vec{a}_i = (\xi^v(\psi_i))_{v=1}^r \in \mathbb{R}^r \quad \text{for each } i = m + 1, \dots, m + q.$$

Consequently, there exist q -many coordinates such that the restrictions $\vec{a}_i|$ of \vec{a}_i ($m + 1 \leq i \leq m + q$) to these coordinates are linearly independent. Without loss of generality, let us assume that they are the first q coordinates; thus,

$$\det (\xi^v(\psi_i))_{1 \leq v \leq q, m+1 \leq i \leq m+q} \neq 0. \tag{3.63}$$

Therefore there exist real numbers $(\lambda'_1, \dots, \lambda'_q)$ such that

$$\sum_{v=1}^q \lambda'_v \xi^v(\psi_i) = -\operatorname{Re} \Phi(\psi_i) \quad \text{for } i = m + 1, \dots, m + q. \tag{3.64}$$

Note that, for $i = 1, 2, \dots, m$, both sides of (3.64) are equal to zero thanks to (3.56) and (3.61). Therefore

$$\sum_{v=1}^q \lambda'_v \xi^v(p) + \operatorname{Re} \Phi(p) = 0 \tag{3.65}$$

for each $p \in \{\psi_1, \dots, \psi_m, \psi_{m+1}, \dots, \psi_{m+q}\}$. In view of (3.65), it is clear that, to complete the proof, it suffices to show that the first summation in (3.65) can be expressed in the form

$$\sum_{v=1}^q \lambda'_v \xi^v(p) = \operatorname{Re} \sum_{j=1}^s p(t''_j) \overline{\tau''_j} \quad \text{for each } p \in \mathcal{P} \tag{3.66}$$

for some $s \leq q$, $\{t''_j\}_{j=1}^s \subseteq Q_E \cup Q_S$, $\{\tau''_j\}_{j=1}^s \subset \mathbb{C} \setminus \{0\}$ such that

$$\tau''_j \in -N_{\Omega_{t''_j}}(p^*(t''_j)) \quad \text{for each } j = 1, 2, \dots, s. \tag{3.67}$$

To do this, we consider, in light of (3.62), v with $1 \leq v \leq q$ for each of the following cases.

(a) $\xi^v = \xi_t$ for some $t \in Q_E$. Then $\tau''_t := \lambda'_v \tau_t \in -N_{\Omega_t}(p^*(t))$ by (3.58), and by (3.59),

$$(\lambda'_v \xi^v)(p) = \lambda'_v \operatorname{Re} \overline{\tau_t} p(t) = \operatorname{Re} p(t) \overline{\tau''_t} \quad \text{for each } p \in \mathcal{P}.$$

(b) $\xi^v = \xi_t$ for some $t \in Q_S$ but $\xi'_t \notin \{\xi^1, \xi^2, \dots, \xi^r\}$. Then $\tau''_t := \lambda'_v \in -N_{\Omega_t}(p^*(t))$ as $\Omega_t = \{p^*(t)\}$, and

$$(\lambda'_v \xi^v)(p) = \lambda'_v \operatorname{Re} p(t) = \operatorname{Re} p(t) \overline{\tau''_t} \quad \text{for each } p \in \mathcal{P}.$$

(c) $\xi^v = \xi'_t$ for some $t \in Q_S$ but $\xi_t \notin \{\xi^1, \xi^2, \dots, \xi^r\}$. Then $\tau''_t := \overline{i\lambda'_v} \in -N_{\Omega_t}(p^*(t))$ and

$$(\lambda'_v \xi^v)(p) = \lambda'_v \operatorname{Re} i p(t) = \operatorname{Re} p(t) \overline{\tau''_t} \quad \text{for each } p \in \mathcal{P}.$$

(d) $\xi^v = \xi_t$ for some $t \in Q_S$ which satisfies an additional property that $\xi'_t \in \{\xi^1, \xi^2, \dots, \xi^r\}$. Assume that $\xi'_t = \xi^v$. Then $\tau''_t := \lambda'_v + \overline{i\lambda'_v} \in -N_{\Omega_t}(p^*(t))$ as $\Omega_t = \{p^*(t)\}$, and

$$\lambda'_v \xi^v(p) + \lambda'_{v'} \xi^{v'}(p) = \lambda'_v \operatorname{Re} p(t) + \lambda'_{v'} \operatorname{Re} \bar{i} p(t) = \operatorname{Re} p(t) \overline{\tau''_t} \quad \text{for each } p \in \mathcal{P}.$$

Combining (a–d) and deleting these terms with the corresponding $\tau''_t = 0$, (3.66) is seen to hold. \square

In the following Theorems 3.1–3.5, we consider relations of the following statements for a fixed pair of functions $f \in C(Q)$ and $p^* \in \mathcal{P}_\Omega$. Recall that $\sigma := f - p^*$.

- (i) p^* is a best-restricted range approximation to f from \mathcal{P} with respect to $\{\Omega_t\}$.
- (ii) For each $p \in \mathcal{P}_R$, there exist $t \in M(\sigma)$, $t' \in B^{rb}(p^*)$ such that

$$\max \left\{ \operatorname{Re} (p(t)\overline{\sigma(t)}), \max_{\tau \in \tau_r^+(t')} \operatorname{Re} (p(t')\overline{\tau}) \right\} \geq 0. \tag{3.68}$$

- (iii) For each $p \in \mathcal{P}_R$, there exist $t \in M(\sigma)$, $t' \in B^{rb}(p^*)$ and $\tau \in \tau_r(t')$ such that

$$\max\{\operatorname{Re} (p(t)\overline{\sigma(t)}), \operatorname{Re} (p(t')\overline{\tau})\} \geq 0. \tag{3.69}$$

- (iv) The origin of \mathbb{R}^m belongs to the convex hull of the \mathcal{W}_r^+ .
- (v) The origin of \mathbb{R}^m belongs to the convex hull of the \mathcal{W}_r .
- (vi) The origin of \mathbb{C}^n belongs to the convex hull of the \mathcal{W} .
- (vii) There exist

$$\{t_i\}_{i=1}^k \subseteq M(\sigma), \quad \{\lambda_i\}_{i=1}^k \subset (0, +\infty)$$

and

$$\{t'_j\}_{j=1}^l \subseteq B^{rb}(p^*), \quad \{\tau'_j\}_{j=1}^l \subset \mathbb{C} \setminus \{0\}$$

with $1 + l \leq k + l \leq m + 1$ such that $\tau'_j \in -N_{\Omega_{t'_j}}(p^*(t'_j))$ for each $j = 1, \dots, l$, and

$$\operatorname{Re} \sum_{i=1}^k \lambda_i p(t_i)\overline{\sigma(t_i)} + \operatorname{Re} \sum_{j=1}^l p(t'_j)\overline{\tau'_j} = 0 \quad \text{for each } p \in \mathcal{P}_R. \tag{3.70}$$

- (viii) There exist

$$\{t_i\}_{i=1}^k \subseteq M(\sigma), \quad \{\lambda_i\}_{i=1}^k \subset (0, +\infty) \tag{3.71}$$

and

$$\{t'_j\}_{j=1}^{l'} \subseteq B(p^*), \quad \{\tau'_j\}_{j=1}^{l'} \subset \mathbb{C} \setminus \{0\} \tag{3.72}$$

with $1 + l' \leq k + l' \leq 2n + 1$ such that $\tau'_j \in -N_{\Omega_{t'_j}}(p^*(t'_j))$ for each $j = 1, \dots, l'$, and

$$\sum_{i=1}^k \lambda_i p(t_i)\overline{\sigma(t_i)} + \sum_{j=1}^{l'} p(t'_j)\overline{\tau'_j} = 0 \quad \text{for each } p \in \mathcal{P}. \tag{3.73}$$

- (ix) $J(\sigma)|_{\mathcal{P}} \cap \left(\sum_{t \in Q} N_{C_t}(p^*)|_{\mathcal{P}} \right) \neq \emptyset$.

Theorem 3.1. *The following implications hold.*

$$\begin{array}{ccccccc} \text{(vii)} & \iff & \text{(viii)} & \iff & \text{(ix)} & \implies & \text{(iv)} \implies \text{(ii)} \iff \text{(iii)} \\ & & & & \downarrow & & \updownarrow \\ & & & & \text{(i)} & & \text{(v)} \implies \text{(vi)} \end{array}$$

Proof. By (3.29), it is easy to verify that (iv) \iff (v). Also, by (3.17) and (3.18), we have (ii) \iff (iii). Applying Lemma 3.6 to the functional Φ on \mathcal{P} defined by

$$\Phi(p) = \sum_{i=1}^k \lambda_i p(t_i) \overline{\sigma(t_i)} + \operatorname{Re} \sum_{j=1}^l p(t'_j) \overline{\tau'_j} \quad \text{for each } p \in \mathcal{P},$$

we have that (vii) \implies (viii) with $l' = l + s$, where s is as in Lemma 3.6. To show (viii) \implies (vii) \implies (v), we suppose that (viii) holds. Thus we assume that (3.73) holds with appropriate $k, l', \{t_i\}, \{\lambda_i\}, \{t'_j\}$ and $\{\tau'_j\}$ as stated in (viii). Without loss of generality, assume that $\{t'_1, \dots, t'_l\} \subseteq B^{rb}(p^*), \{t'_{l+1}, \dots, t'_{l'}\} \subseteq B(p^*) \setminus B^{rb}(p^*)$. Note that if $l + 1 \leq j \leq l'$, then $t'_j \in Q_S \cup Q_E$, and $\Omega_{t'_j}$ is either a singleton or a line-segment containing $p^*(t'_j)$ as an internal point (seeing (3.9)). Hence

$$\operatorname{Re} \overline{\tau'_j} \alpha = 0 \quad \text{for each } \alpha \in \operatorname{span}_R(\Omega_{t'_j} - p^*(t'_j)), \quad j = l + 1, \dots, l'. \tag{3.74}$$

This implies that, for each $p \in \mathcal{P}_R, \operatorname{Re} \overline{\tau'_j} p(t'_j) = 0$ if $l + 1 \leq j \leq l'$ because $p(t'_j) \in \operatorname{span}_R(\Omega_{t'_j} - p^*(t'_j))$ by (3.10). Consequently, (3.73) implies that

$$\operatorname{Re} \sum_{i=1}^k \lambda_i p(t_i) \overline{\sigma(t_i)} + \operatorname{Re} \sum_{j=1}^l p(t'_j) \overline{\tau'_j} = 0 \quad \text{for each } p \in \mathcal{P}_R. \tag{3.75}$$

Replacing λ_i, t'_j by their appropriate positive multipliers if necessary, we can assume that $k + l \leq m + 1$. To see this, we note first that if $\frac{\tau'_j}{|\tau'_j|} \in \tau(t'_j) \setminus \tau_r(t'_j)$, then (3.16), (3.13) and (3.10) imply that $\operatorname{Re} p(t'_j) \overline{\tau'_j} = 0$ for each $p \in \mathcal{P}_R$ and thus the corresponding term in (3.75) can be deleted. Henceforth, we suppose therefore that each $\frac{\tau'_j}{|\tau'_j|} \in \tau_r(t'_j)$ in (3.75). Noting that $k \geq 1$ from the assumption and recalling definitions (3.20) and (3.25), it follows from (3.75) (applied to ψ_1, \dots, ψ_m in place of p) that

$$-\mathbf{b}_r(t_1) \in \operatorname{cone} \{\mathbf{b}_r(t_2), \dots, \mathbf{b}_r(t_k), \mathbf{c}_r(t'_1), \dots, \mathbf{c}_r(t'_l)\} \subseteq \mathbb{R}^m.$$

Consequently, by [19, Corollary 17.1.2], $-\mathbf{b}_r(t_1)$ can be expressed as a linear combination of at most m elements from $\{\mathbf{b}_r(t_2), \dots, \mathbf{b}_r(t_k), \mathbf{c}_r(t'_1), \dots, \mathbf{c}_r(t'_l)\}$ with positive coefficients. Thus, appropriately redefining λ_i and t'_j if necessary, we can assume that, $k + l \leq m + 1$, (3.75) holds for each $p \in \{\phi_1, \dots, \phi_m\}$ and hence for all $p \in \mathcal{P}_R$. Therefore (viii) \implies (vii) and hence (viii) \iff (vii).

For (vii) \implies (v) & (i), suppose that (3.70) holds with appropriate $\{t_i\}, \{\lambda_i\}, \{t'_j\}$ and $\{\tau'_j\}$ given in (vii). By an earlier argument, we may assume that $\{t'_1, \dots, t'_r\} \subseteq Q_N, \{t'_{r+1}, \dots, t'_l\} \subseteq Q_E$ and $\frac{\tau'_j}{|\tau'_j|} \in \tau_r(t'_j)$ for each $r + 1 \leq j \leq l$. Thus, (3.70) means that the origin of \mathbb{R}^m belongs to the convex hull of the \mathcal{W}_r . Consequently, (v) holds. We go on to show that (i)

holds. To this end, let $p \in \mathcal{P}_\Omega$. Then $p^* - p \in \mathcal{P}_R$ and

$$\operatorname{Re} \sum_{i=1}^k \lambda_i (p^* - p)(t_i) \overline{\sigma(t_i)} + \operatorname{Re} \sum_{j=1}^l (p^* - p)(t'_j) \overline{\tau'_j} = 0. \tag{3.76}$$

Since $k \geq 1$ and each $\lambda_i > 0$, we assume without loss of generality that $\sum_{i=1}^k \lambda_i = 1$. Thus, $\|f - p\| \geq \sum_{i=1}^k \lambda_i |(f - p)(t_i)|^2$. Since $p \in P_\Omega$ and $\tau'_j \in -N_{\Omega'_j}(p^*(t'_j))$, one has that

$$\operatorname{Re} (p^* - p)(t'_j) \overline{\tau'_j} \leq 0, \quad j = 1, 2, \dots, l. \tag{3.77}$$

Hence

$$\begin{aligned} \|f - p\|^2 &\geq \sum_{i=1}^k \lambda_i |(f - p)(t_i)|^2 + 2\operatorname{Re} \sum_{j=1}^l (p^* - p)(t'_j) \overline{\tau'_j} \\ &= \sum_{i=1}^k \lambda_i |(f - p^*)(t_i)|^2 + \sum_{i=1}^k \lambda_i |(p^* - p)(t_i)|^2 \\ &\quad + 2\operatorname{Re} \sum_{i=1}^k \lambda_i (p^* - p)(t_i) \overline{(f - p^*)(t_i)} + 2\operatorname{Re} \sum_{j=1}^l (p^* - p)(t'_j) \overline{\tau'_j} \\ &= \sum_{i=1}^k \lambda_i |(f - p^*)(t_i)|^2 + \sum_{i=1}^k \lambda_i |(p^* - p)(t_i)|^2 \\ &\geq \|f - p^*\|^2, \end{aligned}$$

where the second equality holds because of (3.76) while the last inequality holds because $\{t_i\} \subseteq M(\sigma)$. This means that p^* is a best approximation to f from P_Ω and hence (i) holds.

For (v) \implies (vi) & (ii), suppose that there exist nonnegative integers k, l with $k + l \geq 1$ such that

$$0 \in \operatorname{conv} \{\mathbf{b}_r(t_1), \mathbf{b}_r(t_2), \dots, \mathbf{b}_r(t_k), \mathbf{c}_r(t'_1), \dots, \mathbf{c}_r(t'_l)\} \subseteq \mathbb{R}^m \tag{3.78}$$

for some $\{t_i\}_{i=1}^k \subseteq M(\sigma)$ and $\{t'_j\}_{j=1}^l \subseteq B^{rb}(p^*)$. By the Caratheodory Theorem (cf. [2] or [21, p. 73]), we assume without loss of generality that $k + l \leq m + 1$. Moreover, by (3.17), (3.20) and (3.25), there exist $\{\lambda_i\} \subset (0, +\infty)$ and $\{\tau'_j\} \subset \mathbb{C} \setminus \{0\}$ with $\tau'_j \in -N_{\Omega'_j}(p^*(t'_j)) \setminus \{0\}$ for each j such that (3.70) holds for each $p \in \{\psi_1, \dots, \psi_m\}$ and hence for each $p \in \mathcal{P}_R$. (Note: Since k may be zero, we cannot conclude that (vii) holds.) Now by applying Lemma 3.6 to the functional $\Phi: \mathcal{P}_R \rightarrow \mathbb{C}$ defined by

$$\Phi(p) = \sum_{i=1}^k \lambda_i p(t_i) \overline{\sigma(t_i)} + \sum_{j=1}^l p(t'_j) \overline{\tau'_j} \quad \text{for each } p \in \mathcal{P}$$

we conclude that (3.57) holds with appropriate $\{t''_j\}, \{\tau''_j\}$ stated in Lemma 3.6. By the Caratheodory Theorem, we assume that $k + l + s \leq 2n + 1$. Thus we see that (vi) holds (dividing both sides of (3.57) by a positive constant if necessary). Note, in passing, again

that (viii) would hold provided that $k \neq 0$. Moreover, (ii) must also hold because otherwise there exists an element $p_0 \in \mathcal{P}_R$ such that

$$\max\{\operatorname{Re}(p_0(t_i)\overline{\sigma(t_i)}), \operatorname{Re}(p_0(t'_j)\overline{\tau'_j})\} < 0$$

for each $i = 1, \dots, k$ and $j = 1, \dots, l$. (3.79)

This contradicts (3.70) as the number on the left-hand side of (3.70) with $p = p_0$ is negative by (3.79). Hence, the proof of (v) \implies (vi) & (ii) is complete.

Finally we show that (viii) \iff (ix). Suppose first that (ix) holds. Then, there exist $v^* \in J(\sigma)$ and $w_j^* \in N_{C_{t'_j}}(p^*)$, $j = 1, 2, \dots, s$ with $p^* \in \operatorname{bd} C_{t'_j}$ (namely $t'_j \in B(p^*)$) such that

$$\langle v^*, p \rangle = \sum_{j=1}^s \langle w_j^*, p \rangle \quad \text{for all } p \in \mathcal{P}. \tag{3.80}$$

Set $u^* = v^*/\|v^*\|$. Applying [22, Lemma 1.3, p. 169] to the real linear span of $\mathcal{P} \cup \{f\}$, there exist a positive integer r (with $1 \leq r \leq 2(n + 1)$), r extreme points u_1^*, \dots, u_r^* of the unit ball Σ^* of $C(Q)^*$ and positive constants β_i , $i = 1, 2, \dots, r$, with $\sum_{i=1}^r \beta_i = 1$ such that

$$\langle u^*, p \rangle = \sum_{i=1}^r \beta_i \langle u_i^*, p \rangle \quad \text{for all } p \in \operatorname{span}(\mathcal{P} \cup \{f\}). \tag{3.81}$$

By a well-known representation of the extreme points of Σ^* (cf. [22, p. 69]), there exist some $\alpha_i \in \mathbb{C}$ with $|\alpha_i| = 1$ and $t_i \in Q$ such that

$$u_i^* = \alpha_i e_{t_i}, \quad i = 1, 2, \dots, r.$$

By the definition of u^* , $\|u^*\| = 1$ and $\langle u^*, \sigma \rangle = \|\sigma\|$; hence, by (3.81), $t_i \in M(\sigma)$ and $\alpha_i = \overline{\sigma(t_i)}/\|\sigma\|$. Furthermore, by (3.41), for each j , there exists $\tau'_j \in -N_{\Omega_{t'_j}}(p^*(t'_j))$ such that $-w_j^* = \overline{\tau'_j} e_{t'_j}$. Therefore, (3.80) becomes

$$\sum_{i=1}^r \beta_i p(t_i)\overline{\sigma(t_i)} + \sum_{j=1}^s p(t'_j)\overline{\tau'_j} = 0 \quad \text{for all } p \in \mathcal{P}, \tag{3.82}$$

where $\beta'_i = \|v^*\| \beta_i / \|\sigma\|$ for each $i = 1, \dots, r$. Set

$$\mathbf{c}_j = (\phi_1(t), \dots, \phi_n(t))\overline{\tau'_j} \quad \text{for each } j = 1, \dots, s.$$

Then (3.82) implies that

$$-\beta'_1 \mathbf{b}(t_1) \in \operatorname{cone}\{\beta'_2 \mathbf{b}(t_2), \dots, \beta'_r \mathbf{b}(t_r), \mathbf{c}_1, \dots, \mathbf{c}_s\}.$$

Since $\dim_R \mathcal{P} = 2n$, by [19, Corollary 17.1.2], $-\beta'_1 \mathbf{b}(t_1)$ can be expressed as a linear combination of at most $2n$ elements from $\{\beta'_2 \mathbf{b}(t_2), \dots, \beta'_r \mathbf{b}(t_r), \mathbf{c}_1, \dots, \mathbf{c}_s\}$ with positive coefficients. Hence, replacing β'_i and τ'_j by their appropriate positive multipliers we can assume without loss of generality that r, s in (3.82) satisfy the additional property that

$1 + s \leq r + s \leq 2n + 1$. Thus (viii) holds with $(k, l') = (r, s)$. Conversely, suppose that (viii) holds. Hence we have (3.73) with appropriate $\{t_i\}_{i=1}^k$, $\{\lambda_i\}_{i=1}^k$ and $\{t'_j\}_{j=1}^{l'}$, $\{\tau'_j\}_{j=1}^{l'}$ as stated in (viii). We can of course assume that $\sum_{i=1}^k \lambda_i = 1$, and rewrite (3.73) as

$$\sum_{i=1}^k \lambda_i \overline{\sigma(t_i)} e_{t_i} = - \sum_{j=1}^{l'} \overline{\tau'_j} e_{t'_j} \quad (\in \mathcal{P}^*). \tag{3.83}$$

By Lemma 3.2, $\overline{\tau'_j} e_{t'_j} \in N_{C_{t'_j}}(p^*)$ for each $j = 1, 2, \dots, l'$. On the other hand, since $t_i \in M(\sigma)$, we have that $\langle \overline{\sigma(t_i)} e_{t_i}, \sigma \rangle = \|\sigma\|^2$ for each $i = 1, 2, \dots, k$. Therefore the functional expressed by either side of (3.83) belongs to the intersection in (ix). \square

Theorem 3.2. *It holds that (v) \iff (vii) if IC is assumed, and that (vi) \iff (viii) if IC_0 is assumed.*

Proof. Suppose that (v) holds and we proceed as in the proof for (v) \implies (vi) & (ii) of Theorem 3.1. If IC is assumed in addition, $0 \notin \text{conv } \mathcal{U}_r$ by Lemma 3.1. Hence k in (3.78) must be nonzero and so (vii) holds. Similarly, suppose that (vi) holds (thus, with the exception that k is possibly zero, (3.73) holds). Suppose further that IC_0 is assumed (instead of IC). Then $0 \notin \text{conv } \mathcal{U}$ by Lemma 3.1. Hence k in (3.73) must be nonzero. Therefore (viii) holds. \square

Theorem 3.3. *If the system $\{\mathcal{P}, C_t : t \in Q\}$ has the strong CHIP at p^* , then (i) \iff (vii).*

Proof. Note that $\mathcal{P}_\Omega = \mathcal{P} \cap (\bigcap_{t \in Q} C_t)$. By the implication (i) \iff (iv) in Theorem 2.3 and the fact that \mathcal{P} is a vector subspace containing p^* (so $N_{\mathcal{P}}(p^*)|_{\mathcal{P}} = 0$), we now have that (i) \iff (ix) thanks to the strong CHIP assumption. Since (ix) \iff (vii) by Theorem 3.1, (i) \iff (vii) holds. \square

Theorem 3.4. *If both LKC and IC are assumed, then the statements in the list (i)–(ix) except (vi) are equivalent to each other.*

Proof. Suppose that both LKC and IC hold. We will show that the CCS-system $\{\mathcal{P}, C_t : t \in Q\}$ has the strong CHIP at p^* . For this purpose, note that, by Lemma 3.3 and Remark 2.1, the condition LKC implies that the set-valued function $t \mapsto C_t$ is lower semicontinuous on Q while, by Lemma 3.1, the condition IC implies that the system $\{\mathcal{P}, C_t : t \in Q\}$ satisfies the weak-strong interior-point condition with (Q_E, Q_S) . By Theorem 2.1, the system $\{\mathcal{P}, C_t : t \in Q\}$ has the strong CHIP at p^* . By Theorems 3.3, 3.1 and 3.2, it remains to show that (ii) \iff (v). Suppose on the contrary that (ii) holds but (v) is false. Then, by Lemma 3.5, $0 \notin \text{conv } \overline{\mathcal{W}_r^+} (\subseteq \text{conv } \mathcal{W}_r)$. By the Linear Inequality Theorem (see [2]), there exists $z^0 = (\gamma_1^0, \dots, \gamma_m^0) \in \mathbb{R}^m$ such that

$$\langle u, z^0 \rangle < 0 \quad \text{for all } u \in \overline{\mathcal{W}_r^+}. \tag{3.84}$$

Then $\max_{u \in \overline{\mathcal{W}_r^+}} \langle u, z^0 \rangle < 0$ because $\overline{\mathcal{W}_r^+}$ is compact (noting that \mathcal{W}_r^+ is bounded). Let $p_0 = \sum_{i=1}^m \gamma_i^0 \psi_i$. Then $p_0 \in \mathcal{P}_R$. By (3.25) and (3.21), for any $t \in M(\sigma)$, $t' \in B^{rb}(p^*)$ and $\tau \in \tau_r^+(t')$, one has

$$\operatorname{Re}(p_0(t)\overline{\sigma(t)}) = \langle \mathbf{b}_r(t), z^0 \rangle, \quad \operatorname{Re}(p_0(t')\overline{\tau}) = \langle u_\tau, z^0 \rangle,$$

where $u_\tau \in \mathbf{c}_r^+(t')$ is defined by $u_\tau := (\operatorname{Re} \psi_1(t')\overline{\tau}, \dots, \operatorname{Re} \psi_m(t')\overline{\tau})$. Since $\{\mathbf{b}_r(t)\} \cup \mathbf{c}_r^+(t') \subseteq \overline{\mathcal{W}_r^+}$, we have that

$$\begin{aligned} & \max \left\{ \operatorname{Re}(p_0(t)\overline{\sigma(t)}), \max_{\tau \in \tau_r^+(t')} \operatorname{Re}(p_0(t')\overline{\tau}) \right\} \\ &= \max \left\{ \langle \mathbf{b}_r(t), z^0 \rangle, \max_{\tau \in \tau_r^+(t')} \langle u_\tau, z_0 \rangle \right\} \leq \max_{u \in \overline{\mathcal{W}_r^+}} \langle u, z^0 \rangle < 0, \end{aligned}$$

which contradicts (ii). \square

Theorem 3.5. *If both KC and IC_0 are assumed, then the statements (i)–(ix) are mutually equivalent.*

Proof. Suppose that both KC and IC_0 hold. Then \mathcal{W} is compact in \mathbb{C}^n by Lemma 3.4. Using this, and similar arguments as in the proof of (ii) \implies (v) in Theorem 3.4 give that (ii) \iff (vi) (use W , \mathbb{C}^n and $\operatorname{Re} \langle u, z \rangle$ to replace $\overline{W_r^+}$, \mathbb{R}^m and $\langle u, z \rangle$). By Theorem 3.2, (vi) \iff (viii). Thus, by Theorem 3.1, it remains to show that (i) \iff (vii). In view of Theorem 3.3, it suffices to show that the CCS-system $\{\mathcal{P}, C_t : t \in Q\}$ has the strong CHIP at p^* . But this follows easily from Theorem 2.2 which is applicable to this system by Lemma 3.1(i) and Lemma 3.3 (thanks to the assumptions). \square

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