

ON BEST APPROXIMATION BY NONCONVEX SETS AND PERTURBATION OF NONCONVEX INEQUALITY SYSTEMS IN HILBERT SPACES*

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Abstract. By virtue of convexification techniques, we study best approximations to a closed set C in a Hilbert space as well as perturbation conditions relative to C and a nonlinear inequality system. Some results on equivalence of the best approximation and the basic constraint qualification are established.

Key words. best approximation, nonlinear constraint, nonlinear inequality system, strong CHIP, the basic constraint qualification condition, generalized Mangasarian–Fromowitz constraint qualification and regularity

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1. Introduction. Let X, Y be Hilbert spaces over the real field \mathbb{R} (unless specifically stated otherwise), and let C be a closed convex subset of X . Let K consist of all $x \in C$ satisfying the nonconvex inequality system

$$(NIS) \quad A_i(x) \leq 0 \quad \forall i = 1, 2, \dots, m,$$

where each A_i is a composite function of the form $H_i \circ F_i$ with $H_i : Y \rightarrow \mathbb{R}$, $F_i : X \rightarrow Y$ for each i . We assume throughout that, for each i , H_i is continuous convex and F_i is Fréchet differentiable on X with continuous Fréchet derivative denoted by $F'_i(\cdot)$. In general, A_i is nondifferentiable and nonlinear. For each $x \in X$, let $\partial A_i(x)$ denote the subdifferential of A_i at x . Let $x^* \in K$ and $I(x^*)$ denote the set of all active indices $i : I(x^*) = \{i : A_i(x^*) = 0\}$. Let P_C and P_K denote the projection operators from X to C and K , respectively. Because it is generally easier to compute P_C than P_K (noting, in particular, that K is not necessarily convex), we stipulate the following definition: x^* is said to have the perturbation property with respect to C and the above (NIS) if for each $x \in X$,

$$(1.1) \quad x^* = P_K(x) \iff x^* = P_C \left(x - \sum_1^m \lambda_i h_i \right)$$

for some $h_i \in \partial A_i(x^*)$ and $\lambda_i \geq 0$, with $\lambda_i = 0$ for all $i \notin I(x^*)$. Here and throughout, $x^* = P_K(x)$ is read as $x^* \in P_K(x)$ if the operator is multivalued. For the special case in which $Y = \mathbb{R}$ and each A_i is affine, this property has been studied by many authors (see, for example, [2, 4, 5, 9, 11, 17, 18]) and has been shown by Deutsch, Li, and Ward (in [10]) to be equivalent to the strong CHIP (strong conical hull intersection

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property) of $\{C, G_1, G_2, \dots, G_m\}$, where each G_i denotes a half-space defined by A_i . Their result has been extended by Li and Jin in [14] to the cases (a) $X = Y$ and each F_i is the identity mapping and (b) $Y = \mathbb{R}$ and each H_i is the identity mapping. In this paper, we consider the case in which each F_i is a general Fréchet differentiable function and each H_i is a general continuous convex function. For each i , let \tilde{A}_i denote the “convexification” of A_i at x^* . (For a definition, see section 2.) Under a regularity condition (which is automatic in the above case (a)), we show in Theorem 3.7 that x^* has the above perturbation property if and only if the convex inequality system

$$\tilde{A}_i(\cdot) \leq 0, \quad i = 1, 2, \dots, m,$$

satisfies the basic constraint qualification (BCQ) relative to C at x^* (which is equivalent to the strong CHIP of the family $\{C; \tilde{A}_i^{-1}(\mathbb{R}_-), i = 1, 2, \dots, m\}$ in the case in which each \tilde{A}_i is affine). This generalizes a main result of [10] and that of [14]. Moreover, in the case of $Y = \mathbb{R}^n$, the regularity condition mentioned above is shown to be implied by a constraint qualification of Mangasarian–Fromowitz type (see Theorem 3.13). In section 4, some applications are made to study the inequality system with respect to an abstract convex cone in a (real or complex) Hilbert space.

2. Notations and preparatory results. The notation used in this paper is standard (see [1, 6, 13, 20]). In particular, for a set Z in X (or in Y or \mathbb{R}^n), the interior (resp., relative interior, closure, convex hull, convex cone hull, affine space, linear space, negative polar) of Z is defined by $\text{int}Z$ (resp., $\text{ri}Z, \bar{Z}, \text{conv}Z, \text{cone}Z, \text{aff}Z, \text{span}Z, Z^\circ$), and the normal cone of Z at \bar{z} is denoted by $N_Z(\bar{z})$ and defined by $N_Z(\bar{z}) = (Z - \bar{z})^\circ$. \mathbb{R}_- denotes the subset of \mathbb{R} consisting of all nonpositive real numbers. For a proper extended real-valued function f on X , the subdifferential of f at $x \in X$ is denoted by $\partial f(x)$ and defined by

$$\partial f(x) = \{z \in X : f(x) + \langle z, y - x \rangle \leq f(y) \forall y \in X\}.$$

In particular, $N_Z(\bar{z}) = \partial \mathbf{I}_Z(\bar{z})$. Here and throughout, \mathbf{I}_Z denotes the indicator function of Z : $\mathbf{I}_Z(x) = 0$ if $x \in Z$, and $\mathbf{I}_Z(x) = +\infty$ if $x \in X \setminus Z$.

Let m, C, K, H_i, F_i , and A_i be as in the preceding section. Let $x^* \in K$ and $I(x^*) = \{i : A_i(x^*) = 0\}$. For each i , let \tilde{A}_i denote the “convexification” of A_i at x^* defined by

$$(2.1) \quad \tilde{A}_i(x) = H_i(F_i(x^*) + F'_i(x^*)(x - x^*)) \quad \forall x \in X.$$

Note that \tilde{A}_i is continuous and convex (because H_i is, and because $x \mapsto F_i(x^*) + F'_i(x^*)(x - x^*)$ is affine). Note also that

$$(2.2) \quad \tilde{A}_i(x^*) = A_i(x^*), \quad i = 1, 2, \dots, m.$$

DEFINITION 2.1. *An element $d \in X$ is called*

(a) *a convexification feasible direction of (NIS) at x^* if*

$$\tilde{A}_i(x^* + d) \leq 0, \quad i \in I(x^*),$$

(b) *a sequentially feasible direction of K at x^* if there exist sequences $\{d_k\} \rightarrow d$ and a sequence of positive real numbers $\{\delta_k\} \rightarrow 0$ such that $\{x^* + \delta_k d_k\} \subseteq K$.*

Let $\text{CFD}(x^*)$ (resp., $\text{SFD}(x^*)$) denote the set of all d satisfying (a) (resp., (b)). Note that $\text{CFD}(x^*) = [\cap_{i \in I(x^*)} \tilde{A}_i^{-1}(\mathbb{R}_-)] - x^*$ and is a closed convex set containing the

origin (but not necessary a cone), while $SFD(x^*)$ is a closed cone (but not necessarily convex).

DEFINITION 2.2. Let $K_S(x^*)$, $K_C(x^*)$, and $K_L(x^*)$ be, respectively, defined by

$$(2.3) \quad K_S(x^*) = (x^* + \overline{\text{conv}}(SFD(x^*))) \cap C,$$

$$(2.4) \quad K_C(x^*) = (x^* + CFD(x^*)) \cap C$$

and

$$(2.5) \quad K_L(x^*) = (x^* + \overline{\text{cone}}(CFD(x^*))) \cap C.$$

Note that the three sets are closed convex and that

$$(2.6) \quad K_C(x^*) = \bigcap_{i \in I(x^*)} \tilde{A}_i^{-1}(\mathbb{R}_-) \cap C.$$

Note also that

$$(2.7) \quad K_C(x^*) \subseteq K_L(x^*),$$

and that $K_C(x^*) = K_L(x^*)$ when the level set $H_i^{-1}(\mathbb{R}_-)$ is a cone with the vertex $F_i(x^*)$ for all $i \in I(x^*)$. Furthermore, we have the following claim.

PROPOSITION 2.3. Suppose that the level set $H_i^{-1}(\mathbb{R}_-)$ is a cone with the vertex $F_i(x^*)$ for all $i \in I(x^*)$. Then $SFD(x^*) \subseteq CFD(x^*)$, and hence $K_S(x^*) \subseteq K_C(x^*) = K_L(x^*)$.

Proof. The second assertion follows from the first and the fact that $CFD(x^*)$ is closed convex. (It is straightforward to verify that $K_C(x^*) = K_L(x^*)$ under the stated assumption.) To prove the first assertion, let $d \in SFD(x^*)$, and let $\{d_k\}$, $\{\delta_k\}$ be as in Definition 2.1(b). In particular, for each $i \in I(x^*)$, one has $H_i(F_i(x^*) + \delta_k d_k) \leq 0$ and hence that $F_i(x^*) + \delta_k d_k \in V_i$, where $V_i := H_i^{-1}(\mathbb{R}_-)$. Therefore

$$\delta_k F'_i(x^*)d_k + o(\|\delta_k d_k\|) \in V_i - F_i(x^*).$$

By the assumption, $V_i - F_i(x^*)$ is a cone. It follows that

$$F'_i(x^*)d_k + o(\|d_k\|) \in V_i - F_i(x^*);$$

passing to the limits, one has that $F'_i(x^*)d \in V_i - F_i(x^*)$. This implies that $d \in CFD(x^*)$, and the proof is complete. \square

PROPOSITION 2.4. Suppose that $\text{int}(\overline{\text{cone}}(CFD(x^*))) \neq \emptyset$ and that, for each $i \in I(x^*)$, $F'_i(x^*)$ is surjective. Then $SFD(x^*) \subseteq \overline{\text{cone}}(CFD(x^*))$, and hence $K_S(x^*) \subseteq K_L(x^*)$.

Proof. We need only to prove the first assertion. As in the proof of Proposition 2.3, let $d \in SFD(x^*)$, with $\{d_k\}$, $\{\delta_k\}$ as in Definition 2.1(b). Then

$$F'_i(x^*)d_k + o(\|d_k\|) \in V_i - F_i(x^*) \subseteq \overline{\text{cone}}(V_i - F_i(x^*));$$

passing to the limits, one has that $F'_i(x^*)d \in \overline{\text{cone}}(V_i - F_i(x^*))$ for each $i \in I(x^*)$. This shows that

$$(2.8) \quad SFD(x^*) \subseteq \bigcap_{i \in I(x^*)} F'_i(x^*)^{-1}(\overline{\text{cone}}(V_i - F_i(x^*))).$$

We claim that

$$(2.9) \quad \text{cone}(\text{CFD}(x^*)) = \bigcap_{i \in I(x^*)} F'_i(x^*)^{-1}(\text{cone}(V_i - F_i(x^*))).$$

Indeed, it is clear that the set on the left-hand side of (2.9) is contained in the set on the right-hand side. Conversely, let d belong to the set of the right-hand side in (2.9). Then for each $i \in I(x^*)$ there exists $t_i > 0$ such that $\frac{d}{t_i} \in F'_i(x^*)^{-1}(V_i - F_i(x^*))$; that is,

$$F'_i(x^*) \frac{d}{t_i} \in V_i - F_i(x^*) \quad \forall i \in I(x^*).$$

Set $t := \max_i t_i$. Then, since $V_i - F_i(x^*)$ is a cone,

$$F'_i(x^*) \frac{d}{t} \in V_i - F_i(x^*), \quad i \in I(x^*).$$

This implies that $\frac{d}{t} \in \text{CFD}(x^*)$, and so $d \in \text{cone}(\text{CFD}(x^*))$. Therefore, (2.9) holds. In addition, by (2.9) and the assumption $\text{int}(\overline{\text{cone}(\text{CFD}(x^*))}) \neq \emptyset$,

$$\text{int} \overline{\bigcap_{i \in I(x^*)} (F'_i(x^*)^{-1}(\text{cone}(V_i - F_i(x^*)))} \neq \emptyset.$$

This implies that

$$\begin{aligned} \overline{\text{cone}(\text{CFD}(x^*))} &= \overline{\bigcap_{i \in I(x^*)} F'_i(x^*)^{-1}(\text{cone}(V_i - F_i(x^*)))} \\ &= \bigcap_{i \in I(x^*)} \overline{F'_i(x^*)^{-1}(\text{cone}(V_i - F_i(x^*)))} \\ &= \bigcap_{i \in I(x^*)} F'_i(x^*)^{-1}(\overline{\text{cone}(V_i - F_i(x^*))}). \end{aligned}$$

Here the last equality holds by the open mapping theorem and by the assumption that $F'_i(x^*)$ is surjective. Thus, by (2.8), we have the desired result. \square

PROPOSITION 2.5. *Let \tilde{A}_i be defined by (2.1). Then it holds that*

$$(2.10) \quad \partial A_i(x^*) = \partial \tilde{A}_i(x^*) = \partial H_i(F_i(x^*)) \circ F'_i(x^*),$$

where, by definition, $z \in \partial H_i(F_i(x^*)) \circ F'_i(x^*)$ if and only if there is $\zeta \in \partial H_i(F_i(x^*))$ such that

$$\langle z, v \rangle = \langle \zeta, F'_i(x^*)v \rangle \quad \forall v \in X.$$

Proof. Recalling that H_i is regular at $F_i(x^*)$ and F_i is strictly differentiable (see [6, Proposition 2.3.6 and section 2.2]), it follows from the chain rule (Theorem 2.3.10 of [6]) that

$$\partial A_i(x^*) = \partial H_i(F_i(x^*)) \circ F'_i(x^*).$$

Similarly, we also have

$$\partial \tilde{A}_i(x^*) = \partial H_i(F_i(x^*)) \circ F'_i(x^*). \quad \square$$

We shall need the following well-known characterization theorem for the best approximation from a closed convex set G in X ; see [3, 9, 10].

PROPOSITION 2.6. *Let G be a closed convex set in X . Then for any $x \in X$, $P_G(x) = g_0$ if and only if $g_0 \in G$, and for any $g \in G$, $\langle x - g_0, g_0 - g \rangle \geq 0$, that is, $x - g_0 \in N_G(g_0)$.*

DEFINITION 2.7. (a) *Let $\{C_0, \dots, C_m\}$ be a collection of closed convex sets and $x \in \bigcap_0^m C_j$. The collection is said to have the strong CHIP at x if*

$$N_{\bigcap_0^m C_j}(x) = \sum_{j=0}^m N_{C_j}(x).$$

(b) *Let $\{\phi_i : i = 1, 2, \dots, m\}$ be a collection of continuous convex functions on X , and let C be a closed convex set in X . The system of convex inequalities*

$$(2.11) \quad \phi_i(z) \leq 0, \quad i = 1, 2, \dots, m,$$

is said to satisfy the BCQ relative to C at x if (2.11) holds for $Z = x$ and

$$N_{C \cap (\bigcap_{i=1}^m \phi_i^{-1}(\mathbb{R}_-))}(x) = N_C(x) + \text{cone}(\bigcup_{i \in I(x)} \{\partial \phi_i(x)\}),$$

where $I(x) = \{i : \phi_i(x) = 0\}$.

Remark 2.1. (a) It is known (see [14]) and easy to see that if system (2.11) satisfies the BCQ relative to C at x , then $\{C, \phi_1^{-1}(\mathbb{R}_-), \dots, \phi_m^{-1}(\mathbb{R}_-)\}$ has the strong CHIP. For further discussions relating to the strong CHIP, see also [7, 8, 19].

(b) If ϕ_i is affine, it is well known that

$$\text{cone}(\partial \phi_i(x)) = N_{\phi_i^{-1}(\mathbb{R}_-)}(x), \quad i \in I(x),$$

and hence that

$$\text{cone} \left(\bigcup_{i \in I(x)} \partial \phi_i(x) \right) = \sum_{i \in I(x)} \text{cone}(\partial \phi_i(x)) = \sum_{i \in I(x)} N_{\phi_i^{-1}(\mathbb{R}_-)}(x) = \sum_{i=1}^m N_{\phi_i^{-1}(\mathbb{R}_-)}(x).$$

Thus system (2.11) satisfies the BCQ relative to C at x if and only if $\{C, \phi_1^{-1}(\mathbb{R}_-), \phi_2^{-1}(\mathbb{R}_-), \dots, \phi_m^{-1}(\mathbb{R}_-)\}$ has the strong CHIP at x .

(c) When $C = X$, the definition of the BCQ relative to C at x is the same as the BCQ at x considered in [12, 13]. Note that if $x \in \bigcap_{j \in I(x)} \phi_j^{-1}(\mathbb{R}_-)$ and $\phi_i(x) = 0$, then $\text{cone}(\partial \phi_i(x)) \subseteq N_{\phi_i^{-1}(\mathbb{R}_-)}(x)$. In addition, some further properties were investigated in [14].

3. Reformulation of the best approximation. We begin with a key lemma that provides a unified tool for the study of best approximation from nonconvex sets.

LEMMA 3.1. *Let \hat{K} be a closed set, C a closed convex set in X , and let $x^* \in X$ be such that $x^* \in \hat{K} \subseteq C$. Let T be a closed convex cone in X . Then the following statements are equivalent:*

- (i) $\hat{K} \subseteq (x^* + T) \cap C$;
- (i*) $\hat{K} \subseteq x^* + T$;
- (ii) $P_{\hat{K}}(x) = x^*$ whenever $x \in X$ with $P_{(x^*+T) \cap C}(x) = x^*$;
- (iii) $P_{\hat{K}}(x) = x^*$ whenever $x \in X$ with $P_{x^*+T}(x) = x^*$.

Note: By abuse of notations, $P_{\hat{K}}(x) = x^*$ is read as $x^* \in P_K(x)$ when $P_{\hat{K}}(x)$ is multivalued.

Proof. Since $\hat{K} \subseteq C$, (i) \Leftrightarrow (i*). It is trivial that (i) \Rightarrow (ii) \Rightarrow (iii). It remains to show that (iii) \Rightarrow (i*). Suppose that (i*) does not hold; take $\bar{x} \in \hat{K}$ such that $\bar{x} \notin x^* + T$. Let $x^* + e \in P_{x^*+T}(\bar{x})$, where $e \in T$. Write h for $\bar{x} - (x^* + e)$. Then, by Proposition 2.6,

$$\langle \bar{x} - (x^* + e), (x^* + z) - (x^* + e) \rangle \leq 0 \quad \forall z \in T;$$

that is, $\langle h, z - e \rangle \leq 0$ for each $z \in T$. Letting $z = 2e, e/2$ separately, it follows that $\langle h, e \rangle = 0$, and hence that $\langle h, z \rangle \leq 0$ for each $z \in T$. Consequently, $P_{x^*+T}(x_t) = x^*$ for each $t > 0$, where $x_t := x^* + th$; this is because of Proposition 2.6 and

$$\langle \bar{x}_t - x^*, (x^* + z) - x^* \rangle = \langle th, z \rangle \leq 0 \quad \forall z \in T.$$

By (iii), it follows that

$$(3.1) \quad P_{\hat{K}}(x_t) = x^*.$$

On the other hand, for $t > 1$ large enough,

$$\begin{aligned} \|x_t - \bar{x}\|^2 &= \|x^* + th - (h + x^* + e)\|^2 \\ &= (t - 1)^2 \|h\|^2 + \|e\|^2 \\ &< t^2 \|h\|^2. \end{aligned}$$

Since $\bar{x} \in \hat{K}$, this contradicts (3.1). The proof is complete. \square

The following corollary is evident.

COROLLARY 3.2. *Let C be a closed convex set, and let T_1, T_2 be closed convex cones in X ; let $x^* \in C$. Then the following statements are equivalent:*

- (i) $C \cap (x^* + T_1) = C \cap (x^* + T_2)$;
- (ii) for any $x \in X$, $P_{C \cap (x^* + T_1)}(x) = x^*$ if and only if $P_{C \cap (x^* + T_2)}(x) = x^*$.

Theorem 3.7 of [14] follows immediately from Lemma 3.1 by applying to $\hat{K} = K$ defined in section 1 and $T = \overline{\text{conv}}(\text{SFD}(x^*))$. Similarly, by letting $T = \overline{\text{cone}}(\text{CFD}(x^*))$ in Lemma 3.1, we have the following result.

PROPOSITION 3.3. *Let $x^* \in K$. Then the following statements are equivalent:*

- (i) $K \subseteq K_L(x^*)$;
- (ii) for any $x \in X$, $P_{K_L(x^*)}(x) = x^* \implies P_K(x) = x^*$.

DEFINITION 3.4. *Let $x^* \in K$. Then*

(a) x^* is called a regular point of K (more precisely, a regular point of K with respect to C and the system (NIS)) if

$$(3.2) \quad K \subseteq K_C(x^*) \subseteq K_S(x^*);$$

(b) x^* is called a weakly regular point of K (with respect to C and the system (NIS)) if

$$(3.3) \quad K \subseteq K_L(x^*) \quad \text{and} \quad K_C(x^*) \subseteq K_S(x^*).$$

Remark 3.1. (a) Obviously, a regular point of K must be a weakly regular point of K ; the converse is true if the assumption of Proposition 2.3 is satisfied.

(b) If F_i is affine for each $i \in I(x^*)$, then x^* is a regular point of K .

THEOREM 3.5. *Let $x^* \in K$. If x^* is a regular point of K in the above sense, then for any $x \in X$,*

$$(3.4) \quad P_K(x) = x^* \iff P_{K_C(x^*)}(x) = x^*.$$

Furthermore, if $K \subseteq K_S(x^*)$ and $H_i^{-1}(\mathbb{R}_-)$ is a cone with the vertex $F_i(x^*)$ for each $i \in I(x^*)$, then x^* is regular if and only if (3.4) holds.

Proof. Suppose that (3.2) holds. Then

$$(3.5) \quad P_K(x) = x^* \implies P_{K_S(x^*)}(x) = x^* \implies P_{K_C(x^*)}(x) = x^* \implies P_K(x) = x^*,$$

where the first implication holds by [14] (see also Lemma 3.8 below). Hence (3.4) holds. Conversely, suppose that $K \subseteq K_S(x^*)$ and that $H_i^{-1}(\mathbb{R}_-)$ is a cone with the vertex $F_i(x^*)$ (thus $K_C(x^*) = K_L(x^*)$) for each $i \in I(x^*)$. Then it follows from the first implication of (3.5) that, for any $x \in X$, $P_{K_S(x^*)}(x) = x^* \iff P_K(x) = x^*$. Thus, from (3.4), we have

$$P_{K_S(x^*)}(x) = x^* \iff P_{K_C(x^*)}(x) = x^*.$$

By Corollary 3.2 and noting that $K_C(x^*) = K_L(x^*)$ and $K_S(x^*)$ are cones with the vertex x^* , this implies that $K_C(x^*) = K_S(x^*)$, and so $K \subseteq K_C(x^*)$, i.e., (3.2) holds. The proof is complete. \square

If K is convex, then $K \subseteq K_S(x^*)$. We therefore have the following result.

COROLLARY 3.6. *Let $x^* \in K$. Suppose that K is convex and that $H_i^{-1}(\mathbb{R}_-)$ is a cone with the vertex $F_i(x^*)$ for each $i \in I(x^*)$. Then x^* is regular if and only if (3.4) holds.*

We are now ready to present one of our main results. Recall that \tilde{A}_i is defined by (2.1).

THEOREM 3.7. *Let $x^* \in K$. Then the following statements are equivalent:*

- (i) *The system of convex inequalities*

$$(3.6) \quad \tilde{A}_i(z) \leq 0, \quad i \in I(x^*),$$

satisfies the BCQ relative to C at x^ .*

- (ii) *The system of convex inequalities*

$$(3.7) \quad \tilde{A}_i(z) \leq 0, \quad i = 1, 2, \dots, m,$$

satisfies the BCQ relative to C at x^ .*

- (i*) *x^* has the perturbation property with respect to C and the system (3.6).*

- (ii*) *x^* has the perturbation property with respect to C and the system (3.7).*

Moreover, if $x^* \in K$ is a regular point of K with respect to C and the system (NIS), then each of the above statements is also equivalent to the following:

- (iii) *x^* has the perturbation property with respect to C and the system (NIS).*

Proof. The equivalence of (i) \iff (i*) and (ii) \iff (ii*) are given in [14, Theorem 5.1]. By (2.1), $\tilde{A}_i(x^*) = A_i(x^*)$; hence $i \in I(x^*)$ if and only if $\tilde{A}_i(x^*) = 0$. We may assume that $I(x^*)$ is a proper subset of $\{1, 2, \dots, m\}$ and note that $x^* \in \text{int}(\cap_{i \notin I(x^*)} \tilde{A}_i^{-1}(\mathbb{R}_-))$. Writing C_i for $\tilde{A}_i^{-1}(\mathbb{R}_-)$, it follows from [1, Corollary 2.4, p. 113] that

$$\partial(\mathbf{I}_{C \cap (\cap_{i \in I(x^*)} C_i)}(x^*) + \mathbf{I}_{\cap_{i \notin I(x^*)} C_i}(x^*)) = \partial \mathbf{I}_{C \cap (\cap_{i \in I(x^*)} C_i)}(x^*) + \partial \mathbf{I}_{\cap_{i \notin I(x^*)} C_i}(x^*);$$

that is,

$$N_{C \cap (\cap_{i=1}^m C_i)}(x^*) = N_{C \cap (\cap_{i \in I(x^*)} C_i)}(x^*).$$

Therefore (i) \iff (ii). Under the additional assumption that x^* is regular (and thus that Theorem 3.5 is applicable), we will show the equivalence of (i*) \iff (iii). Consider the following statements for $x \in X$:

- (1) $P_K(x) = x^*$;
- (2) $P_{C \cap (\cap_{i \in I(x^*)} \tilde{A}_i^{-1}(\mathbb{R}_-))}(x) = x^*$;
- (3) $P_C(x - \sum_{i \in I(x^*)} \lambda_i h_i) = x^*$ for some $h_i \in \partial A_i(x^*)$ and $\lambda_i \geq 0$.

By Theorem 3.5 and the fact that $K_C(x^*) = C \cap (\cap_{i \in I(x^*)} \tilde{A}_i^{-1}(\mathbb{R}_-))$, (1) \iff (2). Since $\partial A_i(x^*) = \partial \tilde{A}_i(x^*)$ (see Proposition 2.6), (i*) holds if and only if [(2) \iff (3)]. Therefore, (i*) holds if and only if [(1) \iff (3)]; namely, (i*) holds if and only if (iii) holds. \square

Remark 3.2. (a) The statement (ii*) simply means (by definition):

(ii') For any $x \in X$,

$$P_{K_C(x^*)}(x) = x^* \iff P_C \left(x - \sum_{i \in I(x^*)} \lambda_i h_i \right) = x^* \quad \text{for some } h_i \in \partial \tilde{A}_i(x^*), \lambda_i \geq 0,$$

where $\partial \tilde{A}_i(x^*)$ can be replaced by $\partial A_i(x^*)$, by Proposition 2.6.

(b) The sufficient part of (ii') holds in general by Lemma 3.8(ii) below.

(c) The system (3.6) (or (3.7)) may be referred to as a convexification system of (NIS).

LEMMA 3.8. *Let $x^* \in K$ and $x \in X$. The following statements hold:*

- (i) *If $P_K(x) = x^*$, then $P_{K_S(x^*)}(x) = x^*$.*
- (ii) *If*

$$(3.8) \quad P_C \left(x - \sum_{i=1}^m \lambda_i h_i \right) = x^*$$

for some $h_i \in \partial A_i(x^)$ and $\lambda_i \geq 0$ with $\lambda_i = 0$ for all $i \notin I(x^*)$, then $P_{K_L(x^*)}(x) = x^*$, and so $P_{K_C(x^*)}(x) = x^*$.*

Proof. For a proof of (i), see [14]. Next suppose that (3.8) holds. Then, by Proposition 2.6,

$$x - \sum_{i=1}^m \lambda_i h_i - x^* \in N_C(x^*).$$

Hence,

$$\begin{aligned} x - x^* &\in N_C(x^*) + \sum_{i=1}^m \lambda_i h_i \\ &\subseteq N_C(x^*) + \sum_{i \in I(x^*)} \text{cone} \partial A_i(x^*) \\ &\subseteq N_C(x^*) + \sum_{i \in I(x^*)} N_{\tilde{A}_i^{-1}(\mathbb{R}_-)}(x^*) \\ &\subseteq N_C(x^*) + N_{\cap_{i \in I(x^*)} \tilde{A}_i^{-1}(\mathbb{R}_-)}(x^*) \\ &= N_C(x^*) + \left(\bigcap_{i \in I(x^*)} \tilde{A}_i^{-1}(\mathbb{R}_-) - x^* \right)^\circ \\ &= N_C(x^*) + (\overline{\text{coneCFD}}(x^*))^\circ \\ &= N_C(x^*) + N_{(x^* + \overline{\text{coneCFD}}(x^*))}(x^*) \\ &\subseteq N_{C \cap (x^* + \overline{\text{coneCFD}}(x^*))}(x^*) \\ &= N_{K_L(x^*)}(x^*). \end{aligned}$$

This implies that $P_{K_L(x^*)}(x) = x^*$ by Proposition 2.6.

The following theorem shows that the regularity condition in Theorem 3.7 can be replaced by weak regularity if a Slater-type condition is satisfied.

THEOREM 3.9. *Let $x^* \in K$ be a weakly regular point of K , and suppose that*

$$(3.9) \quad \text{ri}(x^* + \text{coneCFD}(x^*)) \cap C \neq \emptyset.$$

Then the following statements are equivalent:

- (i) *System (3.7) satisfies the BCQ relative to C at x^* .*
- (ii) *x^* has the perturbation property with respect to C and the system (NIS).*

Proof. Suppose that (i) holds. Then, by Theorem 3.7, (ii') holds. For each $x \in X$, the following implications hold:

$$\begin{aligned} P_K(x) = x^* &\implies P_{K_S(x^*)} = x^* && \text{(Lemma 3.8)} \\ \implies P_{K_C(x^*)} = x^* &&& (K_C(x^*) \subseteq K_S(x^*) \text{ by weak regularity}) \\ \implies (3.8) \text{ holds} &&& \text{(Theorem 3.7(ii'))} \\ \implies P_{K_L(x^*)} = x^* &&& \text{(Lemma 3.8(ii))} \\ \implies P_K = x^*. &&& (K \subseteq K_L(x^*) \text{ by weak regularity}) \end{aligned}$$

This proves that (i) \implies (ii). ((3.9) is not needed for this implication.)

To prove the opposite implication (ii) \implies (i), note that, since (3.9) is satisfied,

$$(3.10) \quad K_L(x^*) = \overline{(x^* + \text{cone}(\text{CFD}(x^*))) \cap C}.$$

We will show below that

$$(3.11) \quad P_{K_C(x^*)}(x) = x^* \iff P_{K_L(x^*)}(x) = x^*.$$

Indeed, since $K_C(x^*) \subseteq K_L(x^*)$, it is sufficient to show

$$(3.12) \quad P_{K_C(x^*)}(x) = x^* \implies P_{K_L(x^*)}(x) = x^*.$$

Assume that $P_{K_C(x^*)}(x) = x^*$. By Proposition 2.6, we have

$$(3.13) \quad \langle x - x^*, x^* - \bar{z} \rangle \geq 0 \quad \forall \bar{z} \in K_C(x^*).$$

Let $z \in K_L(x^*)$: $z \in C$ and $z = x^* + t(\bar{z} - x^*)$ for some $\bar{z} \in x^* + \text{CFD}(x^*)$ and $t \geq 0$. Without loss of generality, assume that $t > 1$. Thus, $\bar{z} = x^* + (1/t)(z - x^*)$, and so $\bar{z} \in C$ since $z \in C$; consequently, $\bar{z} \in K_C(x^*)$. This, with (3.13), implies that

$$\langle x - x^*, x^* - z \rangle = t \langle x - x^*, x^* - \bar{z} \rangle \geq 0.$$

Hence, by (3.10), $x - x^* \in N_{K_L(x^*)}(x^*)$. By Proposition 2.6 again, (3.12) holds and so does (3.11). For each $x \in X$, the following implications hold:

$$\begin{aligned} P_K(x) = x^* &\iff (3.8) \text{ holds} && \text{(by (ii))} \\ \implies P_{K_L(x^*)}(x) = x^* &&& \text{(Lemma 3.8)} \\ \implies P_K(x) = x^*. &&& (K \subseteq K_L(x^*) \text{ by the weak regularity}) \end{aligned}$$

Combining this with (3.11), (ii') of Remark 3.2(a) is seen to hold. Thus, by Theorem 3.7, (i) holds. \square

Remark 3.3. (a) The implication (i) \implies (ii) of Theorem 3.9 remains true even if the condition (3.9) is dropped. Example 3.1 below shows that we do require condition (3.9) for the implication (ii) \implies (i).

(b) In the case in which the condition (3.9) is satisfied, Theorem 3.9 is a genuine extension of Theorem 3.7 (see Example 3.2 below).

Remark 3.4. If x^* is regular, then

$$(3.14) \quad K \subseteq K_C(x^*) \quad \text{and} \quad K \subseteq K_S(x^*).$$

In the following corollaries, we consider (3.14) instead of the regularity.

COROLLARY 3.10. *Suppose that x^* satisfies (3.14). Then the following statements are equivalent:*

(i) *The system (3.7) satisfies the BCQ relative to C at x^* , and x^* is a regular point of K .*

(ii) *x^* has the perturbation property with respect to C and the system (NIS).*

Proof. By Theorem 3.7, (i) \implies (ii). Conversely, suppose that (ii) holds. We claim that, for every $x \in X$,

$$(3.15) \quad P_{K_S(x^*)}(x) = x^* \iff P_K(x) = x^* \iff P_{K_C(x^*)}(x) = x^*.$$

Indeed, by (3.14), $P_K(x) = x^*$ if either $P_{K_S(x^*)}(x) = x^*$ or $P_{K_L(x^*)}(x) = x^*$. Conversely, let $x \in X$ and $x^* = P_K(x)$. Then $x^* = P_{K_S(x^*)}(x)$ by Lemma 3.8(i), and it follows from (ii) that $x^* = P_C(x - \sum_1^m \lambda_i h_i)$ for some $h_i \in \partial A_i(x^*)$ and $\lambda_i \geq 0$, with $\lambda_i = 0$ for all $i \notin I(x^*)$. By Lemma 3.8(ii), it follows that $x^* = P_{K_C(x^*)}(x)$. Therefore, (3.15) holds. By Lemma 3.1, this implies that $K_C(x^*) \subseteq K_S(x^*)$. Combining this with (3.14), x^* is regular. Now Theorem 3.7 is applicable, and thus (ii) \implies (i). \square

COROLLARY 3.11. *Suppose that the system (3.7) satisfies the BCQ relative to C at x^* and that*

$$(3.16) \quad K_C(x^*) = K_S(x^*).$$

Then (3.14) holds if and only if x^ has the perturbation property with respect to C and the system (NIS).*

Proof. In view of the preceding corollary, the necessity part is clear. Conversely, suppose that x^* has the perturbation property with respect to C and the system (NIS). Then we have the following equivalences:

$$\begin{aligned} P_K(x) = x^* &\iff (3.8) \text{ holds} \\ &\iff P_{K_C(x^*)}(x) = x^* && ((ii) \iff (ii^*) \text{ of Theorem 3.7}) \\ &\iff P_{K_S(x^*)}(x) = x^*. && (\text{by (3.16)}) \end{aligned}$$

Thus $K \subseteq K_S(x^*)$ by Lemma 3.1. Combining this with (3.16), we see that (3.14) holds. \square

A natural question arises from Theorem 3.7: When does the inclusion $K_C(x^*) \subseteq K_S(x^*)$ hold? Apart from the obvious sufficient condition that each F_i , $i \in I(x^*)$, is affine, we give another sufficient condition below in the case when $Y = R^n$. Let $\text{aff}(C)$ denote the linear manifold (i.e., affine subspace) spanned by C . Define

$$E = \{i : \text{int}H_i^{-1}(\mathbb{R}_-) = \emptyset\}, \quad I_0(x^*) = I(x^*) \setminus E.$$

Write

$$F_i := (F_{i_1}, F_{i_2}, \dots, F_{i_n}), \quad i = 1, 2, \dots, m.$$

Note that

$$(3.17) \quad H_i(x) \geq 0 \quad \text{on } X \quad \forall i \in E.$$

Let

$$S^* = \{d \in X : H_i(F_i(x^*) + F'_i(x^*)d) = 0, i \in E; H_i(F_i(x^*) + F'_i(x^*)d) < 0, i \in I_0(x^*)\}.$$

Thus $S^* \subseteq \text{CFD}(x^*)$; moreover, by (3.17),

$$(3.18) \quad (1 - t)d_1 + td_2 \in S^* \quad \forall t \in [0, 1), d_1 \in S^*, d_2 \in \text{CFD}(x^*).$$

In particular (by letting $d_2 = 0$), one has

$$(3.19) \quad (1 - t)d_1 \in S^* \quad \forall t \in [0, 1), d_1 \in S^*.$$

DEFINITION 3.12. *Let $x^* \in K$, and suppose that $Y = R^n$. We say that (NIS) satisfies the generalized MFCQ (Mangasarian–Fromowitz constraint qualification) at x^* if the following conditions are satisfied:*

- (a) *The intersection $(x^* + \text{CFD}(x^*)) \cap \text{ri}C$ is nonempty;*
- (b) *$\{F'_{ij}(x^*) : i \in E, j = 1, 2, \dots, n\}$ are linearly independent on $\text{span}(C - x^*)$;*
- (c) *the intersection $S^* \cap \text{span}(C - x^*)$ is nonempty.*

Remark 3.5. In the special case in which $Y = \mathbb{R}$, each H_i is the identity mapping, and C is a subspace of X , the above (a) is automatic, while (b) and (c) are, respectively, equivalent to the following:

- (b') $\{F'_{ij}(x^*) : i \in E, j = 1\}$ are linearly independent on C ;
- (c') the intersection $S^* \cap C$ is nonempty.

That is, the generalized MFCQ condition coincides with the standard MFCQ on C ([16]; see also [15, 21]).

Our next main result is the following.

THEOREM 3.13. *Let $x^* \in K$ and $Y = R^n$. Suppose that (NIS) satisfies the generalized MFCQ at x^* . Then*

$$(3.20) \quad K_C(x^*) \subseteq K_S(x^*).$$

If, in addition, for each $i \in I(x^*)$,

$$(3.21) \quad \tilde{A}_i(z) \leq A_i(z) \quad \text{for each } z \in C,$$

then x^* is regular.

Proof. It is easy to verify that $K \subseteq K_C(x^*)$ if (3.21) holds. Thus we need only to prove (3.20). By Definition 3.12(a), it is not difficult to verify that

$$(3.22) \quad K_C(x^*) = \overline{(x^* + \text{CFD}(x^*)) \cap \text{ri}C}.$$

Thus, we need only to prove that

$$(3.23) \quad (x^* + \text{CFD}(x^*)) \cap \text{ri}C \subseteq K_S(x^*).$$

Let \bar{x} belong to the set on the left-hand side of (3.23), and let $d = \bar{x} - x^*$. Then

$$(3.24) \quad d \in \text{CFD}(x^*) \cap \text{span}(C - x^*).$$

By (c), pick $\bar{d}_0 \in S^* \cap \text{span}(C - x^*)$. Define

$$(3.25) \quad \bar{d}_k = \left(1 - \frac{1}{k}\right) d + \frac{1}{k} \bar{d}_0 \quad \forall k \in \mathbb{N}.$$

Then, by (3.18) and (3.24), one has

$$(3.26) \quad \bar{d}_k \in S^* \cap \text{span}(C - x^*).$$

By (b), take a family $\{x_{ij} \in \text{span}(C - x^*) : i \in E; j = 1, 2, \dots, n\}$ of vectors in $\text{span}(C - x^*)$, which is dual to $\{F'_{ij}(x^*)\}$ in the sense that

$$(3.27) \quad F'_{ij}(x^*)x_{hl} = \begin{cases} 1 & \text{if } (i, j) = (h, l), \\ 0 & \text{otherwise.} \end{cases}$$

Let Z_k denote the linear subspace of X spanned by \bar{d}_k and the vectors x_{hl} , with $h \in E$ and $l = 1, 2, \dots, n$. We will show that there exist $\bar{\theta}_k \in (0, \frac{1}{k})$ and a continuously differentiable function $\theta \mapsto x_k(\theta)$ defined on $[0, \bar{\theta}_k]$ such that, for each $(i, j) \in E \times \{1, 2, \dots, n\}$ and each $\theta \in [0, \bar{\theta}_k]$,

$$(*) \quad \begin{cases} x_k(\theta) \in Z_k + x^*, \\ x_k(0) = x^*, \\ x'_k(0) = \bar{d}_k, \\ F_{ij}(x(\theta)) = F_{ij}(x^*) + \theta F'_{ij}(x^*)\bar{d}_k, \quad (i, j) \in E \times \{1, 2, \dots, n\}. \end{cases}$$

Granting this, we show below that $x_k(\theta)$ satisfies (NIS) for sufficiently small $\theta > 0$:

$$(3.28) \quad H_i(F_i(x_k(\theta))) \leq 0, \quad i = 1, 2, \dots, m.$$

Since $x_k(0) = x^*$ and by considering smaller θ if necessary, we need only verify the above (3.28) for $i \in I(x^*)$. If $i \in E$, then the last equality in (*) gives

$$H_i(F_i(x(\theta))) = H_i(F_i(x^*) + \theta F'_i(x^*)\bar{d}_k) = 0 \quad \forall \theta \in [0, \bar{\theta}_k],$$

thanks to (3.19) and (3.26). If $i \in I_0(x^*)$, then the Taylor theorem gives

$$F_i(x_k(\theta)) = F_i(x^*) + \theta F'_i(x^*)\bar{d}_k + o(\theta),$$

and thus it follows from the convexity that

$$\begin{aligned} H_i(F_i(x_k(\theta))) &= H_i(F_i(x^*) + \theta(F'_i(x^*)\bar{d}_k + O(\theta))) \\ &\leq \theta H_i(F_i(x^*) + F'_i(x^*)\bar{d}_k) + O(\theta) < 0, \end{aligned}$$

provided that $\theta > 0$ is sufficiently small. Here the last inequality holds because

$$H_i(F_i(x^*) + F'_i(x^*)\bar{d}_k) < 0$$

as $\bar{d}_k \in S^*$ and $i \in I_0(x^*)$. Therefore, by taking smaller $\bar{\theta}_k > 0$ if necessary, (3.28) becomes valid for all $\theta \in [0, \bar{\theta}_k]$. By (*), take θ_k with $0 < \theta_k \leq \bar{\theta}_k \leq 1/k$ such that

$$\left\| \frac{x_k(\theta_k) - x^*}{\theta_k} - \bar{d}_k \right\| < \frac{1}{k}.$$

Then, by (3.25),

$$\left\| \frac{x_k(\theta_k) - x^*}{\theta_k} - d \right\| < \frac{1}{k}(1 + \|d - \bar{d}_0\|).$$

Thus, setting $d_k = \frac{x_k(\theta_k) - x^*}{\theta_k}$, we have $\lim_{k \rightarrow \infty} d_k = d$. To verify (3.23), it suffices to show $d \in \text{SFD}(x^*)$. We will establish this by showing that $x_k(\theta_k) \in K$. To do this, note first that, because

$$d_k + x^* = \frac{x_k(\theta_k)}{\theta_k} + \left(1 - \frac{1}{\theta_k}\right) x^* \in \text{aff}(C),$$

it follows from $\bar{x} \in \text{ri}C$ and $\lim_{k \rightarrow \infty} (d_k + x^*) = \bar{x}$ that $d_k + x^* \in C$ for k large enough. This implies that $x_k(\theta_k) \in C$ as $x_k(\theta_k) = (1 - \theta_k)x^* + \theta_k(d_k + x^*)$. Consequently, it follows from (3.28) that $x_k(\theta_k) \in K$, as required.

To show that there exists x_k with property (*), henceforth we fix k and consider only the special case in which \bar{d}_k is linearly independent from $\{x_{ij}, (i, j) \in E \times \{1, 2, \dots, n\}\}$ (the case in which \bar{d}_k is linearly dependent on $\{x_{ij}\}$ can be dealt with similarly but somewhat more simply); in this case, take a unit vector $x_0 \in Z_k$ such that $\langle x_0, x_{ij} \rangle = 0$ for each $(i, j) \in E \times \{1, 2, \dots, n\}$. Then

$$(3.29) \quad \bar{d}_k = \langle x_0, \bar{d}_k \rangle x_0 + \sum_{ij} \lambda_{ij} x_{ij}$$

for some $\lambda_{ij} \in \mathbb{R}$. We consider the equality system for x in $Z_k + x^*$ near x^* :

$$\begin{cases} F_{ij}(x) = F_{ij}(x^*) + \theta F'_{ij}(x^*) \bar{d}_k, & (i, j) \in E \times \{1, 2, \dots, n\}, \\ \langle x_0, x - x^* \rangle = \theta \langle x_0, \bar{d}_k \rangle. \end{cases}$$

For simplicity of notation, we write \tilde{E} for $E \times \{1, 2, \dots, n\}$ and N for the cardinality $|\tilde{E}|$ of \tilde{E} . Expressing x in the form

$$x = \alpha_0 x_0 + \sum_{ij} \alpha_{ij} x_{ij} + x^*,$$

the above system can be written as for $(\alpha_0, \alpha_{ij}) \in \mathbb{R}^{1+N}$ near the origin:

$$(3.30) \quad \begin{cases} F_{ij}(\alpha_0 x_0 + \sum_{ij} \alpha_{ij} x_{ij} + x^*) = F_{ij}(x^*) + \theta F'_{ij}(x^*) \bar{d}_k, & (i, j) \in \tilde{E}, \\ \alpha_0 = \theta \langle x_0, \bar{d}_k \rangle. \end{cases}$$

The Jacobi matrix J for (3.30) at the origin is nonsingular; in fact, by (3.27),

$$J = \begin{pmatrix} 1 & 0 & \cdots & 0 & F'_{11}(x^*)x_0 \\ 0 & 1 & \cdots & 0 & F'_{12}(x^*)x_0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & F'_{|E|n}(x^*)x_0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

By the implicit function theorem, there exist $\theta_k \in (0, \frac{1}{k})$ and continuously differentiable functions, still denoted by α_0, α_{ij} , such that the preceding equality system is satisfied by these functions on $[-\theta_k, \theta_k]$ and such that each of these functions vanishes at $\theta = 0$. Set

$$x_k(\theta) = \alpha_0(\theta)x_0 + \sum \alpha_{ij}(\theta)x_{ij} + x^*, \quad \theta \in [-\theta_k, \theta_k].$$

Then (*) is seen to hold. Indeed, by differentiating each equation in the preceding system at the origin and making use of the dual property (3.27) of $\{x_{ij}\}$ relative to

$\{F'_{ij}(x^*)\}$, one has

$$J \cdot \begin{pmatrix} \alpha'_{11}(0) \\ \vdots \\ \alpha'_{|E|n}(0) \\ \alpha'_0(0) \end{pmatrix} = \begin{pmatrix} F'_{11}(x^*)\bar{d}_k \\ \vdots \\ F'_{|E|n}(x^*)\bar{d}_k \\ \langle x_0, \bar{d}_k \rangle \end{pmatrix}.$$

Computing the last row gives

$$(3.31) \quad \alpha'_0(0) = \langle x_0, \bar{d}_k \rangle,$$

and computing the other rows gives

$$(3.32) \quad \alpha'_{ij}(0) + F'_{ij}(x^*)x_0 \cdot \alpha'_0(0) = F'_{ij}(x^*)\bar{d}_k \quad \forall (ij) \in \tilde{E}.$$

By the dual property of $\{x_{ij}\}$ relative to $\{F'_{ij}(x^*)\}$, it follows from (3.29), (3.31), and (3.32) that $\alpha'_{ij}(0) = \lambda_{ij}$ for each $(i, j) \in \tilde{E}$. Consequently,

$$\begin{aligned} x'_k(0) &= \alpha'_0(0)x_0 + \sum \alpha'_{ij}(0)x_{ij} \\ &= \langle x_0, \bar{d}_k \rangle x_0 + \sum \lambda_{ij}x_{ij} \\ &= \bar{d}_k, \end{aligned}$$

thanks to (3.29). Therefore (*) holds, and the proof is complete. \square

COROLLARY 3.14. *Let $Y = \mathbb{R}^n$ and $x^* \in K$. Suppose that*

(a) *the intersection $(x^* + \text{CFD}(x^*)) \cap \text{ri}C$ is nonempty;*

(b') *$\{F'_{ij}(x^*) : i \in I(x^*); j = 1, 2, \dots, n\}$ is linearly independent on $\text{span}(C - x^*)$.*

Then $K_C(x^) \subseteq K_S(x^*)$.*

Proof. It is sufficient to show that the condition (c) of Definition 3.12 is satisfied by virtue of the strengthened condition (b') (comparing with (b)). If $I_0(x^*) = \emptyset$, then $0 \in S^* \cap \text{span}(C - x^*)$. Hence, we assume that $I_0(x^*) \neq \emptyset$. For any $i \in I_0(x^*)$, let $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{in}) \neq 0$ be an element of \mathbb{R}^n satisfying $H_i(F_i(x^*) + \alpha_i) < 0$. From assumption (b'), there exists $\{x_{kl} : (k, l) \in I(x^*) \times \{1, 2, \dots, n\}\}$ in $\text{span}(C - x^*)$ such that $F'_{ij}(x^*)x_{kl} = \alpha_{ij}$ if $(i, j) = (k, l) \in I_0(x^*) \times \{1, 2, \dots, n\}$, and $F'_{ij}(x^*)x_{kl} = 0$ otherwise. Then $d_k := \sum_{l=1}^n x_{kl}$ satisfies $F'_i(x^*)d_k = \alpha_i$ if $i = k \in I_0(x^*)$, and $F'_i(x^*)d_k = 0$ otherwise. Let $d := \sum_{k \in I(x^*)} d_k$. Then $d \in S^* \cap \text{span}(C - x^*)$. The proof is complete. \square

Example 3.1. Let $X = Y = \mathbb{R}^2$ and $C = [-1/2, 1] \times \{0\}$. Define

$$A(x_1, x_2) = H(F_1(x_1, x_2), F_2(x_1, x_2)),$$

where

$$F_1(x_1, x_2) = x_1(1 + x_2^2) \quad \forall (x_1, x_2) \in \mathbb{R}^2,$$

$$F_2(x_1, x_2) = x_1^2 + x_2 \quad \forall (x_1, x_2) \in \mathbb{R}^2,$$

and

$$H(u, v) = \begin{cases} u^2 + (v - 1)^2 - 1, & u \geq 0, \\ -u + (v - 1)^2 - 1, & u \leq 0. \end{cases}$$

Then, if $(x_1, x_2) \in C$,

$$A(x_1, x_2) = \begin{cases} x_1^4 - x_1^2, & x_1 \geq 0, \\ x_1(x_1^3 - 2x_1 - 1), & x_1 \leq 0. \end{cases}$$

Since $t^3 - 2t - 1 < 0$ on $[-1/2, 0]$, it follows that $K = [0, 1] \times \{0\}$. Let $x^* = (0, 0)$. Then

$$F_1(x^*) = F_2(x^*) = 0,$$

$$F'_1(x^*) = (1, 0), \quad F'_2(x^*) = (0, 1),$$

and so

$$\tilde{A}(x) = \begin{cases} x_1^2 + (x_2 - 1)^2 - 1, & x_1 \geq 0, \\ -x_1 + (x_2 - 1)^2 - 1, & x_1 \leq 0. \end{cases}$$

Thus,

(3.33)

$$x^* + \text{CFD}(x^*) = \{(x_1, x_2) \in \mathbb{R}^2 : x_2^2 - 2x_2 \leq x_1 \leq \sqrt{1 - (x_2 - 1)^2}, 0 \leq x_2 \leq 2\},$$

which is the set bounded by a parabola Γ_1 and a semicircle Γ_2 whose tangents at x^* are of slopes $1/2$ and 0 , respectively, and hence $x^* + \text{cone}(\text{CFD}(x^*))$ is the polyhedral cone generated by these two tangents. Consequently,

$$K_C(x^*) = \{(0, 0)\}, \quad K_L(x^*) = [0, 1] \times \{0\},$$

so that

$$K_C(x^*) \subseteq K_S(x^*), \quad K \subseteq K_L(x^*);$$

that is, x^* is a weakly regular point of S . Furthermore,

$$\partial\tilde{A}(x^*) = \partial A(x^*) = [-1, 0] \times \{-2\}.$$

For any $x = (x_1, x_2) \in X$, $P_K(x) = x^*$ if and only if $x_1 \leq 0$. Taking $\lambda = -x_1$, $h = -1$, we get that $P_C(x - \lambda h) = x^*$. This implies that x^* has the perturbation property with respect to C and the system (NIS). However, note that

$$N_{K_C(x^*)}(x^*) = \mathbb{R}^2, \quad N_C(x^*) = \{0\} \times \mathbb{R},$$

$$\text{cone}(\partial\tilde{A}(x^*)) = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq 2x_1 \leq 0\}.$$

This implies that the system $\tilde{A}(\cdot) \leq 0$ does not satisfy the BCQ relative to C . Thus (ii) does not imply (i) in Theorem 3.9 if the condition (3.9) is dropped.

Example 3.2. Let H, F, x^* be defined as in Example 3.1, but let C be defined by

$$C = \{(x_1, x_2) : -2x_2 \leq x_1 \leq 1, x_2 \in [0, 1]\}.$$

By Example 3.1, we obtain that $K_L(x^*) = C \supseteq K$. Moreover,

$$(3.34) \quad \text{ri}(x^* + \text{cone}(\text{CFD}(x^*))) \cap \text{ri}C \neq \emptyset.$$

Since $\{F'_1(x^*), F'_2(x^*)\}$ is linearly independent, $K_C(x^*) \subseteq K_S(x^*)$ by Corollary 3.14. It follows that x^* is a weakly regular point of K ; hence, by (3.34), the assumptions of Theorem 3.9 are satisfied, and so (i) and (ii) are equivalent. However, $K \not\subseteq K_C(x^*)$ because $(0, 1] \times \{0\} \subset K$, but $(0, 1] \times \{0\}$ is disjoint from $K_C(x^*)$. Hence, Theorem 3.7 cannot be applied. Therefore, in the case in which (3.9) holds, Theorem 3.9 is a genuine extension of Theorem 3.7.

4. Inequality system with respect to cones. In this section, we will apply the results obtained to study an abstract inequality system. Let X, C be as before. Let W be a closed convex cone in \mathbb{R}^N . Then W defines a partial order \succcurlyeq on \mathbb{R}^N :

$$(4.1) \quad \bar{a} \succcurlyeq \bar{b} \iff \bar{a} - \bar{b} \in W.$$

Let $G = (g_1, g_2, \dots, g_N)$ be a Fréchet differentiable function from X to \mathbb{R}^N , and let $b = (b_1, b_2, \dots, b_N) \in \mathbb{R}^N$. Let \hat{K} consist of all $x \in C$ satisfying the abstract inequality system

$$(AIS) \quad G(x) \succcurlyeq b,$$

namely,

$$(4.2) \quad \hat{K} = C \cap \{x \in X : G(x) \in b + W\}.$$

Let $x^* \in \hat{K}$. This system can be rephrased as a system of the form (NIS) by the following device. Define $H : \mathbb{R}^N \rightarrow \mathbb{R}$ by the Euclidean distance function of W :

$$(4.3) \quad H(y) = \text{dist}(y, W), \quad y \in \mathbb{R}^N.$$

Then $H(\cdot) \geq 0$ on \mathbb{R}^N , $H(G(x^*) - b) = 0$, $W = \{y \in \mathbb{R}^N : H(y) = 0\}$, and, by [13, Example 3.3, p. 259],

$$(4.4) \quad \partial H(y) = N_W(y) \cap \mathbf{B}(0, 1) \quad \forall y \in W,$$

where $\mathbf{B}(0, 1)$ denotes the unit ball of \mathbb{R}^N . Note that x satisfies (AIS) if and only if

$$(4.5) \quad H(G(x) - b) \leq 0.$$

Clearly (4.5) is of the type (NIS) with $m = 1$. According to the notation arrangements in sections 1 and 2, let F, A, \tilde{A} be defined by, for each $x \in X$,

$$(4.6) \quad \left. \begin{aligned} F(x) &= G(x) - b \\ A(x) &= H(G(x)) \\ \tilde{A}(x) &= H(F(x^*) + F'(x^*)(x - x^*)) \\ \tilde{F}(x) &= F(x^*) + F'(x^*)(x - x^*) \end{aligned} \right\}.$$

Let $J(x^*) = \{j : g_j(x^*) = b_j\}$.

THEOREM 4.1. *Let $x^* \in \hat{K}$, and suppose that x^* is regular with respect to C and the system (4.5). Then the following statements are equivalent:*

- (i) $N_{C \cap \tilde{F}^{-1}(W)}(x^*) = N_C(x^*) + N_W(G(x^*) - b) \circ G'(x^*)$;
- (ii) for any $x \in X$, $P_{\hat{K}}(x) = x^* \iff P_C(x - \sum_{i=1}^N y_i g'_i(x^*)) = x^*$ for some $(y_1, y_2, \dots, y_N) \in W^\circ$ with $\sum_{i \notin J(x^*)} y_i (g_i(x^*) - b_i) = 0$.

Proof. Clearly, $\tilde{A}(x^*) = A(x^*) = 0$, $G'(\cdot) = F'(\cdot)$, and $\tilde{A}^{-1}(\mathbb{R}_-) = \tilde{F}^{-1}(W)$. By Proposition 2.5 and (4.4), we have

$$(4.7) \quad \text{cone} \partial \tilde{A}(x^*) = N_W(F(x^*)) \circ F'(x^*).$$

Then (i) holds if and only if the convexification system

$$(4.8) \quad \tilde{A}(x) \leq 0$$

corresponding to (4.5) satisfies the BCQ relative to C at x^* .

On the other hand, it is well known and easy to verify that

$$\begin{aligned}
 (4.9) \quad N_W(F(x^*)) &= \{y \in W^\circ : \langle y, F(x^*) \rangle = 0\} \\
 &= \left\{ (y_1, y_2, \dots, y_N) \in W^\circ : \sum_{i=1}^N y_i(g_i(x^*) - b_i) = 0 \right\} \\
 &= \left\{ (y_1, y_2, \dots, y_N) \in W^\circ : \sum_{i \notin J(x^*)} y_i(g_i(x^*) - b_i) = 0 \right\}.
 \end{aligned}$$

Combining this with (4.7), one has

$$\text{cone} \partial \tilde{A}(x^*) = \left\{ \sum_{i=1}^N y_i g'_i(x^*) : (y_1, y_2, \dots, y_N) \in W^\circ; \sum_{i \notin J(x^*)} y_i(g_i(x^*) - b_i) = 0 \right\}.$$

Thus, (ii) is exactly the perturbation property with respect to C and system (4.5).

Therefore Theorem 4.1 follows from Theorem 3.7. \square

Remark 4.1. Since W is a closed cone, the regularity assumption is equivalent to the weak regularity of x^* (see Proposition 2.3).

An important special case of (AIS) considered above is the following familiar inequality-equality system: $x \in C$ and

$$(4.10) \quad \begin{cases} g_i(x) = b_i, & i = 1, 2, \dots, m_e, \\ g_i(x) \leq b_i, & i = m_e + 1, 2, \dots, m, \end{cases}$$

where $m_e \in \{1, 2, \dots, m\}$. Writing N for m and letting

$$(4.11) \quad W = \{(y_1, y_2, \dots, y_m) : y_i = 0 \ \forall i = 1, \dots, m_e; y_i \leq 0 \ \forall i = m_e + 1, \dots, m\},$$

we see that the system (4.10) is of the type considered in (AIS). Let K consist of all $x \in C$ satisfying (4.10), and let $x^* \in K$. Let $I(x^*)$ consist of all i satisfying $g_i(x^*) = b_i$, and let $I_0(x^*) = I(x^*) \setminus \{1, 2, \dots, m_e\}$; thus $I(x^*) := I_0(x^*) \cup \{1, 2, \dots, m_e\}$. We define

$$\begin{aligned}
 C_i &= \{x \in X : g'_i(x^*)(x - x^*) = 0\}, \quad i \in \{1, 2, \dots, m_e\}, \\
 C_i &= \{x \in X : g'_i(x^*)(x - x^*) \leq 0\}, \quad i \in I_0(x^*).
 \end{aligned}$$

The following facts are well known (and easy to verify):

$$(4.12) \quad \left. \begin{aligned} N_{C_i}(x^*) &= \text{span}\{g'_i(x^*)\}, \quad i \in \{1, 2, \dots, m_e\}, \\ N_{C_i}(x^*) &= \text{cone}\{g'_i(x^*)\}, \quad i \in I_0(x^*), \\ W^\circ &= \{(\lambda_1, \lambda_2, \dots, \lambda_m) : \lambda_i \geq 0 \ \forall i = m_e + 1, \dots, m\}. \end{aligned} \right\}$$

COROLLARY 4.2 (see Theorem 4.1 of [14]). *Let $x^* \in K$ be a regular point with respect to C and the system (4.10). Let*

$$D = \{x \in X : g'_i(x^*)(x - x^*) = 0 \ \forall i \in \{1, 2, \dots, m_e\}\}.$$

Then the following statements are equivalent:

- (i) $\{C, D, C_i : i \in I_0(x^*)\}$ has the strong CHIP at x^* ;

(ii) for any $x \in X$, $P_K(x) = x^* \iff P_C(x - \sum_1^m \lambda_i g'_i(x^*)) = x^*$ for some $\lambda_i, i = 1, \dots, m$, with

$$\begin{cases} \lambda_i \geq 0 & \forall i \in I_0(x^*), \\ \lambda_i = 0 & \forall i \notin I(x^*). \end{cases}$$

Proof. Let $G : x \mapsto (g_1(x), \dots, g_m(x))$, and let $H, F, A, \tilde{A}, \tilde{F}$ be defined as in (4.3), (4.6). Then,

$$N_{C \cap \tilde{F}^{-1}(W)}(x^*) = N_{C \cap (\cap_{i \in I(x^*)} C_i)}(x^*).$$

Moreover, by (4.12) and (4), one has

$$\begin{aligned} N_W(F(x^*)) &= \left\{ (\lambda_1, \lambda_2, \dots, \lambda_m) \in W^\circ : \sum_{i=1}^m \lambda_i (g_i(x^*) - b_i) = 0 \right\} \\ &= \{ (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{R}^m : \lambda_i \geq 0 \forall i \geq m_e + 1; \lambda_i = 0 \forall i \notin I(x^*) \} \end{aligned}$$

and

$$N_W(F(x^*)) \circ F'(x^*) = \left\{ \sum_{i=1}^m \lambda_i g'_i(x^*) : \lambda_i \geq 0 \forall i \geq m_e + 1; \lambda_i = 0 \forall i \notin I(x^*) \right\}.$$

Thus, by (4.12),

$$N_W(F(x^*)) \circ F'(x^*) = N_D(x^*) + \sum_{i \in I_0(x^*)} N_{C_i}(x^*).$$

Hence (i) and (ii) are the same as (i) and (ii), respectively, of Theorem 4.1. Therefore Corollary 4.2 follows from Theorem 4.1. \square

Finally, we should point out that the results in this paper can be applied to the case when our Hilbert space X is over the complex field \mathbb{C} . For the remainder of the paper, let X be a complex Hilbert space and F_j be a Fréchet differentiable complex function defined on X for each $j = 1, 2, \dots, m$. Let V_1, V_2, \dots, V_m be convex closed subsets of the complex plane \mathbb{C} . Let C be a closed convex subset of X , and let K consist of all $x \in C$ satisfying the complex system

$$(CS) \quad F_j(x) \in V_j, \quad j = 1, 2, \dots, m.$$

As usual, \mathbb{C} can be metrically viewed as \mathbb{R}^2 , while X can be regarded as a real Hilbert space with the inner product defined by

$$\langle x, y \rangle_R = \operatorname{Re} \langle x, y \rangle, \quad x, y \in X.$$

Consequently, F_j is a mapping from X into \mathbb{R}^2 , and V_j is a closed convex subset of \mathbb{R}^2 . Let $H_j : \mathbb{R}^2 \rightarrow \mathbb{R}$ denote the distance function to V_j . Then H_j is a real-valued convex function on \mathbb{R}^2 such that

$$V_j = \{y \in \mathbb{C} : H_j(y) \leq 0\},$$

and hence K consists of all $x \in C$ satisfying the real system

$$(RS) \quad H_j(F_j(x)) \leq 0, \quad j = 1, 2, \dots, m.$$

Thus, Theorems 3.7 and 3.9 can then be applied in a manner similar to what we have done for Theorem 4.1; details need not be repeated here. However, it is worth pointing out that the approach of using $\operatorname{Re}F_j$ and $\operatorname{Im}F_j$ does not work here because, for general closed convex sets V_j , the constraint $F_j(x) \in V_j$ cannot be described by $\operatorname{Re}F_j$ and $\operatorname{Im}F_j$ separately.

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