SIAM J. Optim.
Vol. 14, No. 2, pp. 584–607

CONSTRAINT QUALIFICATION, THE STRONG CHIP, AND BEST APPROXIMATION WITH CONVEX CONSTRAINTS IN BANACH SPACES*

CHONG LI† AND K. F. NG‡

Abstract. Several fundamental concepts such as the basic constraint qualification (BCQ), the strong conical hull intersection property (CHIP), and the perturbations for convex systems of inequalities in Banach spaces (over $\mathbb{R}$ or $\mathbb{C}$) are extended and studied; here the systems are not necessarily finite. Their relationships with each other in connection with the best approximations are investigated. As applications, we establish results on the unconstrained reformulation of best approximations with infinitely many constraints in Hilbert spaces; also we give several characterizations of best restricted range approximations in $C(Q)$ under quite general constraints.

Key words. convex inequality system, the strong CHIP, the basic constraint qualification, best approximation, perturbation, best restricted range uniform approximation

AMS subject classifications. Primary, 41A65; Secondary, 41A29

DOI. S1052623402415846

1. Introduction.

For the study of best approximation problems with a finite system of inequality constraints in $\mathbb{R}^N$ (or in Hilbert spaces), the strong CHIP (the strong conical hull intersection property) and other constraint qualification concepts have played important roles in dual reformulation of the best approximation problems. See, e.g., [6, 7, 13, 14, 15, 22, 23, 26, 27]. In this paper these concepts are extended and studied in connection with more general systems. The system (of convex inequalities) that we will focus on is

$$g_i(x) \le 0, \quad i \in I, \tag{CIS}$$

where I is an index set (finite or otherwise), $x \in X$, each g_i is a real continuous convex function on X, and X is a Banach space (say, over the real field $\mathbb{R}$, but later we will also consider the case when X is over the complex field $\mathbb{C}$).

In what follows we always assume that the solution set S of the system (CIS) is nonempty, i.e.,

$$S := \{x \in X : \ g_i(x) \le 0 \quad \text{for all } i \in I\} \ne \emptyset. \tag{1.1}$$

Let $G(\cdot)$ denote the sup-function [18] of $\{g_i\}$:

$$G(x) := \sup_{i\in I} g_i(x) \quad \text{for all } x \in X.$$

Then S is also the solution set of the convex inequality

$$G(x) \le 0. \tag{SCIS}$$

*Received by the editors October 8, 2002; accepted for publication (in revised form) May 14, 2003; published electronically November 6, 2003.
http://www.siam.org/journals/siopt/14-2/41584.html

†Department of Mathematics, Zhejiang University, Hangzhou 310027, People's Republic of China (cli@zju.edu.cn). This author was supported in part by the National Natural Science Foundation of China (grant 10271025).

‡Department of Mathematics, Chinese University of Hong Kong, Hong Kong, People's Republic of China (kfng@math.cuhk.edu.hk). This author was supported by a direct grant (CUHK) and an earmarked grant from the Research Grant Council of Hong Kong.

In this paper we assume throughout that

$$G(x) < +\infty \quad \text{for all } x \in X \tag{1.2}$$

and that G is continuous on X. These blanket assumptions are automatically satisfied if $\{g_i : \ i \in I\}$ is locally uniformly bounded. Moreover, the continuity of G automatically follows from (1.2) if X is of finite dimension.

Let C be a closed convex subset of X and let K consist of all $x \in C$ satisfying the system (CIS). For a subset Z of X, we use P_Z to denote the projection operator defined by

$$P_Z(x) = \{y \in Z : \|x - y\| = d_Z(x)\},$$

where $d_Z(x)$ denotes the distance from x to Z.

Recently, studies have been done on establishing the dual formulation of the best approximation problem in the setting of real Hilbert spaces; see [6, 7, 13, 14, 15, 26, 27] for finite systems of linear inequalities and [22, 23] for finite systems of nonlinear inequalities. However, there are many problems in Banach spaces (over $\mathbb{R}$ or $\mathbb{C}$) that have infinitely many convex constraints. One typical example is the problem of best restricted range approximations in $C(Q)$, the space of all continuous complex-valued functions defined on a compact metric space Q; see [21, 33, 34, 35, 36, 37]; this problem can be reformulated as an approximation problem with constraints defined by an infinite system of convex inequalities. This motivates us to consider the following question: Can the results on the dual formulation of the best constrained approximation in Hilbert spaces for finite systems be extended to infinite systems in general Banach spaces? We shall study the relationships between the basic constraint qualification (BCQ) and the CHIP in Banach spaces (over $\mathbb{R}$ or $\mathbb{C}$) in section 3. As applications, we establish some results on the unconstrained reformulation of best approximations with infinitely many constraints in Hilbert spaces. This is done in section 4, where we begin with a general result (applicable to both real and complex Hilbert spaces) relating the BCQ and the dual formulation of the best approximation problem. Our result, on the complex Hilbert space X, is in a very general setting: $\{\Omega_i : \ i \in I\}$ is a family of closed convex subsets of $\mathbb{C}$, $\{h_i : \ i \in I\} \subseteq X \setminus \{0\}$, C is a closed convex subset of X, and $\hat{C}_i := \{x \in X : \ \langle h_i, x\rangle \in \Omega_i\}$. Theorem 4.2 shows that the family $\{C, \hat{C}_i : i \in I\}$ has the strong CHIP if and only if a dual formulation in terms of the projections P_C and $P_{(\cap_{i\in I}\hat{C}_i)\cap C}$ holds. It is worth noting in particular that $\{\Omega_i\}$ is not necessarily explicitly given by (CIS) at the outset. Another application of our results is given in section 5, where several characterizations of best restricted range approximations in $C(Q)$ are given for a class of quite general constraints.

To end this section, we describe some basic notation, most of which is standard (cf. [8, 18]). In particular, for a set Z in X (or in $\mathbb{R}^n$), the interior (resp., closure, convex hull, convex cone hull, linear hull, negative polar, boundary) of Z is defined by $\operatorname{int} Z$ (resp., $\overline{Z}$, $\operatorname{conv} Z$, $\operatorname{cone} Z$, $\operatorname{span} Z$, $Z^{\ominus}$, $\operatorname{bd}\ Z$); the normal cone of Z at z_0 is denoted by $N_Z(z_0)$ and defined by $N_Z(z_0) = (Z - z_0)^{\ominus}$. Let $\operatorname{ext} Z$ denote the set of all extreme points of Z and let $\mathbb{R}_-$ denote the subset of $\mathbb{R}$ consisting of all nonpositive real numbers. For a proper extended real-valued convex function on X, the subdifferential of f at $x \in X$ is denoted by $\partial f(x)$ and defined by

$$\partial f(x) = \{z^* \in X^* : \ \ f(x) + \langle z^*, y - x\rangle \le f(y) \quad \text{for all } y \in X\},$$

where $\langle z^*, x\rangle$ denotes the value of a functional z^* in X^* at $x \in Z$, i.e., $\langle z^*, x\rangle = z^*(x)$.

Remark 1.1. (a) Let f be a continuous convex function f on X and $x \in X$ with $f(x) = 0$. It is easy to see that $\operatorname{cone}(\partial f(x)) \subseteq N_{f^{-1}(\mathbb{R}_-)}(x)$ and that the equality holds if f is an affine function or if x is not a minimizer of f; see [8, Corollary 1, p. 56].

(b) The directional derivative of the function f at x in the direction d is denoted by $f'_+(x, d)$:

$$f'_+(x, d) := \lim_{t \to 0+} \frac{f(x + td) - f(x)}{t}. \tag{1.3}$$

We recall [8, Proposition 2.2.7] (see also [28]) that

$$\partial f(x) = \{z^* \in X^* : \langle z^*, d\rangle \le f'_+(x, d) \quad \text{for all } d \in X\} \tag{1.4}$$

and

$$f'_+(x, d) = \max\{\langle z^*, d\rangle : z^* \in \partial f(x)\}. \tag{1.5}$$

2. Preliminaries. Let $\{A_i : i \in J\}$ be a family of subsets of X. The set $\sum_{i \in J} A_i$ is defined by

$$\sum_{i \in J} A_i = \begin{cases} \left\{\sum_{i \in J_0} a_i : \quad a_i \in A_i, \quad J_0 \subseteq J \text{ being finite}\right\} & \text{if } J \neq \emptyset, \\ \{0\} & \text{if } J = \emptyset. \end{cases} \tag{2.1}$$

Consider (CIS) as before with the solution set denoted by S. For $x \in X$, let $I(x)$ denote the set of all active indices $i : I(x) = \{i \in I : g_i(x) = G(x) = 0\}$. Following [17, 24], we define

$$N'(x) := \sum_{i \in I(x)} \operatorname{cone}(\partial g_i(x)), \quad x \in X. \tag{2.2}$$

Note that, by (2.1), $N'(x) = \operatorname{cone}\left(\bigcup_{i \in I(x)} \partial g_i(x)\right)$ if $I(x) \neq \emptyset$ and $N'(x) = \{0\}$ if $I(x) = \emptyset$.

In the remainder of this paper, we let $K := C \cap S$, where S denotes the solution set of (CIS). The following concepts are well known in the case when I is finite or X is of finite dimension; see, e.g., [24, 22, 23].

DEFINITION 2.1. *Let $x \in K$. The system* (CIS) *is said to satisfy the BCQ relative to C at x if*

$$N_K(x) = N_C(x) + N'(x). \tag{2.3}$$

Remark 2.1. (CIS) satisfies the BCQ at each $x \in C \cap \operatorname{int} S$ because (2.3) holds trivially in this case.

The following concept of the strong CHIP is due to [13, 14] in the case when I is finite and plays an important role in optimization theory; see, e.g., [1, 2, 9, 12, 32].

DEFINITION 2.2. *Let $\{C_i : i \in I\}$ be a collection of closed convex subsets of X and $x \in \bigcap_{i \in I} C_i$. The collection is said to have the strong CHIP at x if*

$$N_{\bigcap_{i \in I} C_i}(x) = \sum_{i \in I} N_{C_i}(x). \tag{2.4}$$

Remark 2.2. (a) If $g_i(x) < 0$, then $x \in \operatorname{int}(g_i^{-1}(\mathbb{R}_-))$ and $N_{g_i^{-1}(\mathbb{R}_-)}(x) = \{0\}$. Hence

$$\sum_{i \in I(x)} N_{g_i^{-1}(\mathbb{R}_-)}(x) = \sum_{i \in I} N_{g_i^{-1}(\mathbb{R}_-)}(x).$$

(b) Let $x \in C \cap \text{bd } S$. Then

$$\begin{aligned}&\text{The system (CIS) satisfies BCQ relative to } C \text{ at } x\\ &\Longrightarrow \{C, g_i^{-1}(\mathbb{R}_-) : i \in I\} \text{ has the strong CHIP at } x.\end{aligned}$$

(c) Let $x \in C \cap \text{bd } S$ and suppose that, for each $i \in I(x)$, either g_i is affine or there exists $x_i \in C$ such that $g_i(x_i) < 0$ (so $\operatorname{cone}\big(\partial g_i(x)\big) = N_{g_i^{-1}(\mathbb{R}_-)}(x)$ by Remark 1.1). Then

$$\begin{aligned}&\text{The system (CIS) satisfies BCQ relative to } C \text{ at } x\\ &\Longleftrightarrow \{C, g_i^{-1}(\mathbb{R}_-) : i \in I\} \text{ has the strong CHIP at } x.\end{aligned}$$

(This assertion is of course trivial when $x \in C \cap \operatorname{int} S$.)

(d) When each g_i is affine, $\{g_i^{-1}(\mathbb{R}_-) : \ i \in I\}$ has the strong CHIP at x automatically if I is finite. However, this is not necessarily true if I is infinite; see [24, Example 1].

DEFINITION 2.3. *We say that the system* (CIS) *satisfies the Slater condition on* C *if there exists a point* $\bar{x} \in C$ *such that* $G(\bar{x}) < 0$. *In this case,* $\bar{x}$ *is called a Slater point of* (CIS) *on* C.

The following theorem, which is known (cf. [18, 24]) in the special case when X is of finite dimension, will play a key role in section 5.

THEOREM 2.1. *Assume that* I *is a compact metric space and that the function* $i \mapsto g_i(x)$ *is upper semicontinuous on* I *at each* $x \in X$. *Let* C *be a nonempty closed convex subset of* X *such that* $\operatorname{span} C$ *is of finite dimension. Suppose that there exists a Slater point* $\bar{x}$ *of* (CIS) *on* C. *Then the system* (CIS) *satisfies the BCQ relative to* C *at every point* $x \in K$.

Proof. As the result is trivial if $x \in C \cap \operatorname{int} S$, we may assume that $x \in C \cap \text{bd } S$. We divide the proof into two steps. First we show that

$$N_C(x) + \partial G(x) \subseteq N_C(x) + N'(x) \quad \text{for all } x \in C \cap \text{bd } S. \tag{2.5}$$

Let $\tilde{G}$ and $\tilde{g}_i$, respectively, denote the restrictions of G and g_i on $\operatorname{span} C$, where $i \in I$. Then

$$\tilde{G}(z) = \sup_{i \in I} \tilde{g}_i(z) \quad \text{for all } z \in \operatorname{span} C. \tag{2.6}$$

By assumptions and [18, Theorem 4.4.2, p. 267] (see also [24, Theorem 3.1]), for any $x \in C \cap \text{bd } S$, we have that

$$\partial \tilde{G}(x) = \operatorname{conv}\left(\bigcup_{i \in I(x)} \partial \tilde{g}_i(x) \right). \tag{2.7}$$

For any $y^* \in \partial G(x)$, y^* can be viewed as an element of $\partial \tilde{G}(x)$. Thus, by (2.7), there exist $\tilde{y}_j^* \in \partial \tilde{g}_{i_j}(x)$, $\lambda_j \geq 0$, $i_j \in I(x)$, $j = 1, 2, \ldots, m$, such that $\sum_{j=1}^m \lambda_j = 1$ and

$$\langle y^*, z \rangle = \left\langle \sum_{j=1}^m \lambda_j \tilde{y}_j^*, z \right\rangle \quad \text{for all } z \in \operatorname{span} C. \tag{2.8}$$

Noting

$$\langle \tilde{y}_j^*, z\rangle \le \tilde{g}'_{i_j+}(x,z) = g'_{i_j+}(x,z) \quad \text{for all } z \in \operatorname{span} C, \tag{2.9}$$

and making use of the Hahn–Banach extension theorem, there exists $y_j^* \in X^*$ satisfying

$$\langle y_j^*, z\rangle = \langle \tilde{y}_j^*, z\rangle \quad \text{for all } z \in \operatorname{span} C \tag{2.10}$$

and such that $\langle y_j^*, z\rangle \le g'_{i_j+}(x,z)$ holds for all z in X. This implies that $y_j^* \in \partial g_{i_j}(x)$. Let $y_0^* = y^* - \sum_{j=1}^m \lambda_j y_j^*$. Then, by (2.8) and (2.10), one has

$$\langle y_0^*, z\rangle = \left\langle y^* - \sum_{j=1}^m \lambda_j y_j^*, z \right\rangle = 0 \quad \text{for all } z \in C - x,$$

in particular, $y_0^* \in N_C(x)$. This implies that $y^* = y_0^* + \sum_{j=1}^m \lambda_j y_j^* \in N_C(x) + \operatorname{cone}(\bigcup_{i\in I(x)} \partial g_i(x))$; hence (2.5) is established.

Next, by the assumed Slater condition, it follows from [14, Proposition 2.3] (the proof given there is valid for an arbitrary Banach space although it was stated for Hilbert spaces) that $\{C, S\}$ has the strong CHIP at every point $x \in C \cap \operatorname{bd} S$:

$$N_K(x) = N_C(x) + N_S(x) \quad \text{for all } x \in C \cap \operatorname{bd} S. \tag{2.11}$$

Since $G(\bar{x}) < 0$, Remark 1.1(a) implies

$$N_S(x) = \operatorname{cone}\big(\partial G(x)\big) \quad \text{for all } x \in C \cap \operatorname{bd} S.$$

Then, by (2.11) and (2.5),

$$N_K(x) = N_C(x) + \operatorname{cone}\big(\partial G(x)\big) = N_C(x) + N'(x).$$

Thus Theorem 2.1 is proved. □

In the remainder of this paper, we will assume that X is a Banach space over the complex field $\mathbb{C}$ or the real field $\mathbb{R}$. When X is a Banach space over the complex field $\mathbb{C}$, let X_R denote the corresponding real Banach space by restricting the scalar multiplication to the reals. In this case, for any subset Z of X and $z_0 \in X$, one has two different versions for normal cones:

$$\tilde{N}_Z(z_0) = \{z^* \in X_R^* :\ \langle z^*, x - z_0\rangle \le 0 \quad \text{for all } x \in X\}, \tag{2.12}$$

$$N_Z(z_0) = \{z^* \in X^* :\ \operatorname{Re}\langle z^*, x - z_0\rangle \le 0 \quad \text{for all } x \in X\}. \tag{2.13}$$

Likewise, if f is a proper convex function on X and $x \in X$, then one can define

$$\tilde{\partial} f(x) = \{z^* \in X_R^* :\ \ f(x) + \langle z^*, y - x\rangle \le f(y) \quad \text{for all } y \in X\}, \tag{2.14}$$

$$\partial f(x) = \{z^* \in X^* :\ \ f(x) + \operatorname{Re}\langle z^*, y - x\rangle \le f(y) \quad \text{for all } y \in X\}. \tag{2.15}$$

Finally in addition to (2.2), one can define $\tilde{N}'(x)$ in the above manner. In view of the Bohnenblust–Sobczyk theorem ($x^* \mapsto \operatorname{Re} x^*$ is a real-isometry from X^* onto X_R^*; cf. [39, p. 192]), such distinctions are immaterial; for example, regarding Definition 2.1, the system (CIS) in X satisfies the BCQ relative to C at x in the sense of (2.3) if and only if it does in X_R. Thus, the results in this section, such as Theorem 2.1, can be applied to spaces over $\mathbb{C}$.

We now introduce some new concepts. Recall that $K := C \cap S$, where S denotes the solution set of (CIS). The index set I is not assumed to have any topological structure.

DEFINITION 2.4. *Let $x \in K$. An element $d \in X$ is called*

(a) *a linearized feasible direction of* (CIS) *at x if*

$$\operatorname{Re}\langle z^*, d\rangle \le 0 \quad \text{for all } z^* \in \bigcup_{i\in I(x)} \operatorname{ext}\partial g_i(x); \tag{2.16}$$

(b) *a sequentially feasible direction of K at x if there exist a sequence $d_k \to d$ and a sequence of positive real numbers $\delta_k \to 0$ such that $\{x + \delta_k d_k\} \subseteq K$.*

Remark 2.3. When I is finite and each g_i is differentiable at x, the definition of a linearized feasible direction of (CIS) at x in a real space X coincides with the corresponding definition introduced in [25, 38]; see also [22].

Let LFD(x) (resp., SFD(x)) denote the set of all d satisfying (a) (resp., (b)) in Definition 2.4. Note that LFD(x) is a closed convex cone (so it contains the origin) while SFD(x) is a closed cone (but not necessarily convex). Note also that $\mathrm{LFD}(x) = X$ if $I(x) = \emptyset$.

DEFINITION 2.5. *Let $x \in K$. Let $K_S(x)$ and $K_L(x)$ be defined, respectively, by*

$$K_S(x) = \left(x + \overline{\operatorname{conv}(\mathrm{SFD}(x))}\right) \bigcap C \tag{2.17}$$

and

$$K_L(x) = (x + \mathrm{LFD}(x)) \bigcap C. \tag{2.18}$$

Note that the two sets are closed convex sets. We have the following well-known inclusion relationship.

PROPOSITION 2.1. *Let $x \in C \cap S$. Then* $\mathrm{SFD}(x) \subseteq \mathrm{LFD}(x)$ *and*

$$K \subseteq K_S(x) \subseteq K_L(x). \tag{2.19}$$

Let $x_0 \in K$ and suppose that $I(x_0) \neq \emptyset$. In the study of the system (CIS), it would be useful to consider the following associated (linearized) system on X:

$$\operatorname{Re}\langle z^*, x - x_0\rangle \le 0, \quad z^* \in \bigcup_{i\in I(x_0)} \operatorname{ext}\partial g_i(x_0). \tag{2.20}$$

Let

$$\hat{S}_{z^*}(x_0) := \{x \in X : \operatorname{Re}\langle z^*, x - x_0\rangle \le 0\} \quad \text{for all } z^* \in \bigcup_{i\in I(x_0)} \operatorname{ext}\partial g_i(x_0)$$

and

$$\hat{S}(x_0) := \bigcap \left\{ \hat{S}_{z^*}(x_0) : z^* \in \bigcup_{i\in I(x_0)} \operatorname{ext}\partial g_i(x_0) \right\}. \tag{2.21}$$

Moreover, we define $\hat{S}(x_0) = X$ if $I(x_0) = \emptyset$. Then

$$x_0 + \mathrm{LDF}(x_0) = \hat{S}(x_0) \quad \text{and} \quad K_L(x_0) = \hat{S}(x_0) \cap C, \tag{2.22}$$

whether or not $I(x_0) \neq \emptyset$. For our convenience we state the following elementary lemma. We omit its proof as it is straightforward.

LEMMA 2.1. *Let $z^* \in X^*$, $x_0 \in X$, and let $\varphi : X \to \mathbb{R}$ be defined by*

$$\varphi(x) = \mathrm{Re}\,\langle z^*, x - x_0\rangle \quad \text{for all } x \in X.$$

Then $\partial\varphi(x_0) = z^*$. Consequently, $N'(x_0)$ defined by (2.2) *with respect to the system* (CIS) *coincides with the corresponding one with respect to the system* (2.20).

Recall that the duality map J from X to 2^{X^*} is defined by

$$J(x) := \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2, \ \|x^*\| = \|x\|\}. \tag{2.23}$$

In fact, $J(x) = \partial\phi(x)$, where $\phi(x) := \frac{1}{2}\|x\|^2$. Thus a Banach space X is smooth if and only if for each $x \in X$ the duality map is single-valued.

The following proposition will be useful later. This result was established independently by Deutsch [10] and Rubenstein [29] (see also [3]). We thank the two anonymous referees for their helpful comments. One of the referees kindly suggested the above references as well as the formulation of Corollary 4.3.

PROPOSITION 2.2. *Let Z be a closed convex set in X. Then for any $x \in X$, $z_0 \in P_Z(x)$ if and only if $z_0 \in Z$ and there exists $x^* \in J(x - z_0)$ such that $\mathrm{Re}\,\langle x^*, z - z_0\rangle \leq 0$ for any $z \in Z$; that is, $J(x - z_0) \cap N_Z(z_0) \neq \emptyset$. In particular, when X is smooth, $z_0 \in P_Z(x)$ if and only if $z_0 \in Z$ and $J(x - z_0) \in N_Z(z_0)$.*

3. Best constrained approximations in Banach spaces.

Before proving the main theorem of this section we recall two lemmas. These two lemmas were stated in the Hilbert space setting in [22, 23]. The proof given in [22, Theorem 3.1] for the first lemma is valid for Banach spaces, while the proof of the second lemma given in [23, Lemma 3.1] for Hilbert space will need to be modified to suit our purpose here.

LEMMA 3.1. *Let K be a nonempty closed convex subset of X, and let $x_0 \in K$. Then, for any $x \in X$, we have*

$$x_0 \in P_K(x) \Longleftrightarrow x_0 \in P_{K_S(x_0)}(x). \tag{3.1}$$

LEMMA 3.2. *Suppose that X is reflexive and smooth. Let C be a closed convex set, let $x_0 \in C$, and let T_1, T_2 be closed convex cones in X. Then the following statements are equivalent:*

(i) $C \cap (x_0 + T_1) \subseteq C \cap (x_0 + T_2)$.

(ii) $x_0 \in P_{C\cap(x_0+T_1)}(x)$ *whenever $x \in X$ and* $x_0 \in P_{C\cap(x_0+T_2)}(x)$.

Proof. We modify the proof that is given in [23] for the special case when X is a Hilbert space. Since X is assumed smooth, the map $x \mapsto J(x)$ is a (single-valued) weak*-continuous map from X to X^*.

Suppose that (i) does not hold; take $\bar{x} \in C \cap (x_0 + T_1)$ such that $\bar{x} \notin x_0 + T_2$. Let $x_0 + e \in P_{x_0+T_2}(\bar{x})$, where $e \in T_2$. Denote $h = \bar{x} - (x_0 + e)$. Then, by Proposition 2.2,

$$\langle J(h), (x_0 + z) - (x_0 + e)\rangle \leq 0 \quad \text{for all } z \in T_2.$$

Therefore,

$$\langle J(h), e\rangle = 0, \tag{3.2}$$

and $P_{x_0+T_2}(x_t) = x_0$ for each $t > 0$, where $x_t := x_0 + th$. By (ii), it follows that

$$P_{C\cap(x_0+T_1)}(x_t) = x_0. \tag{3.3}$$

Let $\bar{x}_t = (x_t - \bar{x})/t$ for $t > 0$. Then $\bar{x}_t = (1-1/t)h - e/t$ and $\lim_{t\to+\infty} \bar{x}_t = h$; hence,

$$\lim_{t\to+\infty} \langle J(\bar{x}_t) - J(h), h + e\rangle = 0. \tag{3.4}$$

Consequently, by (3.2) and (3.4),

$$\begin{aligned}
\|\bar{x}_t\|^2 &= \langle J(\bar{x}_t), \bar{x}_t\rangle \\
&= \langle J(\bar{x}_t), h\rangle - \langle J(\bar{x}_t) - J(h), (h+e)/t\rangle - \langle J(h), (h+e)/t\rangle \\
&\le \|\bar{x}_t\| \cdot \|h\| + |\langle J(\bar{x}_t) - J(h), (h+e)\rangle|/t - \|h\|^2/t \\
&< \|\bar{x}_t\| \cdot \|h\|,
\end{aligned}$$

and so $\|x_t - \bar{x}\| < t\|h\|$ for $t > 1$ large enough. Since $\bar{x} \in C\cap(x_0+T_1)$, this contradicts (3.3). The proof is complete. □

Remark 3.1. The result of Lemma 3.2 characterizes the smoothness of X (among reflexive Banach spaces). Indeed, suppose that there exists a unit vector $x_0 \in X$ such that $J(x_0)$ contains two distinct elements $x_1^*,\ x_2^*$. Write $x_0^* = \frac{x_1^*+x_2^*}{2},\ x_3^* = \frac{2}{3}x_1^* + \frac{1}{3}x_2^*$, and define

$$T_1 = \{x \in X :\ \langle x_3^*, x\rangle \ge 0\}, \quad T_2 = \{x \in X :\ \langle x_0^*, x\rangle \ge 0\}.$$

Then $x_0+T_1 \not\subseteq x_0+T_2$ although, for each $x \in X$, $x_0 \in P_{x_0+T_2}(x) \Longrightarrow x_0 \in P_{x_0+T_1}(x)$. In fact, if $x_0 \in P_{x_0+T_2}(x)$, then, by Proposition 2.2, there exists $x^* \in J(x - x_0)$ such that $\langle x^*, z\rangle \le 0$ for all $z \in T_2$. This implies that $x^* = -\|x - x_0\|x_0^*$; hence $\langle x_0^*, x_0 - x\rangle = \|x - x_0\|$. Consequently, $\langle x_i^*, x_0 - x\rangle = \|x - x_0\|$ for $i = 1, 2, 3$. Thus, for each $z \in T_1$,

$$\|x - (x_0 + z)\| \ge \langle x_3^*, x_0 + z - x\rangle \ge \langle x_3^*, x_0 - x\rangle = \|x - x_0\|;$$

hence $x_0 \in P_{x_0+T_1}(x)$, as claimed.

Let Z^* be a subset of X^* and $Z \subseteq X$. Let $z^*|_Z$ denote the restriction of z^* on Z; i.e., $z^*|_Z$ is viewed as a functional defined on Z instead of X. Set

$$Z^*|_Z = \{z^*|_Z :\ z^* \in Z^*\}. \tag{3.5}$$

Recall that $K := C \cap S$, where S denotes the solution set of (CIS). Let $x_0 \in K$, and let $\hat{S}(x_0)$ and $\hat{S}_{z^*}$ be defined as in (2.21). By Remark 2.2(c) (applied to the system (2.20) in place of (CIS)), we have the following equivalence:

(3.6) The system (2.20) satisfies the BCQ relative to C at x_0 if and only if the family $\{C, \hat{S}_{z^*}(x_0) :\ z^* \in \bigcup_{i\in I(x_0)} \mathrm{ext}\,\partial g_i(x_0)\}$ has the strong CHIP at x_0.

Thus one has (ii)⟺(ii*) in the following theorem.

THEOREM 3.1. *Let $x_0 \in K$. Consider the following statements:*

(i) *the system* (CIS) *satisfies the BCQ relative to C at x_0;*

(ii) *$K_S(x_0) = K_L(x_0)$, and the family $\{C, \hat{S}_{z^*}(x_0) :\ z^* \in \bigcup_{i\in I(x_0)} \mathrm{ext}\,\partial g_i(x_0)\}$ has the strong CHIP at x_0;*

(ii*) *$K_S(x_0) = K_L(x_0)$, and the system* (2.20) *satisfies the BCQ relative to C at x_0;*

(iii) *for each* $x \in X$, $x_0 \in P_K(x)$ *if and only if*

$$J(x - x_0) \cap \big(N_C(x_0) + N'(x_0)\big) \neq \emptyset; \tag{3.7}$$

(iv) *for each* $x \in X$, $x_0 \in P_K(x)$ *if and only if*

$$J(x - x_0)|_{C-x_0} \cap \big(N_C(x_0)|_{C-x_0} + N'(x_0)|_{C-x_0}\big) \neq \emptyset. \tag{3.8}$$

Then the following implications hold:

(1) (i)$\Longrightarrow$(iii)$\Longrightarrow$(iv); (ii)$\Longleftrightarrow$(ii*)$\Longrightarrow$(iii)$\Longrightarrow$(iv);

(2) (i)$\Longleftrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv) *if* X *is reflexive;*

(3) (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iii)$\Longrightarrow$(iv) *if* X *is both reflexive and smooth.*

Proof. The results are trivial when $x_0 \in C \cap \operatorname{int} S$ since each of (i)–(iv) in Theorem 3.1 holds automatically. Hence we assume that $x_0 \in C \cap \operatorname{bd} S$.

(1) Suppose that (i) holds. Then (3.7) can be rewritten as $J(x-x_0) \cap N_K(x_0) \neq \emptyset$; hence (iii) holds by Proposition 2.2. Therefore (i)$\Longrightarrow$(iii). Thus assuming that (ii*) holds, and applying this implication to the system (2.20) in place of (CIS), one has, for each $x \in X$,

$$x_0 \in P_{C \cap \hat{S}(x_0)}(x) \Longleftrightarrow J(x - x_0) \bigcap (N_C(x_0) + N'(x_0)) \neq \emptyset \tag{3.9}$$

(see Lemma 2.1). Consequently, by (2.22),

$$x_0 \in P_{K_L(x_0)}(x) \Longleftrightarrow J(x - x_0) \bigcap (N_C(x_0) + N'(x_0)) \neq \emptyset. \tag{3.10}$$

Further, by (3.1) and the assumption $K_S(x_0) = K_L(x_0)$ in (ii), we have that, for each $x \in X$,

$$x_0 \in P_K(x) \Longleftrightarrow x_0 \in P_{K_L(x_0)}(x). \tag{3.11}$$

Therefore, combining (3.10) and (3.11), we have established that (ii)$\Longleftrightarrow$(ii*)$\Longrightarrow$(iii).

Since (3.7) implies (3.8), to prove that (iii) implies (iv), it suffices to show that if (3.8) holds, then $x_0 \in P_K(x)$. By (3.8) and $N_C(x_0)|_{C-x_0} + N'(x_0)|_{C-x_0} \subseteq N_K(x_0)|_{C-x_0}$, we obtain that there exists $x^* \in J(x - x_0)$ such that

$$\operatorname{Re} \langle x^*, x' - x_0 \rangle \leq 0 \quad \text{for all } x' \in K. \tag{3.12}$$

Hence, for any $x' \in K$, we have that

$$\|x^*\| \cdot \|x - x_0\| = \operatorname{Re} \langle x^*, x - x_0 \rangle \leq \operatorname{Re} \langle x^*, x - x' \rangle \leq \|x^*\| \cdot \|x - x'\|.$$

This shows that $x_0 \in P_K(x)$, as required. Therefore (iii)$\Longrightarrow$(iv).

(2) Suppose that (3) is valid, and that X is reflexive. Then, by a known result in Banach space theory (cf. [16, p. 186]), there exists an equivalent norm on X such that X is smooth under the new norm. Then (3) implies that (i) and (ii) are equivalent. Other implications in (2) have already been proved in (1).

(3) By statement (1), we only need to show that (iii) implies (i) and (ii*). Suppose that (iii) holds. Let $z^* \in N_K(x_0)$. By the reflexivity of X, there exists $\bar{x} \in X$ such that $\langle z^*, \bar{x} \rangle = \|z^*\| \|\bar{x}\| = \|z^*\|^2$. Let $x = \bar{x} + x_0$. Then $z^* = J(x - x_0)$ by the smoothness, and $x_0 \in P_K(x)$ by Proposition 2.2. It follows from (iii) that $z^* \in N_C(x_0) + N'(x_0)$. This shows that $N_K(x_0) \subseteq N_C(x_0) + N'(x_0)$ and so (i) holds. Therefore (iii)$\Longrightarrow$(i).

To prove (iii)$\Longrightarrow$(ii*), noting from (2.19) that $K \subseteq K_S(x_0) \subseteq K_L(x_0)$, we have, for each $x \in X$,

$$x_0 \in P_{K_L(x_0)}(x) \Longrightarrow x_0 \in P_{K_S(x_0)}(x) \Longrightarrow x_0 \in P_K(x). \tag{3.13}$$

Conversely, let $x_0 \in P_K(x)$. Then, by (iii), $J(x-x_0) \in N_C(x_0) + N'(x_0)$. By (2.22) and (2.16), one has $N'(x_0) \subseteq N_{K_L(x_0)}(x_0)$. Since $K_L(x_0) \subseteq C$, it follows that $J(x-x_0) \in N_{K_L(x_0)}(x_0)$. Consequently, by Proposition 2.2, $x_0 \in P_{K_L(x_0)}(x)$. Hence, we have proved that, for each $x \in X$,

$$x_0 \in P_{K_S(x_0)}(x) \Longleftrightarrow x_0 \in P_{K_L(x_0)}(x) \Longleftrightarrow x_0 \in P_K(x). \tag{3.14}$$

It follows from Lemma 3.2 that $K_S(x_0) = K_L(x_0)$. Furthermore, by (3.14) and (iii), we obtain that $x_0 \in P_{K_L(x_0)}(x) \Longleftrightarrow J(x-x_0) \in N_C(x_0) + N'(x_0)$. Applying the just proved implication (iii)$\Longrightarrow$(i), we see that the system (2.20) satisfies the BCQ relative to C at x_0. This completes the proof of (iii)$\Longrightarrow$(ii*). ◻

Remark 3.2. The proof given for Theorem 3.1 is valid even if $I(x_0) = \emptyset$.

Remark 3.3. Example 3.1 (a) and (b) below show that neither the condition that X is smooth nor the condition that X is reflexive can be dropped for the implication (iii)$\Longrightarrow$(i) in Theorem 3.1.

Example 3.1 (cf. [24, Example 1]). (a) Let X be the Banach space $\mathbb{R}^2$ endowed with the l_1 norm defined as follows:

$$\|x\| = |t_1| + |t_2| \quad \text{for all } x = (t_1, t_2) \in \mathbb{R}^2. \tag{3.15}$$

Let $C = X,\ I = \{1, 2, \dots\}$, and define

$$g_i(x) = t_1 + \frac{1}{i} t_2 \quad \text{for all } x = (t_1, t_2) \in \mathbb{R}^2,\ \ i \in I.$$

Then, for any $x = (t_1, t_2) \in \mathbb{R}^2$,

$$G(x) := \sup_{i \in I} g_i(x) = \begin{cases} t_1 & \text{if } t_2 \le 0, \\ t_1 + t_2 & \text{if } t_2 \ge 0; \end{cases}$$

in particular, G is continuous. Furthermore,

$$K := C \cap S = S = \{x = (t_1, t_2) \in X :\ t_1 \le 0,\ t_1 + t_2 \le 0\}.$$

Take $x_0 = (0, 0)$. Then

$$N_K(x_0) = \{(t_1, t_2) \in \mathbb{R}^2 :\ 0 \le t_2 \le t_1\},$$

$$N'(x_0) = \{(t_1, t_2) \in \mathbb{R}^2 :\ 0 < t_2 \le t_1\} \cup \{(0, 0)\}.$$

Hence, the system (CIS) does not satisfy the BCQ relative to C at x_0. On the other hand, for any $x = (t_1, t_2) \in X$, from (3.15), $x_0 \in P_K(x)$ if and only if x lies in the first quadrant W of $\mathbb{R}^2$. Moreover, one has

$$J(x - x_0) = \begin{cases} [-1, 1] \times \operatorname{sgn} t_2 & \text{if } x = (0, t_2) \ne 0, \\ \operatorname{sgn} t_1 \times [-1, 1] & \text{if } x = (t_1, 0) \ne 0, \\ \big(\operatorname{sgn} t_1, \operatorname{sgn} t_2\big) & \text{if } x = (t_1, t_2),\ t_1 \ne 0, t_2 \ne 0, \end{cases}$$

where $\operatorname{sgn} t$ denotes the sign of t. Hence (3.7) holds if and only if $x \in W$. Thus, (iii) of Theorem 3.1 holds.

(b) Let X be any nonreflexive Banach space. By the well-known James theorem (cf. [19]; see also [31, Corollary 2.4, p. 99]), there exists a nonzero functional $x_0^* \in X^*$ such that it does not attain its norm on the unit ball of X. Set $C = \{x \in X : \langle x, x_0^* \rangle \leq 2\}$, $I = \{0, 1, \dots\}$. Define

$$g_0(x) = -\langle x, x_0^* \rangle, \quad x \in X,$$

and

$$g_i(x) = \langle x, x_0^* \rangle - \frac{1}{i}, \quad x \in X,\ i = 1, 2, \dots.$$

Then

$$K = C \cap S = \{x \in X : \langle x, x_0^* \rangle = 0\}.$$

Taking $x_0 = 0$, we have that $I(x_0) = \{0\}$ and that

$$N_K(x_0) = \{x^* \in X^* : \langle x, x^* \rangle = 0 \quad \text{for all } x \in K\} = \operatorname{span}\{x_0^*\},$$

$$N_C(x_0) = 0, \quad N'(x_0) = \operatorname{cone}\big(\partial g_0(x_0)\big) = \{-\lambda x_0^* : \lambda \geq 0\}. \tag{3.16}$$

In particular,

$$N_C(x_0) + N'(x_0) \neq N_K(x_0),$$

and hence the system

$$g_i(x) \leq 0, \quad i = 0, 1, 2, \dots,$$

does not satisfy the BCQ relative to C at x_0. Moreover, by our choice of x_0^* and (3.16), it is easy to see that (3.7) holds if and only if $x - x_0 = 0$. Recalling from [31, p. 100] that $P_K(z) \neq \emptyset$ implies $z \in K$, it follows that (iii) holds. $\square$

Remark 3.4. When I is finite and g_i is both convex and differentiable for each $i \in I$, the equivalence of (i) and (ii) in Theorem 3.1 was established in [22] for Hilbert spaces. Theorem 3.1 is new even in the case when $C = X = \mathbb{R}^n$. Two new features here are worth noting: I is not necessarily finite and g_i is not necessarily smooth. Moreover, our treatments are in the general Banach space setting.

4. Best constrained approximation in Hilbert spaces. Throughout this section, let X denote a Hilbert space (over $\mathbb{R}$ or $\mathbb{C}$). Let C be a closed convex subset of X and let K be the set of $x \in C$ that satisfies (CIS). Since X is a Hilbert space, $X^* = X$. In particular, (2.15) can be redefined as

$$\partial f(x) = \{z \in X : f(x) + \operatorname{Re}\langle z, y - x \rangle \leq f(y) \quad \text{for all } y \in X\}.$$

Similarly, $N_Z(z_0) = \{y \in X : \operatorname{Re}\langle y, z - z_0 \rangle \leq 0 \quad \text{for all } z \in Z\}$.

Dual formulation of the constrained best approximation problem in Hilbert spaces has been extensively investigated for finite systems of linear inequality constraints, e.g., [6, 7, 13, 14, 15, 26, 27], and for that of nonlinear inequalities, e.g., [22, 23]. In this section, we will establish similar results for infinite systems of convex inequalities. The first main result is as follows. Notation is as in the preceding sections (see (2.17), (2.18), and (2.21) in particular).

THEOREM 4.1. *Let $x_0 \in K$. Then the following statements are equivalent:*

(i) *the system* (CIS) *satisfies the BCQ relative to C at x_0;*

(ii) $K_S(x_0) = K_L(x_0)$ *and the family* $\{C, \hat{S}_{z^*}(x_0) :\ z^* \in \bigcup_{i\in I(x_0)} \operatorname{ext}\partial g_i(x_0)\}$ *has the strong CHIP at x_0;*

(iii) *for any $x \in X$, $P_K(x) = x_0$ if and only if there exists a finite (possibly empty) set $I_0 \subseteq I(x_0)$ such that $P_C(x - \sum_{i\in I_0} \lambda_i h_i) = x_0$ for some $\lambda_i \geq 0$ and $h_i \in \partial g_i(x_0)$ with each $i \in I_0$.*

Proof. By Theorem 3.1, it suffices to show that (3.7) holds if and only if there exists a finite set $I_0 \subseteq I(x_0)$ such that $P_C(x - \sum_{i\in I_0} \lambda_i h_i) = x_0$ for some $\lambda_i \geq 0$ and $h_i \in \partial g_i(x_0)$ with each $i \in I_0$. In view of the definition of $N'(x_0)$ and since $J(x - x_0) = x - x_0$ in a Hilbert space, $J(x - x_0) \in N_C(x_0) + N'(x_0)$ if and only if there exist a finite set $I_0 \subseteq I(x_0)$, $\lambda_i \geq 0$, and $h_i \in \partial g_i(x_0)$ such that

$$x - \sum_{i\in I_0} \lambda_i h_i - x_0 \in N_C(x_0). \tag{4.1}$$

By Proposition 2.2, (4.1) holds if and only if $P_C(x - \sum_{i\in I_0} \lambda_i h_i) = x_0$. Thus the result is clear. □

COROLLARY 4.1. *Consider the system* (CIS) *as before but suppose that, for each $i \in I$, g_i is an affine function defined by*

$$g_i(x) = \operatorname{Re}\langle h_i, x\rangle - b_i \quad \text{for all } x \in X, \tag{4.2}$$

where $\{h_i :\ i \in I\} \subset X \setminus \{0\}$ and $\{b_i\} \subseteq \mathbb{R}$. Let $C_i \subseteq X$ be defined by

$$C_i = \{x \in X :\ \operatorname{Re}\langle h_i, x\rangle \leq b_i\}. \tag{4.3}$$

Let C be a closed convex set in X and let $x_0 \in C \bigcap \left(\bigcap_{i\in I} C_i\right)$. Then the following statements are equivalent:

(i) *the family $\{C, C_i :\ i \in I\}$ has the strong CHIP at x_0;*

(ii) *for any $x \in X$, $P_K(x) = x_0$ if and only if there exists a finite (possibly empty) set $I_0 \subseteq I(x_0)$ such that $P_C(x - \sum_{i\in I_0} \lambda_i h_i) = x_0$ for some $\lambda_i \geq 0$ with each $i \in I_0$.*

More generally, let C be a closed convex set in X, $\{h_i :\ i \in I\} \subset X \setminus \{0\}$, and let $\{\Omega_i :\ i \in I\}$ be a family of nonempty closed convex subsets of the scalar field. Define

$$\hat{C}_i = \{x \in X :\ \langle h_i, x\rangle \in \Omega_i\}, \quad i \in I, \tag{4.4}$$

and

$$\hat{K} = C \bigcap \left(\bigcap_{i\in I} \hat{C}_i \right). \tag{4.5}$$

Let $x_0 \in \hat{K}$, and define

$$\hat{I}(x_0) := \{i \in I :\ \langle h_i, x_0\rangle \in \operatorname{bd}\Omega_i\}.$$

For convenience, we shall write $\tilde{h}_i(\cdot)$ for the function $\langle h_i, \cdot\rangle$ on X, and h_i^0 for the scalar $\langle h_i, x_0\rangle$. Then we have the following perturbation theorem.

THEOREM 4.2. *Let X be a Hilbert space (over $\mathbb{C}$ or $\mathbb{R}$), and let $x_0 \in \hat{K}$. Then the following statements are equivalent:*

(i) *the collection of convex sets $\{C, \hat{C}_i :\ i \in I\}$ has the strong CHIP at x_0;*

(ii) *for any* $x \in X$, $P_{\hat{K}}(x) = x_0$ *if and only if there exists a finite (possibly empty) set* $I_0 \subseteq \hat{I}(x_0)$ *such that* $P_C(x - \sum_{i\in I_0} \overline{\alpha}_i h_i) = x_0$ *for some* $\alpha_i \in N_{\Omega_i}(h_i^0)$ *with each* $i \in I_0$.

Proof. We may assume that X is over $\mathbb{C}$ (the case when X is over $\mathbb{R}$ is similar). For each $i \in I$, let $F_i(\cdot)$ be any (real-valued) convex function on $\mathbb{C}$ such that

$$\Omega_i = \{x \in \mathbb{C} : \ F_i(x) \le 0\} \tag{4.6}$$

(see (4.9) below, for example). Then we have that

$$\partial(F_i \circ \tilde{h}_i)(x_0) = \{\overline{\alpha} h_i : \ \alpha \in \partial F_i(h_i^0)\}. \tag{4.7}$$

In fact, it is easy to verify that the set on the right-hand side of (4.7) is contained in the set on the left-hand side. Conversely, let $x^* \in \partial(F_i \circ \tilde{h}_i)(x_0)$: $\mathrm{Re}\,\langle x^*, x - x_0\rangle \le (F_i \circ \tilde{h}_i)(x) - (F_i \circ \tilde{h}_i)(x_0)$ for all $x \in X$. Treating the corresponding real space X_R as in section 2, it follows that the real part $\mathrm{Re}\, x^* \in \tilde{\partial}(F_i \circ \tilde{h}_i)(x_0)$, where $\mathrm{Re}\, x^* : x \mapsto \mathrm{Re}\,\langle x^*, x\rangle$. Thus, by [23, Proposition 2.3], there exists $\alpha \in \partial F_i(h_i^0)$ such that

$$\mathrm{Re}\,\langle x^*, x\rangle = \mathrm{Re}\ \overline{\alpha}\langle h_i, x\rangle \quad \text{ for all } x \in X. \tag{4.8}$$

This implies that $x^* = \overline{\alpha} h_i$; hence (4.7) is proved.

Define

$$\hat{g}_i(x) = d_{\Omega_i}(\langle h_i, x\rangle) \quad \text{ for all } x \in X,\ i \in I, \tag{4.9}$$

where $d_{\Omega_i}(\cdot)$ denotes the distance function from the set Ω_i. Note that $\hat{g}_i^{-1}(\mathbb{R}_-) = \hat{C}_i$. Also, by (4.7) and [18, Example 3.3, p. 259], we get

$$\partial \hat{g}_i(x_0) = \{\overline{\alpha} h_i : \ \alpha \in N_{\Omega_i}(h_i^0),\ |\alpha| \le 1\}. \tag{4.10}$$

Consequently, by Theorem 4.1, (ii) holds if and only if the following system on X,

$$\hat{g}_i(x) \le 0, \quad i \in I, \tag{4.11}$$

satisfies the BCQ relative to C at x_0, that is,

$$N_{C\cap\left(\cap_{i\in I}\hat{g}_i^{-1}(\mathbb{R}_-)\right)}(x_0) = N_C(x_0) + \sum_{i\in\hat{I}(x_0)} \mathrm{cone}\left(\partial \hat{g}_i(x_0)\right) = N_C(x_0) + \sum_{i\in I} \mathrm{cone}\left(\partial \hat{g}_i(x_0)\right),$$

where the last equality holds because, for each $i \in I \setminus \hat{I}(x_0)$, one has $N_{\Omega_i}(h_i^0) = 0$ (and hence, by (4.10), that $\partial \hat{g}_i(x_0) = 0$). Note also that $N_{\hat{C}_i}(x_0) = 0$ for each $i \in I \setminus \hat{I}(x_0)$. Thus, to complete the proof, it suffices, by (4.10), to prove that

$$N_{\hat{C}_i}(x_0) = \{\overline{\alpha} h_i : \ \alpha \in N_{\Omega_i}(h_i^0)\} \quad \text{ for all } i \in \hat{I}(x_0). \tag{4.12}$$

Let $i \in \hat{I}(x_0)$ and divide the case in two: $\mathrm{int}\,\Omega_i \neq \emptyset$ and $\mathrm{int}\,\Omega_i = \emptyset$. In the first case, take a convex function F_i on $\mathbb{C}$ such that $\Omega_i = \{z \in \mathbb{C} : \ F_i(z) \le 0\}$ and $\mathrm{int}\,\Omega_i = \{z \in \mathbb{C} : \ F_i(z) < 0\}$ (e.g., $F_i(\cdot) = \hat{q}_i(\cdot - \hat{z}_i) - 1$, where $\hat{q}_i$ denotes the Minkowski functional (cf. [30, p. 24]) of the set $\Omega_i - \hat{z}_i$ for some $\hat{z}_i \in \mathrm{int}\,\Omega_i$). Then, by Remark 1.1(a),

$$N_{\Omega_i}(h_i^0) = \mathrm{cone}\left(\partial F_i(h_i^0)\right). \tag{4.13}$$

Similarly, note that $\hat{C}_i = \{x \in X : (F_i \circ \tilde{h}_i)(x) \le 0\}$ and that x_0 is not a minimizer of the convex function $F_i \circ \tilde{h}_i$ on X; again, by Remark 1.1, we have that

$$N_{\hat{C}_i}(x_0) = \text{cone}\left(\partial(F_i \circ \tilde{h}_i)(x_0)\right). \tag{4.14}$$

Hence, by (4.7), (4.13), and (4.14), (4.12) holds. It remains to consider the second case: Ω_i is of empty interior. Then the convex set Ω_i in $\mathbb{C}$ must be of one dimension and hence can be expressed as the intersection of at most four real half-spaces in $\mathbb{C}$ (e.g., a bounded closed line-segment in $\mathbb{R}^2$ is the intersection of four half-spaces). Thus there are affine functionals, say $\hat{F}_j$, $j = 1, \dots, m$ with $m \le 4$, such that $\Omega_i = \bigcap_{j=1}^m \hat{F}_j^{-1}(\mathbb{R}_-)$. Write $\hat{f}_j$ for the function $\hat{F}_j \circ \tilde{h}_i$ $(j = 1, \dots, m)$ and denote $J_0 := \{j : \hat{f}_j(x_0) = 0, \ j = 1, \dots, m\} = \{j : \hat{F}_j(h_i^0) = 0, \ j = 1, \dots, m\}$. Then by Remark 1.1(a) we have that, for each $j \in J_0$,

$$N_{\hat{F}_j^{-1}(\mathbb{R}_-)}(h_i^0) = \text{cone}\left(\partial \hat{F}_j(h_i^0)\right) \tag{4.15}$$

and

$$N_{\hat{f}_j^{-1}(\mathbb{R}_-)}(x_0) = \text{cone}\left(\partial \hat{f}_j(x_0)\right). \tag{4.16}$$

In addition, it is clear that $\hat{C}_i = \bigcap_{j=1}^m \hat{f}_j^{-1}(\mathbb{R}_-)$. It follows from Remark 2.2(d) and (4.16) that

$$N_{\hat{C}_i}(x_0) = \sum_{j=1}^m N_{\hat{f}_j^{-1}(\mathbb{R}_-)}(x_0) = \sum_{j \in J_0} \text{cone}\left(\partial \hat{f}_j(x_0)\right). \tag{4.17}$$

Similarly, we also have that

$$N_{\Omega_i}(h_i^0) = \sum_{j \in J_0} \text{cone}\left(\partial \hat{F}_j(h_i^0)\right). \tag{4.18}$$

Thus, by (4.7), (4.17), and (4.18), we get

$$N_{\hat{C}_i}(x_0) = \sum_{j \in J_0} \{\overline{\alpha} h_i : \alpha \in \text{cone}\left(\partial \hat{F}_j(h_i^0)\right)\} = \{\overline{\alpha} h_i : \alpha \in N_{\Omega_i}(h_i^0)\}, \tag{4.19}$$

and so (4.12) holds. The proof is complete. □

Let g_i be defined by

$$g_i(x) = \langle h_i, x\rangle - b_i \quad \text{for all } x \in X, \tag{4.20}$$

where $\{h_i : i \in I\} \subset X \setminus \{0\}$ and $\{b_i\} \subseteq \mathbb{C}$, and let $\tilde{S} = \bigcap_{i \in I} S_i$, where

$$S_i = \{x \in X : \langle h_i, x\rangle = b_i\}, \quad i \in I. \tag{4.21}$$

Applying Theorem 4.2 to the case when $\Omega_i = \{b_i\}$ for each i, we have the following corollary.

COROLLARY 4.2. *Let X be a Hilbert space over $\mathbb{R}$ (resp., $\mathbb{C}$) and let $x_0 \in C \cap \tilde{S}$. Then the following statements are equivalent:*

(i) *$\{C, S_i : i \in I\}$ has the strong CHIP at x_0;*

(ii) *for each* $x \in X$, $P_{C\cap\tilde{S}}(x) = x_0$ *if and only if there exists a finite (possibly empty) set* $I_0 \subseteq I(x_0)$ *such that* $P_C(x - \sum_{i\in I_0} \lambda_i h_i) = x_0$ *for some* $\lambda_i \in \mathbb{R}$ *(resp.,* $\mathbb{C}$*) with each* $i \in I_0$.

Remark 4.1. In the case when I is finite, each of (i) and (ii) of Corollary 4.2 is equivalent to the condition (cf. [11, 13, 14]) that

(i*) $\{C, \cap_{i\in I} S_i\}$ has the strong CHIP at x_0.

This is no longer true if I is infinite, as shown by the following example.

Example 4.1. Let X be the (real or complex) Hilbert space l^2 consisting of all infinite (real or complex) sequences (x_i) satisfying $\sum_{i=1}^{\infty} |x_i|^2 < \infty$. Let C be the closed unit ball of X. Let $I = \{2, 3, \dots\}$, and define

$$g_i(x) = x_i \quad \text{for all } x = (x_1, x_2, \dots) \in X, \quad i \in I.$$

Then $\tilde{S} = \{(x_1, 0, \dots) : x_1 \in \mathbb{R}\}$. Let $x_0 = 0$. Since $\operatorname{int} C \cap \tilde{S} \neq \emptyset$, $\{C, \tilde{S}\}$ has the strong CHIP at x_0. However, since $N_{S_i}(x_0) = \{x = (x_1, x_2, \dots) \in X : x_j = 0, j \neq i\}$ for each $i \in I$, $\sum_{i\in I} N_{S_i}(x_0)$ is not closed, and hence $\{C, S_i : i \in I\}$ does not have the strong CHIP at x_0. By Corollary 4.2, (ii) of Corollary 4.2 does not hold. □

Remark 4.2. Note that $\operatorname{int} C \cap (\cap_{i\in I} S_i) \neq \emptyset$ in Example 4.1. Thus Proposition 2.3(2) of [14] is not longer true if the index set I is infinite. Moreover, it is easy to verify that C itself is the only extremal subset of C containing $C\cap\tilde{S}$. Consequently the extremal subset C_b of C introduced in [15, Definition 4.1] is equal to C. Therefore the perturbation results in [15, Theorem 4.5] cannot be extended directly to the infinite case.

Remark 4.3. Results in this section have been presented as local ones; namely, we characterize conditions that hold at a single point x_0 of the set $C \cap (\cap_{i\in I} S_i)$. It is simple but sometimes desirable to describe the global analogue of the local results. For example, corresponding to Corollary 4.2, we have the following.

COROLLARY 4.3. *Let* X *be a Hilbert space. We write* $\tilde{S}$ *for* $\cap_{i\in I} S_i$. *Then the following statements are equivalent:*

(i) $\{C, S_i : i \in I\}$ *has the strong CHIP at each point of the intersection* $C \cap \tilde{S}$*;*

(ii) *for each* $x \in X$, *there exist a finite (possibly empty) set* I_x *of* I *and scalars* λ_i *such that*

$$P_{C\cap\tilde{S}}(x) = P_C\left(x - \sum_{i\in I_x} \lambda_i h_i\right).$$

Remark 4.4. By considering the whole space X in place of the unit ball in Example 4.1, we have a family $\{S_i : i \in I\}$ of polyhedra (in fact, maximal subspaces) which does not have the strong CHIP. In Example 4.2, we exhibit an infinite collection of polyhedra that has the strong CHIP.

Example 4.2. Let X be the real Hilbert space l^2 and let $I = \{1, 2, \dots\}$. Define, for each $i \in I$,

$$C_i = \{x = (x_n) \in X : x_i \leq 1\}.$$

Let $C = \cap_{i\in I} C_i$. Then $\{C_i : i \in I\}$ has the strong CHIP at each point x of C. Indeed, since $x = (x_n) \in l^2$, there exists an $N \in \mathbb{N}$ such that $|x_n| \leq 1/2$ for all $n \geq N$. Let U denote the ball with center x and radius 1/2. Then $U \subset \cap_{i\geq N} C_i$. This shows that $x \in \operatorname{int}(\cap_{i\geq N} C_i)$ and hence that $N_{\cap_{i\geq N} C_i}(x) = 0$. Since

$$N_C(x) = N_{\cap_{i\leq N} C_i}(x) + N_{\cap_{i\geq N} C_i}(x)$$

and $\{C_1, C_2, \ldots, C_N\}$ has the strong CHIP, we have

$$N_C(x) = \sum_{i=1}^{N} N_{C_i}(x) = \sum_{i=1}^{\infty} N_{C_i}(x).$$

5. Best constrained approximation in $C(Q)$. Let $C(Q)$ denote the Banach space of all complex-valued continuous functions on a compact metric space Q endowed with the uniform norm:

$$\|f\| = \max_{t \in Q} |f(t)| \quad \text{for all } f \in C(Q).$$

Let $\mathcal{P}$ be a finite-dimensional subspace of $C(Q)$, and let $\{\Omega_t : \ t \in Q\}$ be a family of nonempty closed convex sets in the complex plane $\mathbb{C}$. For brevity, we write $\{\Omega_t\}$ for $\{\Omega_t : \ t \in Q\}$. Set

$$\mathcal{P}_\Omega = \{p \in \mathcal{P} : \ p(t) \in \Omega_t \text{ for all } t \in Q\}. \tag{5.1}$$

The problem considered here is that of finding an element $p^* \in \mathcal{P}_\Omega$ for $f \in C(Q)$ such that

$$\|f - p^*\| = \inf_{p \in \mathcal{P}_\Omega} \|f - p\| \tag{5.2}$$

(such p^* is called a best restricted range approximation to f from $\mathcal{P}$ with respect to $\{\Omega_t\}$). This problem was first presented and formulated by Smirnov and Smirnov in [33, 34]; their approach followed the standard path for the corresponding issue in the real-valued continuous function space theory (see, for example, [5, 20] and the relevant references therein). In [34], while it was pointed out that this problem for the general class of restrictions was quite difficult, they took up the special case when each Ω_t is a disk in $\mathbb{C}$. Later, in [35, 36, 37], a more general case was considered in that the family $\{\Omega_t\}$ was assumed to have the following properties:

(i) there exists an element $p_0 \in \mathcal{P}$ satisfying $p_0(t) \in \operatorname{int} \Omega_t$ for each $t \in Q$ (such an element p_0 of $\mathcal{P}$ will be called an interior point with respect to $\mathcal{P}$ and $\{\Omega_t\}$);

(ii) Ω_t is a strictly convex set with "smooth" boundary for each $t \in Q$;

(iii) the set-valued map $t \mapsto \Omega_t$ is continuous with respect to the Hausdorff metric.

It was pointed out in [21] that (i) and (iii) imply that there exists a function F on the product space $\mathbb{C} \times Q$ with the following properties:

(C1) $F(\cdot, t)$ is convex on $\mathbb{C}$ for each $t \in Q$;

(C2) $\operatorname{bd} \Omega_t = \{z \in \mathbb{C} : \ F(z, t) = 0\}$ for all $t \in Q$;

(C3) $\operatorname{int} \Omega_t = \{z \in \mathbb{C} : \ F(z, t) < 0\}$ for all $t \in Q$;

(C4) F is continuous on $\mathbb{C} \times Q$.

This observation led the first author of the present paper to study, in [21], a more general setting in that a function with properties (C1)–(C4) is given, $\Omega_t := \{z \in \mathbb{C} : F(z, t) \leq 0\}$, and an interior point (in the above sense) exists. Thus (ii) and (iii) need not be satisfied.

For the remainder of this section, let $\mathcal{P}$ be a finite-dimensional subspace of $C(Q)$, let Q be a compact metric space, and let $\{\Omega_t : \ t \in Q\}$ be a family of nonempty closed convex subsets of $\mathbb{C}$ satisfying the following:

(D1) the set-valued function $t \mapsto \Omega_t$ is lower semicontinuous on Q;

(D2) there exists $p_0 \in \mathcal{P}$ such that

$$0 \in \operatorname{int}\left(\bigcap_{t\in Q}(\Omega_t - p_0(t))\right) \tag{5.3}$$

(such an element p_0 of $\mathcal{P}$ will be called a strong interior point with respect to $\mathcal{P}$ and $\{\Omega_t\}$).

The following remarks show in particular that the present setting is more general than that of [21] (and [33, 34, 35, 36, 37]).

Remark 5.1. (a) In the case when (C1)–(C4) are satisfied, the map $t \mapsto \Omega_t$ is both upper (in the sense of Kuratowski; see [24, p. 37]) and lower semicontinuous on Q. In fact, the upper semicontinuity is trivial while the lower semicontinuity holds because for any $t_0 \in Q$ and $x_0 \in \operatorname{int}\Omega_{t_0}$ there exists an open neighborhood $V(t_0)$ of t_0 such that $x_0 \in \operatorname{int}\Omega_t$ for all $t \in V(t_0)$.

(b) One can prove that properties (C1)–(C4) imply that $p_0 \in \mathcal{P}$ is a strong interior point if and only if $\sup_{t\in Q} F(p_0(t), t) < 0$. Hence, in this case, $p_0 \in \mathcal{P}$ is an interior point if and only if it is a strong interior point.

(c) For a family $\{\Omega_t\}$ satisfying (D1) and (D2), there exist many functions $F(\cdot,\cdot)$ on $\mathbb{C}\times Q$ with properties (C1)–(C3). One such function which is given below has additional properties that will be useful for us. Let $t \in Q$ and $p_0 \in \mathcal{P}$ be such that (5.3) holds. Define $\hat{F} : \mathbb{C}\times Q \to \mathbb{R}$ by

$$\hat{F}(z,t) = \hat{q}_t(z - p_0(t)) - 1, \tag{5.4}$$

where $\hat{q}_t$ denotes the Minkowski functional (cf. [30, p. 24]) of the closed convex set $\hat{\Omega}_t$ in $\mathbb{C}$ defined by

$$\hat{\Omega}_t = \Omega_t - p_0(t); \tag{5.5}$$

thus $\hat{q}_t(z) = \inf\{\lambda > 0 : z \in \lambda\hat{\Omega}_t\}$. It is easy to verify that $\hat{F}$ does have the properties (C1)–(C3) stated for F. On the other hand, there are many examples for which (C1)–(C3) are satisfied but without any associated function F with the properties (C1)–(C4).

Before giving the main theorem of this section, we need some preliminary results.

LEMMA 5.1. *For each $t \in Q$, let q_t be defined by*

$$q_t(z) := \hat{q}_t(z - p_0(t)) \quad \textit{for all } z \in \mathbb{C}; \tag{5.6}$$

that is, $q_t(\cdot) = \hat{F}(\cdot, t) + 1$. Then

(i) *there exists a constant $\gamma > 0$ such that*

$$|q_t(z) - q_t(z')| \le \gamma|z - z'| \quad \textit{for all } t \in Q,\ \ z, z' \in \mathbb{C}; \tag{5.7}$$

(ii) *for each $z \in \mathbb{C}$, the function $t \mapsto q_t(z)$ is upper semicontinuous.*

Proof. (i) By (D2), there exists a ball $B(0,\delta)$ in $\mathbb{C}$ with center 0 and radius $\delta > 0$ such that

$$B(0,\delta) \subseteq \hat{\Omega}_t \quad \text{for all } t \in Q. \tag{5.8}$$

By the definition of Minkowski functionals (cf. [30, p. 24]), it follows that

$$\hat{q}_t(z) \le \frac{1}{\delta}\|z\| \quad \text{for all } t \in Q,\ z \in \mathbb{C}. \tag{5.9}$$

Hence, by the subadditivity of $\hat{q}_t$ and (5.6), (5.7) holds with $\gamma := \frac{1}{\delta}$.

(ii) Let $z \in \mathbb{C}$ and $t_0 \in Q$. We have to show that $\limsup_{t \to t_0} \hat{q}_t(z) \le \hat{q}_{t_0}(z)$. Take a sequence $(t_n) \to t_0$ such that $\lim_{t_n \to t_0} \hat{q}_{t_n}(z) = l$ for some $l \in \mathbb{R}$. It suffices to show that $l \le \hat{q}_{t_0}(z)$. To this end, let $\varepsilon > 0$. Then, by the definition of Minkowski functionals, $z \in (\hat{q}_{t_0}(z) + \varepsilon)\hat{\Omega}_{t_0}$. Let $\lambda = \hat{q}_{t_0}(z) + \varepsilon$. Then $\frac{z}{\lambda} \in \hat{\Omega}_{t_0}$. By the lower semicontinuity, considering subsequences if necessary, we may assume that there exists $(z_n) \to \frac{z}{\lambda}$ with $z_n \in \hat{\Omega}_{t_n}$ for each n; we may assume further that $|z_n - \frac{z}{\lambda}| \le \frac{\varepsilon}{\lambda\gamma}$ for each n. Then it follows from (i) that

$$\hat{q}_{t_n}\left(\frac{z}{\lambda}\right) = \hat{q}_{t_n}\left(\frac{z}{\lambda}\right) - \hat{q}_{t_n}(z_n) + \hat{q}_{t_n}(z_n) \le \gamma\left|\frac{z}{\lambda} - z_n\right| + 1 \le 1 + \frac{\varepsilon}{\lambda}, \tag{5.10}$$

and so $\hat{q}_{t_n}(z) \le \varepsilon + \hat{q}_{t_0}(z) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, letting $\epsilon \to 0$, we have $l \le \hat{q}_{t_0}(z)$, as required. □

Let $\mathcal{P}$, p_0, and $\{\Omega_t :\ t \in Q\}$ be given with the properties (D1) and (D2). A key step to establishing our main result in this section is to apply Theorem 2.1 to (CIS) with $I = Q$, $X = C(Q)$, and g_t, where $g_t : C(Q) \to \mathbb{R}$ defined by

$$g_t(u) = q_t(u(t)) - 1 \quad \text{for all } u \in C(Q), \quad t \in Q. \tag{5.11}$$

Note, by (5.4) and (5.6), that

$$g_t(u) = \hat{q}_t(u(t) - p_0(t)) - 1 = \hat{F}(u(t), t) \quad \text{for all } t \in Q,\ u \in C(Q). \tag{5.12}$$

Thus, each g_t is a continuous convex function on $C(Q)$. Let $\hat{S}$ denote the solution set of the following system of inequalities:

$$g_t(\cdot) \le 0, \quad t \in Q. \tag{5.13}$$

Then $\hat{S}$ is nonempty, since p_0 is a Slater point for (5.13) as $g_t(p_0) = -1$ for each $t \in Q$. Note also that, by definition,

$$\hat{S} = \{u \in C(Q) :\ u(t) \in \Omega_t \quad \text{for all } t \in Q\}. \tag{5.14}$$

Let $\hat{G}$ denote the sup-function of $\{g_t :\ t \in Q\}$:

$$\hat{G}(u) = \sup_{t \in Q} g_t(u). \tag{5.15}$$

In a lemma below, we will show that $\hat{G}$ is continuous and that, for each $u \in C(Q)$, the function $t \mapsto g_t(u)$ is upper semicontinuous on Q. Granting these and applying Theorem 2.1, we immediately obtain the following proposition.

PROPOSITION 5.1. *Let $\mathcal{P}$ be a finite-dimensional subspace of $C(Q)$, $p_0 \in \mathcal{P}$, and let $\{\Omega_t :\ t \in Q\}$ be a family of closed convex subsets of $\mathbb{C}$ such that* (D1) *and* (D2) *are satisfied. Then the system* (5.13) *satisfies the BCQ relative to $\mathcal{P}$ at any point p of $\mathcal{P} \cap \hat{S}$.*

For each $t \in Q$, e_t denotes the point-valued functional on $C(Q)$ defined by

$$\langle e_t, u\rangle = u(t) \quad \text{for all } u \in C(Q). \tag{5.16}$$

LEMMA 5.2. *The function $\hat{G}$ and the set $\{g_t :\ t \in Q\}$ defined above have the following properties:*

(i) *for each $u \in C(Q)$, the function $t \mapsto g_t(u)$ is upper semicontinuous;*

(ii) *the sup-function $\hat{G}(\cdot) = \sup_{t \in Q} g_t(\cdot)$ is continuous;*

(iii) *for each* $u \in C(Q),\ t \in Q$,

$$\partial g_t(u) = \{\overline{\alpha}\, e_t \in C(Q)^* : \ \ \alpha \in \partial q_t(u(t))\}. \tag{5.17}$$

Proof. (i) Let $u \in C(Q)$ and let $t_0 \in Q$. By (5.7),

$$g_t(u) = [q_t(u(t)) - q_t(u(t_0))] + q_t(u(t_0)) - 1 \leq \gamma |u(t) - u(t_0)| + q_t(u(t_0)) - 1.$$

Then, by Lemma 5.1(ii), we have that

$$\limsup_{t\to t_0} g_t(u) \leq \limsup_{t\to t_0} [q_t(u(t_0)) - 1] = q_{t_0}(u(t_0)) - 1 = g_{t_0}(u). \tag{5.18}$$

Thus (i) is proved.

(ii) This follows from the inequalities

$$|\hat{G}(u) - \hat{G}(v)| \leq \sup_{t\in Q} |g_t(u) - g_t(v)| \leq \sup_{t\in Q} |q_t(u(t)) - q_t(v(t))| \leq \gamma \|u - v\| \tag{5.19}$$

for any $u, v \in C(Q)$, where the last inequality holds because of (5.7).

(iii) It is easy to check that $\partial g_t(u)$ contains the set on the right-hand side of (5.17). To show the reverse inclusion, let $u^* \in \partial g_t(u)$. Then $u^* \in C(Q)^*$ and there exists a complex Radon measure μ with bounded variation on Q such that

$$\langle u^*, v\rangle = \int_Q \overline{v}\, \mathrm{d}\mu \quad \text{ for all } v \in C(Q) \tag{5.20}$$

(cf. [39, p. 350]). Write $Q_t = Q \setminus \{t\}$ and $\mu = \mu_R + \mathrm{i}\mu_I$, where $\mu_R,\ \mu_I$ are real Radon measures on Q. Let $E_i \subseteq Q_t,\ i = 1, 2$, be such that $E_1 \cup E_2 = Q_t$, $E_1 \cap E_2 = \emptyset$, μ_R is nonnegative on E_1, and μ_R is nonpositive on E_2. Then $|\mu_R|(Q_t) = \mu_R(E_1) - \mu_R(E_2)$. For any $\varepsilon > 0$, let $F_i \subseteq E_i,\ i = 1, 2$, be closed and satisfy $|\mu_R|(E_i \setminus F_i) < \frac{\varepsilon}{4},\ i = 1, 2$. By Urysohn's lemma, there exists a real continuous function w on Q satisfying $\|w\| \leq 1$ and

$$w(s) = \begin{cases} 1, & s \in F_1, \\ -1, & s \in F_2, \\ 0, & s = t. \end{cases}$$

Define $v = w + u$. Since $w = 0$ at t, $g_t(w + u) = g_t(u)$, and hence

$$0 = g_t(v) - g_t(u) \geq \mathrm{Re}\,\langle u^*, v - u\rangle = \mathrm{Re}\, \int_Q (\overline{v - u})\, \mathrm{d}\mu = \int_Q (v - u)\, \mathrm{d}\mu_R.$$

This implies that

$$\mu_R(F_1) - \mu_R(F_2) < \frac{\varepsilon}{2}. \tag{5.21}$$

Consequently,

$$|\mu_R|(Q_t) = \mu_R(E_1) - \mu_R(E_2) \leq \mu_R(F_1) - \mu_R(F_2) + \frac{\varepsilon}{2} < \varepsilon. \tag{5.22}$$

Hence, $|\mu_R|(Q_t) = 0$. Similarly, we have $|\mu_I|(Q_t) = 0$. Therefore μ must be a point-measure and hence $u^* = \overline{\alpha}\, e_t$ with some $\alpha \in \mathbb{C}$. Since $u^* \in \partial g_t(u),\ \alpha \in \partial q_t(u(t))$ and (5.17) is proved. ∎

Let $F(\cdot,\cdot)$ be any fixed function on $\mathbb{C}\times Q$ satisfying (C1)–(C3). Let $f \in C(Q)$, $p^* \in \mathcal{P}_\Omega$. Following [21, 33, 34], define

$$M(f) = \{t \in Q : \ |f(t)| = \|f\|\}, \quad B(p^*) = \{t \in Q : \ p^*(t) \in \operatorname{bd}\Omega_t\}, \tag{5.23}$$

$$\sigma(t) = f(t) - p^*(t) \quad \text{ for all } t \in Q, \tag{5.24}$$

and, for each $t \in B(p^*)$, let $-\tau(t)$ denote the subdifferential of the convex function $F(\cdot,t)$ at $p^*(t)$, that is,

$$\tau(t) = -\partial F(\cdot,t)|_{p^*(t)} \quad \text{ for all } t \in B(p^*). \tag{5.25}$$

Thus $\sigma(t) \in \mathbb{C}$ and $\tau(t) \subseteq \mathbb{C}$. Note that

$$B(p^*) = \{t \in Q : \ g_t(p^*) = \hat{G}(p^*) = 0\}; \tag{5.26}$$

that is, $B(p^*)$ is exactly the active index set for p^* with respect to the system (5.13). Furthermore, assume that $\mathcal{P}$ has dimension n and is spanned by, say, $\phi_1, \phi_2, \ldots, \phi_n$. For each $t \in Q$, by abuse of notation, let $\mathbf{c}(t) \subseteq \mathbb{C}^n$ be defined by

$$\mathbf{c}(t) = (\overline{\phi_1(t)}, \ldots, \overline{\phi_n(t)})\tau(t);$$

more precisely,

$$\mathbf{c}(t) = \{(\eta\overline{\phi_1(t)}, \ldots, \eta\overline{\phi_n(t)}) : \ \eta \in \tau(t)\}.$$

Similarly, we define $\mathbf{d}(t) \in \mathbb{C}^n$ by

$$\mathbf{d}(t) = (\overline{\phi_1(t)}, \ldots, \overline{\phi_n(t)})\sigma(t).$$

Define

$$\mathcal{U} = \mathcal{U}_1 \bigcup \mathcal{U}_2,$$

where

$$\mathcal{U}_1 = \{\mathbf{d}(t) : \ t \in M(f - p^*)\} \quad \text{ and } \quad \mathcal{U}_2 = \bigcup_{t \in B(p^*)} \mathbf{c}(t).$$

Note that, by continuity and compactness, $\mathcal{U}_1$ is compact. Furthermore, we have the following lemma. We assume that F in (5.25) is the function F given in (5.4).

LEMMA 5.3. *$\mathcal{U}$ is compact in $\mathbb{C}^n$.*

Proof. Note first that $t \in B(p^*)$ if and only if $q_t(p^*(t)) = 1$, where q_t is given by (5.6). Let $\{t_k\} \subseteq B(p^*)$ be a convergent sequence with limit t_0. By Lemma 5.2(i), we have that

$$q_{t_0}(p^*(t_0)) \geq \limsup_k q_{t_k}(p^*(t_k)) = 1. \tag{5.27}$$

Since $p^*(t_0) \in \Omega_{t_0}$, it follows that $q_{t_0}(p^*(t_0)) = 1$. Hence $t_0 \in B(p^*)$ and $B(p^*)$ is closed. By assumption,

$$F(z,t) = q_t(z) - 1 \quad \text{ for all } z \in \mathbb{C}, \ t \in Q. \tag{5.28}$$

Then, by Lemma 5.1(i), one can show (as in [21, Theorem 3.1]) that $\mathcal{U}_2$ is compact and so is $\mathcal{U}$. □

Now we are ready to give the main theorem of this section, which gives characterizations of the best restricted range approximation in $C(Q)$. The properties stated in (ii)–(iv) are standard and well known in approximation theory; see, e.g., [4, 5, 20]. Note that, by Remark 1.1, for any function $F(\cdot,\cdot)$ on $\mathbb{C}\times Q$ satisfying (C1)–(C3), we have that

$$\operatorname{cone}\partial F(\cdot,t)|_{p^*(t)} = N_{\Omega_t}(p^*(t)) \quad \text{for all } t\in B(p^*). \tag{5.29}$$

THEOREM 5.1. *Let $f\in C(Q)$, $p^*\in\mathcal{P}_\Omega$. Then the following four statements are equivalent:*

(i) *p^* is a best restricted range approximation to f from $\mathcal{P}$ with respect to $\{\Omega_t\}$;*

(ii)

(5.30)
$$\max\left\{\max_{t\in M(f-p^*)}\operatorname{Re}\left(p(t)\overline{\sigma(t)}\right),\ \max_{t\in B(p^*)}\max_{\tau\in\tau(t)}\operatorname{Re}\left(p(t)\overline{\tau}\right)\right\}\ge 0 \quad \textit{for all } p\in\mathcal{P};$$

(iii) *the origin of the space $\mathbb{C}^n$ belongs to the convex hull of the set $\mathcal{U}$;*

(iv) *there exist sets $\{t_1,\dots,t_k\}\subseteq M(f-p^*)$, $\{t_1',\dots,t_m'\}\subseteq B(p^*)$, $\tau_j'\in\tau(t_j')$, $i=1,\dots,m$ $(1+m\le k+m\le 2n+1)$, and positive constants $\lambda_1,\dots,\lambda_k$, $\lambda_1',\dots,\lambda_m'$ such that the following condition holds:*

$$\sum_{i=1}^{k}\lambda_i p(t_i)\overline{\sigma(t_i)}+\sum_{j=1}^{m}\lambda_j' p(t_j')\overline{\tau'}_j=0 \quad \textit{for all } p\in\mathcal{P}. \tag{5.31}$$

Proof. Since the result is trivial in the case when $f\in P_\Omega$, we assume that $f\ne p^*$. By (5.29), we may assume, without loss of generality, that F in (5.25) is simply the function given by (5.28). Let $t\in B(p^*)$ and $\eta\in\tau(t)$. Then $q_t(p^*(t))=1$; hence

$$-\eta\in\partial q_t(p^*(t)) \text{ and } -\operatorname{Re}\left(p_0(t)-p^*(t)\right)\overline{\eta}\le q_t(p_0(t))-q_t(p^*(t))=-1. \tag{5.32}$$

Therefore, the case when $k=0$ will not occur in (5.31) because otherwise (5.31) would entail that

$$\sum_{j=1}^{m}\lambda_j'(p_0-p^*)(t_j')\overline{\tau'}_j=0 \tag{5.33}$$

(with p replaced by p_0-p^* as $\mathcal{P}$ is a vector subspace) and (5.33) contradicts (5.32) (applied to $t=t_j'$ and $\eta=\tau_j'$) as each $\lambda_j'>0$. Thus (iii)$\Longleftrightarrow$(iv) by Carathéodory's theorem (cf. [4] and [30, p. 73]). Also, since $\mathcal{P}$ is spanned by $\phi_1,\dots,\phi_n$, it is easy to verify that (ii) does not hold if and only if there exists $z=(\gamma_1,\dots,\gamma_n)\in\mathbb{C}^n$ such that $\operatorname{Re}\langle u,z\rangle<0$ for all $u\in\mathcal{U}$. Thus, as $\mathcal{U}$ is compact by Lemma 5.3, (ii)$\Longleftrightarrow$(iii) by the linear inequality theorem (see [4]). To show that (i)$\Longleftrightarrow$(iv), note that $\mathcal{P}_\Omega=\mathcal{P}\cap\hat{S}$, where $\hat{S}$ denotes the solution set of the convex inequality system in $C(Q)$ defined by (5.13). By Proposition 5.1, this system satisfies the BCQ relative to $\mathcal{P}$ at p^*. By the implication (i)$\Longrightarrow$(iv) in Theorem 3.1 and the fact that $\mathcal{P}$ is a vector subspace containing p^* (so $N_{\mathcal{P}}(p^*)|_{\mathcal{P}} = 0$), the following statements are equivalent:

(i*) p^* is a best approximation to f from $\mathcal{P} \cap \hat{S}$;

(iv*) $J(f - p^*)|_{\mathcal{P}} \bigcap N'(p^*)|_{\mathcal{P}} \neq \emptyset$.

Since (i) and (i*) are the same, it remains to show (iv)$\Longleftrightarrow$(iv*).

(iv)$\Longrightarrow$(iv*). Suppose that (iv) holds. Without loss of generality, assume that $\sum_{i=1}^k \lambda_i = 1$ in (5.31). Then $\sum_{i=1}^k \lambda_i \overline{\sigma(t_i)} e_{t_i} \in J(f - p^*)$. By (5.17) and (5.32), each $-\overline{\tau'}_j e_{t'_j} \in \partial g_{t'_j}(p^*)$ and so, by (5.26), $\sum_{j=1}^m \lambda'_j\big(-\overline{\tau'}_j e_{t'_j}\big) \in N'(p^*)$. Therefore (5.31) implies (iv*).

(iv*)$\Longrightarrow$(iv). Suppose that (iv*) holds. Then there exist $v^* \in J(f - p^*)$, $w_j^* \in \partial g_{t'_j}(p^*)$, and $\lambda'_j > 0$, $j = 1, 2, \ldots, m$, with each $t'_j \in B(p^*)$ such that

$$\langle v^*, p\rangle = \sum_{j=1}^m \lambda'_j \langle w_j^*, p\rangle \quad \text{for all } p \in \mathcal{P}. \tag{5.34}$$

Set $u^* = v^*/\|v^*\|$. Applying [31, Lemma 1.3, p. 169] to the real linear span of $\mathcal{P} \cup \{f\}$, there exist a positive integer l (with $1 \le l \le 2n + 2$), l extreme points $u_1^*, \ldots, u_l^*$ of the unit ball Σ^* of $C(Q)^*$, and positive constants β_i, $i = 1, 2, \ldots, l$, with $\sum_{i=1}^l \beta_i = 1$ such that

$$\langle u^*, p\rangle = \sum_{i=1}^l \beta_i \langle u_i^*, p\rangle \quad \text{for all } p \in \mathcal{P} \cup \{f\}. \tag{5.35}$$

By a well-known representation of the extreme points of Σ^* (cf. [31, p. 69]), there exist some $\alpha_i \in \mathbb{C}$ with $|\alpha_i| = 1$ and $t_i \in Q$ such that

$$u_i^* = \alpha_i e_{t_i}, \quad i = 1, 2, \ldots, l. \tag{5.36}$$

By the definition of u^*, $\|u^*\| = 1$ and $\langle u^*, f - p^*\rangle = \|f - p^*\|$; hence, by (5.35), $t_i \in M(f - p^*)$ and $\alpha_i = \overline{(f - p^*)(t_i)}/\|f - p^*\|$. Furthermore, by (5.17), for each j, there exists $\alpha'_j \in \partial q_{t'_j}(p^*(t'_j))$ such that $w_j^* = \overline{\alpha'}_j e_{t'_j}$. Therefore, (5.34) becomes

$$\sum_{i=1}^l \beta'_i \overline{\sigma(t_i)} \langle e_{t_i}, p\rangle = \sum_{j=1}^m \lambda'_j \overline{\alpha}'_j \langle e_{t'_j}, p\rangle \quad \text{for all } p \in \mathcal{P}, \tag{5.37}$$

where $\beta'_i = \|v^*\|\beta_i/\|f - p^*\|$ for each $i = 1, \ldots, l$. This implies that (iii) holds and so (iv) holds by (iii)$\Longleftrightarrow$(iv). The proof is complete. □

REFERENCES

[1] A. Baken, F. Deutsch, and W. Li, *Strong CHIP, Normality, and Linear Regularity of Convex Sets*, available online from http://www.math.odu.edu/~wuli, May, 2002.

[2] H. Bauschke, J. Borwein, and W. Li, *Strong conical hull intersection property, bounded linear regularity, Jameson's property (G), and error bounds in convex optimization*, Math. Program. Ser. A, 86 (1999), pp. 135–160.

[3] D. Braess, *Nonlinear Approximation Theory*, Springer-Verlag, New York, 1986.

[4] E. W. Cheney, *Introduction to Approximation Theory*, McGraw-Hill, New York, 1966.
[5] B. L. Chalmers and G. D. Taylor, *Uniform approximation with constraints*, Jahresber. Deutsch. Math.-Verein., 81 (1978/1979), pp. 49–86.
[6] C. Chui, F. Deutsch, and J. Ward, *Constrained best approximation in Hilbert space*, Constr. Approx., 6 (1990), pp. 35–64.
[7] C. Chui, F. Deutsch, and J. Ward, *Constrained best approximation in Hilbert space* II, J. Approx. Theory, 71 (1992), pp. 231–238.
[8] F. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley, New York, 1983.
[9] F. Deutsch, *The role of the strong conical hull intersection property in convex optimization and approximation*, in Approximation Theory IX, Vol. I: Theoretical Aspects, C. Chui and L. Schumaker, eds., Vanderbilt University Press, Nashville, TN, 1998, pp. 105–112.
[10] F. Deutsch, *Some Applications of Functional Analysis to Approximation Theory*, Doctoral dissertation, Brown University, Providence, RI, 1965.
[11] F. Deutsch, *Best Approximation in Inner Product Spaces*, Springer-Verlag, New York, 2001.
[12] F. Deutsch, W. Li, and J. Swetits, *Fenchel duality and the strong conical intersection property*, J. Optim. Theory Appl., 102 (1997), pp. 681–695.
[13] F. Deutsch, W. Li, and J. Ward, *A dual approach to constrained interpolation from a convex subset of Hilbert space*, J. Approx. Theory, 90 (1997), pp. 385–444.
[14] F. Deutsch, W. Li, and J. D. Ward, *Best approximation from the intersection of a closed convex set and a polyhedron in Hilbert space, weak Slater conditions, and the strong conical hull intersection property*, SIAM J. Optim., 10 (1999), pp. 252–268.
[15] F. Deutsch, V. Ubhaya, J. Ward, and Y. Xu, *Constrained best approximation in Hilbert space* III: *Application to n-convex functions*, Constr. Approx., 12 (1996), pp. 361–384.
[16] J. Diestel, *Geometry of Banach Spaces—Selected Topics*, Lecture Notes in Math. 485, Springer-Verlag, New York, 1975.
[17] M. A. Goberna and M. A. Lopez, *Linear Semi-infinite Optimization*, Wiley, Chichester, UK, 1998.
[18] J. Hiriart-Urruty and C. Lemarechal, *Convex Analysis and Minimization Algorithms* I, Grundlehren Math. Wiss. 305, Springer-Verlag, New York, 1993.
[19] R. C. James, *Characterization of reflexivity*, Studia Math., 23 (1964), pp. 205–216.
[20] A. Kroo and D. Schmidt, *A Haar-type theory of best uniform approximation with constraints*, Acta Math. Hungar., 58 (1991), pp. 351–374.
[21] C. Li, *On best uniform restricted range approximation in complex-valued continuous function spaces*, J. Approx. Theory, 120 (2003), pp. 71–84.
[22] C. Li and X.-Q. Jin, *Nonlinearly constrained best approximation in Hilbert spaces: The strong CHIP and the basic constraint qualification*, SIAM J. Optim., 13 (2002), pp. 228–239.
[23] C. Li and K. F. Ng, *On best approximation by nonconvex sets and perturbation of nonconvex inequality systems in Hilbert spaces*, SIAM J. Optim., 13 (2002), pp. 726–744.
[24] W. Li, C. Nahak, and I. Singer, *Constraint qualification for semi-infinite systems of convex inequalities*, SIAM J. Optim., 11 (2000), pp. 31–52.
[25] O. Mangasarian, *Nonlinear Programming*, McGraw-Hill, New York, 1969.
[26] C. Micchelli, P. Smith, J. Swetits, and J. Ward, *Constrained L_p-approximation*, Constr. Approx., 1 (1985), pp. 93–102.
[27] C. A. Micchelli and F. I. Utreras, *Smoothing and interpolation in a convex subset of a Hilbert space*, SIAM J. Sci. Statist. Comput., 9 (1988), pp. 728–746.
[28] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Math. 1364, Springer-Verlag, New York, 1989.
[29] G. Sh. Rubenstein, *On an extremal problem in a normed linear space*, Sibirskii Mat. Zh., 6 (1965), pp. 711–714 (in Russian).
[30] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1981.
[31] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1970.
[32] I. Singer, *Duality for optimization and best approximation over finite intersection*, Numer. Funct. Anal. Optim., 19 (1998), pp. 903–915.
[33] G. S. Smirnov and R. G. Smirnov, *Best uniform restricted range approximation of complex-valued functions*, C. R. Math. Rep. Acad. Sci. Canada, 19 (2) (1997), pp. 58–63.
[34] G. S. Smirnov and R. G. Smirnov, *Best uniform approximation of complex-valued functions by generalized polynomials having restricted range*, J. Approx. Theory, 100 (1999), pp. 284–303.
[35] G. S. Smirnov and R. G. Smirnov, *Kolmogorov-type theory of best restricted approximation*, East J. Approx., 6 (3) (2000), pp. 309–329.
[36] G. S. Smirnov and R. G. Smirnov, *Best uniform restricted ranges approximation*. II, C. R. Acad. Sci. Paris Sér. I Math., 330 (2000), pp. 1059–1064.

[37] G. S. Smirnov and R. G. Smirnov, *Theory of best restricted ranges approximation revisited: A characterization theorem*, in Approximation Theory and Its Applications, Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos 31, Natsīonal Akad. Nauk Ukraïni, Īnst. Mat., Kiev, 2000, pp. 436–445.

[38] Y. Yuan and W. Sun, *Optimization Theory and Methods*, Science Press, Beijing, 1997 (in Chinese).

[39] A. C. Zaanen, *Integration*, North-Holland, Amsterdam, 1967.