

On Well-posed Mutually Nearest and Mutually Furthest Point Problems in Banach Spaces

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Abstract Let G be a non-empty closed (resp. bounded closed) boundedly relatively weakly compact subset in a strictly convex Kadec Banach space X . Let $\mathcal{K}(X)$ denote the space of all non-empty compact convex subsets of X endowed with the Hausdorff distance. Moreover, let $\mathcal{K}_G(X)$ denote the closure of the set $\{A \in \mathcal{K}(X) : A \cap G = \emptyset\}$. We prove that the set of all $A \in \mathcal{K}_G(X)$ (resp. $A \in \mathcal{K}(X)$), such that the minimization (resp. maximization) problem $\min(A, G)$ (resp. $\max(A, G)$) is well posed, contains a dense G_δ -subset of $\mathcal{K}_G(X)$ (resp. $\mathcal{K}(X)$), thus extending the recent results due to Blasi, Myjak and Papini and Li.

Keywords Mutually nearest point, Mutually furthest point, Well posedness, Dense G_δ -subset

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1 Introduction

Let X be a real Banach space. We denote by $\mathcal{B}(X)$ the space of all non-empty closed bounded subsets of X . For a closed subset G of X and $A \in \mathcal{B}(X)$, we consider the minimization problem

$$\lambda_{AG} := \inf\{\|z - x\| : x \in A, z \in G\},$$

and the maximization problem

$$\mu_{AG} := \sup\{\|z - x\| : x \in A, z \in G\},$$

if $G \in \mathcal{B}(X)$. They are denoted, respectively, by $\min(A, G)$ and $\max(A, G)$.

Let G be a non-empty closed subset (resp. closed bounded subset) of X . A pair (x_0, z_0) with $x_0 \in A, z_0 \in G$ is called a solution of the minimization problem $\min(A, G)$ (resp. maximization

problem $\max(A, G)$, if $\|x_0 - z_0\| = \lambda_{AG}$ (resp. $\|x_0 - z_0\| = \mu_{AG}$). Moreover, any sequence $\{(x_n, z_n)\}, x_n \in A, z_n \in G$, such that $\lim_{n \rightarrow \infty} \|x_n - z_n\| = \lambda_{AG}$ (resp. $\lim_{n \rightarrow \infty} \|x_n - z_n\| = \mu_{AG}$), is called a minimizing (resp. maximizing) sequence. A minimization (resp. maximization) problem is said to be well posed if it has a unique solution (x_0, z_0) , and every minimizing (resp. maximizing) sequence converges strongly to (x_0, z_0) .

Let

$$\mathcal{C}(X) = \{A \in \mathcal{B}(X) : A \text{ is convex}\},$$

and let $\mathcal{C}(X)$ be endowed with the Hausdorff distance, h , defined as follows:

$$h(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\}, \quad \forall A, B \in \mathcal{C}(X).$$

As is well known, under such metric, $\mathcal{C}(X)$ is complete.

Furthermore, let

$$\mathcal{K}(X) = \{A \in \mathcal{C}(X) : A \text{ is compact}\}$$

and $\mathcal{K}_G(X)$ be the closure of the set $\{A \in \mathcal{K}(X) : A \cap G = \emptyset\}$. Clearly, X can be embedded as a subset of $\mathcal{K}(X)$ in a natural way that, for any $x \in X$, $A_x \in \mathcal{K}(X)$ is defined by $A_x = \{x\}$.

The well-posedness of the minimization and maximization problems was first considered in uniformly convex spaces by De Blasi, Myjak and Papini [1]. Recently, one of the authors investigated this problem in the strongly convex Banach spaces. He proved that if X is a strongly convex Banach space, then the set of all $A \in \mathcal{K}_G(X)$ (resp. $A \in \mathcal{K}(X)$), such that the minimization (resp. maximization) problem $\min(A, G)$ (resp. $\max(A, G)$) is well posed, is a dense G_δ -subset of $\mathcal{K}_G(X)$ (resp. $\mathcal{K}(X)$). Thus one question arises quite naturally: Whether the generic results on the well-posedness problems remain true without the reflexivity assumption? It should be pointed out that the reflexivity plays a key role in the study. In the present paper we will relax the reflexivity assumption but make a further restriction on the set G . More precisely, we will prove that if G is a non-empty closed (bounded closed) boundedly relatively weakly compact subset in a strictly convex Kadec Banach space X , then the set of all $A \in \mathcal{K}_G(X)$ (resp. $A \in \mathcal{K}(X)$), such that the minimization problem $\min(A, G)$ (resp. maximization problem $\max(A, G)$) is well posed, contains a dense G_δ -subset of $\mathcal{K}_G(X)$ (resp. $\mathcal{K}(X)$), which extends some results due to De Blasi, Myjak and Papini [1] and Li [2].

The problems considered here are in the spirit of Stechkin [3]. Some further developments of Stechkin's ideas and some generic results in spaces of convex sets can be found in [4–20] and [4, 5, 21, 19, 22, 23], respectively.

The notation we employ is, for the most part, standard; however, a partial list is provided for the readers' convenience. Let X^* denote the dual of X . We use $B(x, r)$ to denote the closed ball with center at x and radius r . As usual, if $A \subset X$, by \overline{A} and $\text{diam}A$ we mean the closure and the diameter of A , respectively, while $\overline{\text{co}}A$ stands for the closed convex hull of A .

Definition 1.1 *Let D be an open subset of X . A real-valued function f on D is said to be Frechet differentiable at $x \in D$ if there exists an $x^* \in X^*$ such that*

$$\lim_{y \rightarrow x} \frac{f(y) - f(x) - \langle x^*, y - x \rangle}{\|y - x\|} = 0.$$

x^* is called the Frechet differential at x which is denoted by $Df(x)$.

The following proposition on the Frechet differentiability of Lipschitz functions due to [24] is useful.

Proposition 1.1 *Let f be a locally Lipschitz continuous function on an open set D of a Banach space with an equivalent Frechet differentiable norm (in particular, X reflexive will do). Then f is Frechet differentiable on a dense subset of D .*

Definition 1.2 *A Banach space X is said to be (sequentially) Kadec provided that, for each sequence $\{x_n\} \subset X$ which converges weakly to x with $\lim_{n \rightarrow \infty} \|x_n\| = \|x\|$, we have $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$.*

2 Minimization Problems

Let $x \in X, A \in \mathcal{K}(X)$ and G be a closed subset of X . We set

$$d_G(x) = \inf_{z \in G} \|x - z\|,$$

$$d_G(A) = \inf_{x \in A} d_G(x) = \lambda_{AG},$$

and

$$P_A(G) = \{x \in A : d_G(x) = d_G(A)\}.$$

Then

$$|d_G(A) - d_G(B)| \leq h(A, B), \quad \forall A, B \in \mathcal{K}(X).$$

Let

$$L_n(G) = \left\{ A \in \mathcal{K}_G(X) : \begin{array}{l} \inf\{\langle x^*, x - z \rangle : z \in G \cap B(x, d_G(x) + \delta), x \in P_A(G)\} \\ > (1 - 2^{-n})d_G(A), \quad \text{for some } \delta > 0, x^* \in X^* \\ \text{with } \|x^*\| = 1. \end{array} \right\}.$$

Also let

$$L(G) = \bigcap_n L_n(G).$$

Lemma 2.1 *Suppose that G is a closed, boundedly relatively weakly compact non-empty subset of X . Then $L(G)$ is a dense G_δ -subset of $\mathcal{K}_G(X)$.*

Proof Exactly as in the proof of Lemma 2.3 of [2], we have that $L(G)$ is a G_δ -subset of $\mathcal{K}_G(X)$.

In order to prove the density of $L(G)$ in $\mathcal{K}_G(X)$, let

$$\Omega(G) = \left\{ A \in \mathcal{K}_G(X) : \begin{array}{l} \text{there is } x^* \in X^* \text{ with } \|x^*\| = 1 \text{ such that} \\ \text{for each } \epsilon > 0 \text{ there is } \delta > 0 \text{ so that} \\ \inf\{\langle x^*, x - z \rangle : z \in G \cap B(x, d_G(x) + \delta), x \in P_A(G)\} \\ > (1 - \epsilon)d_G(A). \end{array} \right\}.$$

Clearly, $\Omega(G) \subset L(G)$. It suffices to prove that $\Omega(G)$ is dense in $\mathcal{K}_G(X)$. To this end, let $A_0 \in \mathcal{K}_G(X)$, and suppose $d_G(A_0) > \epsilon > 0$. Now fix $N > \sup_{a \in A_0} \|a\| + d_G(A_0) + 2\epsilon$ and

let K denote the weak closure of the set $(B(0, N) \cap G) \cup A_0$. Then K is weakly compact in $Y = \overline{\text{span}K}$. From the factorization theorem of Davis, Figiel, Johnson and Pelczynski [10], there exist a reflexive Banach space R and a one-to-one continuous linear mapping $T : R \rightarrow Y$ such that $T(B(0, 1)) \supseteq K$. Define

$$f_G(u) = d_G(A_0 + Tu), \quad \forall u \in R.$$

It follows from Proposition 1.1 that f_G is Frechet differentiable on a dense subset of R so that there exists a point of differentiability $v \in R$ with $y = Tv \in B(0, \epsilon)$. Let $v^* = Df_G(v)$. Then

$$\lim_{h \rightarrow 0} \frac{d_G(A_0 + T(v+h)) - d_G(A_0 + Tv) - \langle v^*, h \rangle}{\|h\|} = 0,$$

and hence

$$\lim_{h \rightarrow 0} \frac{d_G(A_0 + y + Th) - d_G(A_0 + y) - \langle v^*, h \rangle}{\|h\|} = 0.$$

For each $u \in R$, substituting tu for h in the above expression as $t \rightarrow 0$, and using the non-expansiveness of f_G , we have

$$\langle v^*, u \rangle \leq \|Tu\|, \quad \forall u \in R.$$

Then the Hahn–Banach theorem implies that there is $y^* \in Y^*$ satisfying $v^* = T^*y^*$. Note that

$$\langle y^*, Tu \rangle = \langle T^*y^*, u \rangle \leq \|Tu\|, \quad \forall u \in R,$$

and T has dense range. We have that $\|y^*\| \leq 1$ and y^* can be extended to $x^* \in X^*$ with $\|x^*\| \leq 1$. Set $A = A_0 + y$ and take $x \in P_A(G)$. Then $x \in K + Tv \subset TR$. Now let $\{z_n\}$ be any minimizing sequence in G for x , i.e., $\lim_{n \rightarrow \infty} \|x - z_n\| = d_G(x)$. Clearly, $z_n \in K$ for n large enough so that $z_n - x \in K - x \subset TR$. Thus, for each $t \in (0, 1)$,

$$\begin{aligned} & d_G(A + t(z_n - x)) - d_G(A) \\ & \leq \|x + t(z_n - x) - z_n\| - d_G(A) = (1-t)\|x - z_n\| - d_G(A) \\ & = -t\|x - z_n\| + [\|x - z_n\| - d_G(A)]. \end{aligned}$$

Let $t_n = 2^{-n} + [\|x - z_n\| - d_G(A)]^{\frac{1}{2}}$. Observe that for each $r > 0$,

$$\lim_{t \rightarrow 0} \frac{d_G(A + tTh) - d_G(A) - t\langle x^*, Th \rangle}{t} = 0$$

holds uniformly for all $h \in R$ with $\|h\| \leq r$. This implies that

$$\lim_{n \rightarrow \infty} \left[\frac{d_G(A + t_n(z_n - x)) - d_G(A)}{t_n} - \langle x^*, z_n - x \rangle \right] = 0,$$

so that

$$\liminf_{n \rightarrow \infty} [-\|x - z_n\| + \langle x^*, x - z_n \rangle] \geq 0$$

and

$$d_G(A) = \lim_{n \rightarrow \infty} \|x - z_n\| \leq \liminf_{n \rightarrow \infty} \langle x^*, x - z_n \rangle,$$

which shows that $\|x^*\| \geq 1$ and consequently, $\|x^*\| = 1$. Furthermore we also have that

$$\lim_{n \rightarrow \infty} \|x - z_n\| \geq \lim_{n \rightarrow \infty} \|x^*\| \|x - z_n\| \geq \limsup_{n \rightarrow \infty} \langle x^*, x - z_n \rangle.$$

Comparison of the last two inequalities shows that

$$d_G(A) = \lim_{n \rightarrow \infty} \langle x^*, x - z_n \rangle.$$

Thus for each $\epsilon > 0$, there is $\delta > 0$ such that, whenever $z \in G \cap B(x, d_G(x) + \delta)$, it follows that

$$\langle x^*, x - z \rangle > (1 - \epsilon)d_G(A),$$

so that $A \in \Omega(G)$ as $x \in P_A(G)$ is arbitrary and $P_A(G)$ is compact. We establish the density since $h(A, A_0) \leq \epsilon$, thus proving the lemma.

We still need the following two lemmas, the proofs of which are similar to those of Lemma 2.4 and Lemma 2.5 of [2] and so they are omitted.

Lemma 2.2 *Suppose X is a Kadec Banach space. Let G be a closed, boundedly relatively weakly compact non-empty subset of X . Let $A \in L(G)$. Then any minimizing sequence $\{(x_n, z_n)\}$ with $x_n \in A, z_n \in G$ has a subsequence which converges strongly to a solution of the minimization problem $\min(A, G)$.*

Let

$$Q_n(G) = \left\{ A \in \mathcal{K}_G(X) : \text{diam}P_A(G) < \frac{1}{n} \right\},$$

and let

$$Q(G) = \bigcap_n Q_n(G).$$

Lemma 2.3 *Let X and G be as in Lemma 2.2. Then $Q(G)$ is a dense G_δ -subset of $\mathcal{K}_G(X)$.*

Now we are ready to give the main theorem of this section.

Theorem 2.1 *Suppose that X is a strictly convex Kadec Banach space. Let G be a closed, boundedly relatively weakly compact non-empty subset of X . Then the set of all $A \in \mathcal{K}_G(X)$, such that the minimization problem $\min(A, G)$ is well posed, contains a dense G_δ -subset of $\mathcal{K}_G(X)$.*

Proof It suffices to prove that $\min(A, G)$ is well posed if $A \in Q(G) \cap L(G)$, as $Q(G) \cap L(G)$ is a dense G_δ -subset of $\mathcal{K}_G(X)$ from Lemmas 2.1 and 2.3.

We first show that $\min(A, G)$ has a unique solution. Suppose there is $A \in Q(G) \cap L(G)$ such that $\min(A, G)$ has two solutions $(x_0, z_0), (x_1, z_1)$. Clearly $x_1 = x_0$ because $A \in Q(G)$. On the other hand, since $A \in L(G)$, for each n , there exists $x_n^* \in X, \|x_n^*\| = 1$ satisfying

$$\langle x_n^*, x_0 - z_i \rangle > (1 - 2^{-n})d_G(A), \quad i = 0, 1,$$

so that

$$\|x_0 - z_0 + x_0 - z_1\| \geq \limsup_{n \rightarrow \infty} \langle x_n^*, x_0 - z_0 + x_0 - z_1 \rangle \geq 2d_G(A).$$

Thus, using the strict convexity of X , we have $z_0 = z_1$, thus proving the uniqueness.

Now let (x_n, z_n) with $x_n \in A, z_n \in G$ be any minimizing sequence. Then from the uniqueness and Lemma 2.2 it follows that (x_n, z_n) converges strongly to the unique solution of the minimization problem $\min(A, G)$. The proof is complete.

3 Maximization Problems

Let $x \in X, A \in \mathcal{K}(X)$ and G be a bounded closed subset of X . Define

$$F_G(x) = \sup_{z \in G} \|x - z\|, \quad \forall x \in E,$$

$$F_G(A) = \sup_{x \in A} F_G(x) = \mu_{AG},$$

and

$$R_G(A) = \{z \in G : F_A(z) = F_A(G)\}.$$

Then

$$|F_G(A) - F_G(B)| \leq h(A, B), \quad \forall A, B \in \mathcal{K}(X).$$

Lemma 3.1 *The set-valued map $R_A(G)$ with respect to A is upper semi-continuous in the sense that, for each $A_0 \in \mathcal{K}(X)$ and any open set U with $R_{A_0}(G) \subset U$, there exists $\delta > 0$ such that for any $A \in \mathcal{K}(X)$ with $h(A, A_0) < \delta$, $R_A(G) \subset U$.*

Proof Suppose to the contrary that there exist $\{A_n\} \subset \mathcal{K}(X)$ and $A \in \mathcal{K}(X)$ with $\lim_{n \rightarrow \infty} h(A_n, A) = 0$, such that $R_{A_n}(G) \not\subset U$ for some open subset U with $R_A(G) \subset U$ and each n . Let $x_n \in R_{A_n}(G) \setminus U$ for any n . Note that $\cup_n A_n$ is relatively compact and $\{x_n\} \subset \cup_n A_n$. It follows that there exists a subsequence, denoted by itself, such that $\lim_{n \rightarrow \infty} \|x_n - x_0\| = 0$ for some $x_0 \in X$. Clearly, $x_0 \notin U$. However, by $\lim_{n \rightarrow \infty} h(A_n, A) = 0$, there exists $\{a_n\} \subset A$ such that $\lim_{n \rightarrow \infty} \|x_n - a_n\| = 0$, so that

$$\limsup_{n \rightarrow \infty} \|a_n - x_0\| \leq \lim_{n \rightarrow \infty} \|x_n - a_n\| + \lim_{n \rightarrow \infty} \|x_n - x_0\| = 0$$

and $x_0 \in A$. Furthermore, for each n ,

$$\begin{aligned} \sup_{z \in G} \|z - x_0\| &\geq \sup_{z \in G} \|z - x_n\| - \|x_n - x_0\| \\ &\geq F_G(A) - h(A_n, A) - \|x_n - x_0\|, \end{aligned}$$

which shows that $x_0 \in R_A(G)$, contradicting that $x_0 \notin U$. The proof is completed.

Let

$$H_n(G) = \left\{ A \in \mathcal{K}(X) : \begin{array}{l} \inf\{\langle x^*, x - z \rangle : z \in G \setminus B(x, F_G(x) - \delta), x \in R_A(G)\} \\ > (1 - 2^{-n})F_G(A), \quad \text{for some } \delta > 0, x^* \in X^* \\ \text{with } \|x^*\| = 1 \end{array} \right\}.$$

Also let

$$H(G) = \cap_n H_n(G).$$

Lemma 3.2 *Suppose that G is a closed, relatively weakly compact non-empty bounded subset of X . Then $H(G)$ is a dense G_δ -subset of $\mathcal{K}(X)$.*

Proof To show that $H(G)$ is a G_δ -subset of $\mathcal{K}(X)$, we need to prove only that $H_n(G)$ is open for each n . Let $A \in H_n(G)$. Then there exist $x^* \in X^*$ with $\|x^*\| = 1$ and $\delta > 0$ such that

$$\beta = \inf\{\langle x^*, x - z \rangle : x \in R_A(G), z \in G \setminus B(x, F_G(x) - \delta)\} - (1 - 2^{-n})F_G(A) > 0.$$

Let $\lambda > 0$ be such that $\lambda < \min\{\frac{\delta}{2}, \frac{\beta}{2}\}$. It follows from Lemma 3.1 that there exists $0 < \epsilon < \lambda$ such that, for any $E \in \mathcal{K}(X)$ with $h(E, A) < \epsilon$ and each $y \in R_E(G)$, there exists $x \in R_A(G)$ satisfying $\|y - x\| < \lambda$. For $\delta^* = \delta - 2\lambda$ we have

$$I = G \setminus B(y, F_G(y) - \delta^*) \subset G \setminus B(x, F_G(x) - \delta).$$

Thus if $z \in I$, we get

$$\langle x^*, x - z \rangle \geq \beta + (1 - 2^{-n})F_G(A).$$

Hence,

$$\langle x^*, y - z \rangle > \beta + (1 - 2^{-n})F_G(E) - 2\lambda.$$

Since $z \in I$ and $y \in R_E(G)$ are arbitrary,

$$\inf\{\langle x^*, y - z \rangle : z \in I, y \in R_E(G)\} > (1 - 2^{-n})F_G(E)$$

and $E \in H_n(G)$ for all $E \in \mathcal{K}(X)$ with $h(E, A) < \epsilon$, which implies that $H_n(G)$ is open in $\mathcal{K}(X)$.

In order to prove the density of $H(G)$ in $\mathcal{K}(X)$, let

$$M(G) = \left\{ A \in \mathcal{K}(X) : \begin{array}{l} \text{there is } x^* \in X^* \text{ with } \|x^*\| = 1 \text{ such that} \\ \text{for each } \epsilon > 0 \text{ there is } \delta > 0 \text{ so that} \\ \inf\{\langle x^*, x - z \rangle : z \in G \setminus B(x, F_G(x) - \delta), x \in R_A(G)\} \\ > (1 - \epsilon)F_G(A) \end{array} \right\}.$$

From $M(G) \subset H(G)$, it suffices to prove that $M(G)$ is dense in $\mathcal{K}(X)$. Take $A_0 \in \mathcal{K}(X)$ and let K denote the weak closure of the set $G \cup A_0$. Then K is weakly compact in $Y = \overline{\text{span}K}$. From the factorization theorem of Davis, Figiel, Johnson and Pelczynski [10], there exist a reflexive Banach space R and a one-to-one continuous linear mapping $T : R \rightarrow Y$ such that $T(B(0, \cdot)) \supseteq K$. Define

$$f_G(u) = F_G(A_0 + Tu), \quad \forall u \in R.$$

Then exactly as in the proof of Lemma 2.1, there exist $y \in TR$ with $\|y\| < \epsilon$ and $x^* \in X^*$ with $\|x^*\| \leq 1$ such that for each $r > 0$,

$$\lim_{t \rightarrow 0} \frac{F_G(A + tTh) - F_G(A) - t\langle x^*, Th \rangle}{t} = 0$$

holds uniformly for all $h \in R$ with $\|h\| \leq r$, where $A = A_0 + y$.

Clearly, $R_A(G) \subset TR$. Thus if $x \in R_A(G)$ and $\{z_n\}$ is any maximizing sequence in G for x , i.e., $\lim_{n \rightarrow \infty} \|x - z_n\| = F_G(x)$, $z_n - x \in K - x \subset TR$ for n large enough. Hence for each $t \in (-1, 0)$,

$$\begin{aligned} & F_G(A + t(z_n - x)) - F_G(A) \\ & \geq \|x + t(z_n - x) - z_n\| - F_G(A) = (1 - t)\|x - z_n\| - F_G(A) \\ & = -t\|x - z_n\| + [\|x - z_n\| - F_G(A)]. \end{aligned}$$

Take $t_n \in (-1, 0)$, $t_n \rightarrow 0$ with $t_n^2 > F_G(A) - \|x - z_n\|$. Then we have

$$\lim_{n \rightarrow \infty} \left[\frac{F_G(A + t_n(z_n - x)) - F_G(A)}{t_n} - \langle x^*, z_n - x \rangle \right] = 0,$$

so that

$$\liminf_{n \rightarrow \infty} [-\|x - z_n\| + \langle x^*, x - z_n \rangle] \geq 0$$

and

$$F_G(A) = \lim_{n \rightarrow \infty} \|x - z_n\| \leq \liminf_{n \rightarrow \infty} \langle x^*, x - z_n \rangle,$$

which shows that $\|x^*\| \geq 1$ and consequently, $\|x^*\| = 1$. Furthermore we also have that

$$\lim_{n \rightarrow \infty} \|x - z_n\| \geq \lim_{n \rightarrow \infty} \|x^*\| \|x - z_n\| \geq \limsup_{n \rightarrow \infty} \langle x^*, x - z_n \rangle.$$

Comparison of the last two inequalities shows that

$$F_G(A) = \lim_{n \rightarrow \infty} \langle x^*, x - z_n \rangle.$$

Thus for each $\epsilon > 0$, there is $\delta > 0$ such that whenever $z \in G \setminus B(x, F_G(x) - \delta)$ it follows that

$$\langle x^*, x - z \rangle > (1 - \epsilon)F_G(A).$$

Since $R_A(G)$ is compact and $x \in R_A(G)$ is arbitrary, $A \in M(G)$. Note that $h(A, A_0) \leq \epsilon$. We establish the density, thus proving the lemma.

Lemma 3.3 *Suppose X is a Kadec Banach space and G is a closed, relatively weakly compact bounded subset of X . Let $A \in H(G)$. Then any maximizing sequence $\{(x_n, z_n)\}$ with $x_n \in A$, $z_n \in G$ has a subsequence which converges strongly to a solution of the maximization problem $\max(A, G)$.*

Proof Let $A \in H(G)$. Then $A \in H_m(G)$ for any $m = 1, 2, \dots$. By the definition of $H_m(G)$, there exist $\delta_m > 0$, $x_m^* \in X^*$, $\|x_m^*\| = 1$ such that

$$\inf\{\langle x_m^*, x - z \rangle : z \in G \setminus B(x, F_G(x) - \delta_m), x \in R_A(G)\} > (1 - 2^{-m})F_G(A).$$

Let $\{(x_n, z_n)\}$ with $x_n \in A$, $z_n \in G$ be any maximizing sequence. Without loss of generality, we assume that $x_n \rightarrow x_0$ strongly and $z_n \rightarrow z_0$ weakly as $n \rightarrow \infty$ for some $x_0 \in R_A(G)$, $z_0 \in X$, since A is compact and G is relatively weakly compact. Then we have that

$$\|x_0 - z_0\| \leq \liminf_{n \rightarrow \infty} \|x_0 - z_n\| = F_G(A).$$

We also assume that $\delta_n \leq \delta_m$ if $m < n$ and $z_n \in G \setminus B(x_0, F_G(x_0) - \delta_m)$ for all $n > m$. Thus,

$$\langle x_m^*, x_0 - z_n \rangle > (1 - 2^{-m})F_G(A), \quad \forall n > m$$

and

$$\langle x_m^*, x_0 - z_0 \rangle > (1 - 2^{-m})F_G(A), \quad \forall m.$$

Hence we have

$$\|x_0 - z_0\| \geq \limsup_{m \rightarrow \infty} \langle x_m^*, x_0 - z_0 \rangle \geq F_G(A).$$

This shows that $\|x_0 - z_0\| = F_G(A)$. Now the fact that X is Kadec implies that $\lim_{n \rightarrow \infty} \|z_n - z_0\| = 0$ and $z_0 \in G$. Clearly, (x_0, z_0) is a solution of the maximization problem $\max(A, G)$, which completes the proof.

Let

$$V_n = \left\{ A \in \mathcal{K}(X) : \text{diam}R_A(G) < \frac{1}{n} \right\},$$

and let

$$V(G) = \bigcap_n V_n(G).$$

Lemma 3.4 *Let X and G be as in Lemma 3.3. Then $V(G)$ is a dense G_δ -subset of $\mathcal{K}(X)$.*

Proof The proof is similar to that of Lemma 3.6 of [2].

The main theorem of this section is stated as follows:

Theorem 3.1 *Suppose that X is a strictly convex Kadec Banach space. Let $G \in \mathcal{B}(X)$ be relatively weakly compact. Then the set of all $A \in \mathcal{K}(X)$ such that the maximization problem $\max(A, G)$ is well posed contains a dense G_δ -subset of $\mathcal{K}(X)$.*

Proof It suffices to prove that $\max(A, G)$ is well posed if $A \in V(G) \cap H(G)$, as $V(G) \cap H(G)$ is a dense G_δ -subset of $\mathcal{K}(X)$ from Lemmas 3.2 and 3.4.

We first show that $\max(A, G)$ has a unique solution. Suppose there is $A \in V(G) \cap H(G)$ such that $\max(A, G)$ has two solutions $(x_0, z_0), (x_1, z_1)$. Clearly $x_1 = x_0$ because $A \in V(G)$. On the other hand, since $A \in H(G)$, for each n , there exists $x_n^* \in X, \|x_n^*\| = 1$ satisfying

$$\langle x_n^*, x_0 - z_i \rangle > (1 - 2^{-n})F_G(A), \quad i = 0, 1,$$

so that

$$\|x_0 - z_0 + x_0 - z_1\| \geq \limsup_{n \rightarrow \infty} \langle x_n^*, x_0 - z_0 + x_0 - z_1 \rangle \geq 2F_G(A).$$

Thus, using the strict convexity of X , we have $z_0 = z_1$, thus proving the uniqueness.

Now let (x_n, z_n) with $x_n \in A, z_n \in G$ be any maximizing sequence. Then from the uniqueness and Lemma 3.3 it follows that (x_n, z_n) converges strongly to the unique solution of the maximization problem $\max(A, G)$. The proof is complete.

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