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STRONG CHIP FOR INFINITE SYSTEM OF CLOSED CONVEX SETS IN NORMED LINEAR SPACES*

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Abstract. For a general (possibly infinite) system of closed convex sets in a normed linear space we provide several sufficient conditions for ensuring the strong conical hull intersection property. One set of sufficient conditions is given in terms of the finite subsystems while the other sets are in terms of the relaxed interior-point conditions together with appropriate continuity of the associated set-valued function on the (topologized) index set I. In the special case when I is finite and X is finite dimensional, one of these results reduces to a classical result of Rockafellar.

Key words. system of closed convex sets, interior-point condition, strong conical hull intersection property

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1. Introduction.

The notion of the strong CHIP (conical hull intersection property) was introduced by Deutsch, Li, and Ward in [12, 13] for a finite family of closed convex sets in a Euclidean space (or a Hilbert space) and has been successfully applied in the reformulation of some best approximation problems. This notion closely relates other fundamental concepts such as bounded linear regularity, G-property of Jameson, error bounds in convex optimization [1, 3], and the BCQ (basic constraint qualification) as well as the perturbations for finite convex systems of inequalities. See [5, 6, 8, 12, 13, 14, 18, 19, 24, 25] and references therein, especially in [20], where the strong CHIP was defined for an arbitrary family of closed convex sets in a Banach space and utilized in the study of general systems of infinite convex inequalities, such as the system that naturally arises from the problem of best restricted range approximation in the space $C(Q)$ of complex-valued continuous functions on a compact metric space Q under quite general constraints. This problem was first presented and formulated by Smirnov and Smirnov in [31, 32], where each Ω_t was assumed to be a disk in $\mathbb{C}$. Later in [33, 34, 35] and also more recently in [17, 20], the constraint sets Ω_t have been relaxed but still remain to assume the strong interior-point condition (in particular, $\operatorname{int}\Omega_t \neq \emptyset$ for each $t \in Q$). This unfortunately excludes the interesting case when some Ω_t is a line segment or a singleton in $\mathbb{C}$. As demonstrated in an accompanying paper [22], the results obtained in the present paper have enabled us to study the restricted range approximation problem under much less restrictive assumptions by allowing the case that $\operatorname{int}\Omega_t = \emptyset$ for some $t \in Q$. The present paper is devoted to providing sufficient conditions for a (finite or infinite) family $\{C, C_i : i \in I\}$ of closed convex sets in a Banach (or normed linear) space to have the strong CHIP.

In expanding and improving the known results on the sufficient conditions for the

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strong CHIP for $\{C, C_i : i \in I\}$ from the case when the index set I is finite to the case when I may be infinite, this paper presents two types of results. One type is on the natural approach to answer the question of whether or not the following implication is valid:

$$\{C, C_j : j \in J\} \text{ has the strong CHIP for each finite subset } J \text{ of } I$$
$$\Longrightarrow \{C, C_i : i \in I\} \text{ has the strong CHIP.}$$

While the answer to this question is negative in general (see [13, Example 1]), we provide some reasonable conditions in section 5 to ensure the validity of the above implication. Another type of sufficient conditions presented in this paper is given more directly (in terms of the system itself rather than via its finite subsystems). In this connection, the starting point of our study is the following theorem. "DLW" refers to the authors Deutsch, Li, and Ward of [12, 13], where the assertions regarding the sufficiency for (a) and for (b) were stated and proved in the Hilbert space setting, but the arguments can be modified to suit the Banach space setting. For the sake of completeness and also for more convenient applications, we will present a direct proof for a slightly more general form in the next section (see also [26] for another approach).

THEOREM DLW. *Let I be a finite index set and $\{C, C_i : i \in I\}$ be a finite family of nonempty closed convex sets in a Banach space X. Let $x_0 \in C \cap (\cap_{i\in I} C_i)$. Then the family $\{C, C_i : i \in I\}$ has the strong CHIP at x_0 provided that at least one of the following conditions is satisfied:*

(a) $C \bigcap (\operatorname{int} \bigcap_{i\in I} C_i) \neq \emptyset$.

(b) $\operatorname{ri} C \bigcap (\bigcap_{i\in I} C_i) \neq \emptyset$ *and each C_i is a polyhedron (where "ri" means "relative interior").*

(c) *There exists a subset I_0 of I such that C_i is a polyhedron for each $i \in I \setminus I_0$ and*

$$\operatorname{ri} C \bigcap \left(\operatorname{int} \bigcap_{i\in I_0} C_i\right) \bigcap \left(\bigcap_{i\in I\setminus I_0} C_i\right) \neq \emptyset. \tag{1.1}$$

The sufficiency of (c) follows directly from (a) and (b). The condition (a) is sometimes referred to as the strong interior-point condition (or Slater condition; see, e.g., [13]) which is equivalent (as I is finite) to the following interior-point condition:

(a′) $C \bigcap (\bigcap_{i\in I} \operatorname{int} C_i) \neq \emptyset$.

As shown in [20], when I is infinite, the above (a), (b), and (c) are no longer sufficient for the strong CHIP. A natural condition that one would like to impose is the continuity assumption for the set-valued mapping $i \mapsto C_i$; thus it is judicious for us to assume henceforth that

$$\text{the set } I \text{ is a compact metric space.} \tag{1.2}$$

(When I is finite, it will be regarded as a compact metric space under the discrete metric; needless to say, in this case the continuity assumption is automatically satisfied.)

Under an appropriate continuity assumption we show in Theorem 4.1 that (a) implies the strong CHIP at $x_0 \in C \cap (\cap_{i\in I} C_i)$ provided that C is finite dimensional or the set $I_C^{\mathrm{rb}}(x_0)$ of "C-relative boundary indices" for x_0 is finite. We remark that even in the case when C is finite dimensional, our results are genuinely an extension of Theorem DLW as some (or all) sets C_i can be infinite dimensional. In a similar fashion other parts of Theorem DLW are extended in section 4. In fact, we use the following condition, somewhat weaker than (c), to establish a sufficient condition result in Theorem 4.3. The family $\{C, C_i : i \in I\}$ is said to satisfy the weak-strong interior-point condition with the pair (I_1, I_2) if there exist disjoint finite subsets I_1, I_2 of I satisfying the following two properties:

$$\operatorname{ri} C \bigcap \left(\operatorname{int} \bigcap_{i \in I \setminus (I_1 \cup I_2)} C_i \right) \bigcap \left(\bigcap_{i \in I_1} \operatorname{ri} C_i \right) \bigcap_{i \in I_2} C_i \neq \emptyset; \tag{1.3}$$

$$C_i \text{ is a polyhedron for each } i \in I_2. \tag{1.4}$$

This condition, in contrast to the interior-point condition, enables us to consider the case when some C_i neither is a polyhedron nor has an interior point. Specializing to the case when $I = I_1 \cup I_2$ (thus $\operatorname{int}(\cap_{i \in I \setminus (I_1 \cup I_2)} C_i)$, to be read as X by convention), a corollary of Theorem 4.3 is the following infinite dimension extension of a result of Rockafellar [27, Corollary 23.8.1, p. 223]:

Let $I = J \cup K$ be finite such that C_k is a polyhedron for each $k \in K$ and suppose that

$$\operatorname{ri} C \bigcap \left(\bigcap_{j \in J} \operatorname{ri} C_j \right) \bigcap \left(\bigcap_{k \in K} C_k \right) \neq \emptyset. \tag{1.5}$$

Then the system $\{C, C_i : i \in I\}$ has the strong CHIP if at least one of the following conditions is satisfied.

(a) At least one of $\{C, C_j : j \in J\}$ is finite dimensional.

(b) C_j is finite codimensional for each $j \in J$.

2. Notations and preliminary results. The notations used in the present paper are standard (cf. [7, 16]). In particular, we assume that X is a normed linear space throughout the whole paper, unless we explicitly state otherwise. We use $\mathbf{B}(x, \epsilon)$ to denote the closed ball with center x and radius ϵ. For a set Z in X (or in $\mathbb{R}^n$), the interior (resp., relative interior, closure, convex hull, convex cone hull, linear hull, affine hull, boundary, relative boundary) of Z is denoted by $\operatorname{int} Z$ (resp., $\operatorname{ri} Z$, $\overline{Z}$, $\operatorname{conv} Z$, $\operatorname{cone} Z$, $\operatorname{span} Z$, $\operatorname{aff} Z$, $\operatorname{bd} Z$, $\operatorname{rb} Z$), and the negative polar cone $Z^\ominus$ is the set defined by

$$Z^\ominus = \{x^* \in X^* \ : \ \langle x^*, z \rangle \leq 0 \quad \text{for all } z \in Z\}.$$

The normal cone of Z at z_0 is denoted by $N_Z(z_0)$ and defined by $N_Z(z_0) = (Z - z_0)^\ominus$. For convenience of printing we sometimes use $N(z_0; Z)$ in place of $N_Z(z_0)$. Let A be a closed convex nonempty subset of X. The interior and boundary of Z relative to A are denoted by $\operatorname{rint}_A Z$ and $\operatorname{bd}_A Z$, respectively; they are defined to be, respectively, the interior and boundary of the set $\operatorname{aff} A \cap Z$ in the metric space $\operatorname{aff} A$. Thus, a point $z \in \operatorname{rint}_A Z$ if and only if there exists $\varepsilon > 0$ such that

$$z \in (\operatorname{aff} A) \cap \mathbf{B}(z, \varepsilon) \subseteq Z, \tag{2.1}$$

while $z \in \operatorname{bd}_A Z$ if and only if $z \in \operatorname{aff} A$ and, for any $\varepsilon > 0$, $(\operatorname{aff} A) \cap \mathbf{B}(z, \varepsilon)$ intersects Z and its complement. Let $\mathbb{R}_-$ denote the subset of $\mathbb{R}$ consisting of all nonpositive real numbers. For a proper extended real-valued convex function on X, the subdifferential of f at $x \in X$ is denoted by $\partial f(x)$ and defined by

$$\partial f(x) = \{z^* \in X^* : \ f(x) + \mathrm{Re}\,\langle z^*, y - x\rangle \le f(y) \quad \text{for all } y \in X\},$$

where $\langle z^*, x\rangle$ denotes the value of a functional z^* in X^* at $x \in X$, i.e., $\langle z^*, x\rangle = z^*(x)$.

For simplicity of notations, we will usually assume (with the exception of Proposition 2.1 and section 5) that the scalar field of X is $\mathbb{R}$ and that $\mathrm{Re}\,\langle x^*, x\rangle$ is to be replaced by $\langle x^*, x\rangle$.

Remark 2.1. (a) Let f be a continuous convex function on X and $x \in X$ with $f(x) = 0$. It is easy to see that $\mathrm{cone}(\partial f(x)) \subseteq N_{f^{-1}(\mathbb{R}_-)}(x)$, and that the equality holds if f is an affine function or if x is not a minimizer of f; see [7, Corollary 1, p. 56].

(b) The directional derivative of the function f at x in the direction d is denoted by $f'_+(x,d)$:

$$f'_+(x,d) := \lim_{t\to 0+} \frac{f(x+td) - f(x)}{t}. \tag{2.2}$$

We recall [7, Proposition 2.2.7] that, if x is a continuity point of f,

$$\partial f(x) = \{z^* \in X^* : \ \langle z^*, d\rangle \le f'_+(x,d) \quad \text{for all } d \in X\} \tag{2.3}$$

and

$$f'_+(x,d) = \max\{\langle z^*, d\rangle : \ z^* \in \partial f(x)\}. \tag{2.4}$$

Let $\{A_i : \ i \in J\}$ be a family of subsets of X. The set $\sum_{i\in J} A_i$ is defined by

$$\sum_{i\in J} A_i = \begin{cases} \left\{\sum_{i\in J_0} a_i : \quad a_i \in A_i, \quad J_0 \subseteq J \text{ being finite}\right\} & \text{if } J \ne \emptyset, \\ \{0\} & \text{if } J = \emptyset. \end{cases} \tag{2.5}$$

The following concept of the strong CHIP plays an important role in optimization theory (see [1, 3, 8, 10, 11, 30]) and is due to [12, 13] in the case when I is finite and to [20] in the case when I is infinite.

Definition 2.1. *Let $\{C_i : \ i \in I\}$ be a collection of convex subsets of X and $x \in \bigcap_{i\in I} C_i$. The collection is said to have*

(a) *the strong CHIP at x if $N_{\bigcap_{i\in I} C_i}(x) = \sum_{i\in I} N_{C_i}(x)$, that is,*

$$\left(\bigcap_{i\in I} C_i - x\right)^{\ominus} = \sum_{i\in I} (C_i - x)^{\ominus}; \tag{2.6}$$

(b) *the strong CHIP if it has the strong CHIP at each point of $\cap_{i\in I} C_i$.*

Consider a convex inequality system (*CIS*) defined by

$$g_i(x) \le 0, \quad i \in I, \tag{2.7}$$

where $x \in X$ and each g_i is a real continuous convex function on X. We always assume that the solution set S of the system (*CIS*) is nonempty, i.e.,

$$S := \{x \in X : \ g_i(x) \le 0 \quad \text{for all } i \in I\} \ne \emptyset. \tag{2.8}$$

Let $G(\cdot)$ denote the sup-function [16] of $\{g_i\}$:

$$G(x) := \sup_{i \in I} g_i(x) \quad \text{for all } x \in X.$$

Then S is also the solution set of the convex inequality

$$G(x) \leq 0. \tag{2.9}$$

In this paper we assume throughout that

$$G(x) < +\infty \quad \text{for all } x \in X \tag{2.10}$$

and that G is continuous on X. These blanket assumptions are automatically satisfied if $\{g_i : \ i \in I\}$ is locally uniformly bounded. Moreover, the continuity of G automatically follows from (2.10) if X is finite dimensional.

Let $I(x)$ denote the set of all active indices i: $I(x) = \{i \in I : \ g_i(x) = G(x)\}$. Following [15, 23], we define

$$D'(x) := \operatorname{conv} \bigcup_{i \in I(x)} \partial g_i(x), \quad x \in X. \tag{2.11}$$

Note that, by (2.5), $D'(x) = \{0\}$ if $I(x) = \emptyset$.

The following theorem will play a key role in section 4. It is a known result; see, for example, [16, 23] for the special case when X is finite dimensional and [21] for the general case (the proof presented in [21] is valid for normed linear spaces though the result was stated in the Banach space setting).

THEOREM 2.1. *Suppose that I is a compact metric space and that the function $i \mapsto g_i(x)$ is upper semicontinuous for each $x \in X$. Let $x_0 \in C$. Then $I(x_0) \neq \emptyset$ and the following assertions hold.*

(i) *If* $\operatorname{span} C$ *is finite dimensional, then*

$$N_C(x_0) + \partial G(x_0) = N_C(x_0) + D'(x_0). \tag{2.12}$$

(ii) $\partial G(x_0) = D'(x_0)$ *provided that $I(x_0)$ is finite.*

Theorem 2.2 below is a slight extension (applicable to convex, but not necessarily closed, sets in a normed space). To prepare for the proof we begin with a simple lemma.

LEMMA 2.1. *Assume that C is a polyhedron in X defined by*

$$C = \bigcap_{i=1}^{k} \{x \in X : \ \langle h_i, x \rangle \leq d_i\}, \tag{2.13}$$

where $h_i \in X^* \setminus \{0\}$ and d_i is a real number for each $i = 1, \ldots, k$. Let $x_0 \in \operatorname{bd} C$ and let $I(x_0) = \{i \in \{1, \ldots, k\} : \ \langle h_i, x_0 \rangle = d_i\}$. Then

$$N_C(x_0) = \operatorname{cone}\{h_i : \ i \in I(x_0)\}. \tag{2.14}$$

Consequently, $\{C_i : \ i = 1, 2, \ldots, n\}$ has the strong CHIP if each C_i is a polyhedron of X.

Proof. We need only prove that the set on the left-hand side of (2.14) is contained in that on the right-hand side. To do this, suppose on the contrary that

$y^* \in N_C(x_0) \setminus \text{cone}\{h_i : i \in I(x_0)\}$. Since $\text{cone}\{h_i : i \in I(x_0)\}$ is closed as I is finite, by the separation theorem, there exists an element $x^{**} \in X^{**}$ such that

$$\langle x^{**}, y^* \rangle > 0 \geq \sup\{\langle x^{**}, h \rangle : h \in \text{cone}\{h_i : i \in I(x_0)\}\}. \tag{2.15}$$

Moreover, as $I(x_0)$ is a finite set, there exists $x \in X$ such that

$$\langle y^*, x \rangle = \langle x^{**}, y^* \rangle \quad \text{and} \quad \langle h_i, x \rangle = \langle x^{**}, h_i \rangle \quad \text{for each } i \in I(x_0). \tag{2.16}$$

Hence, by (2.15) and (2.16), we have that $tx + x_0 \in C$ for some $t > 0$ small enough. But $\langle y^*, (tx + x_0) - x_0 \rangle = t\langle y^*, x \rangle > 0$, which contradicts that $y^* \in N_C(x_0)$, and the lemma is proved. ☐

Let Y be a subspace of X. We use N^Y to represent the normal cone operator taken in Y; namely, for any subset A of Y, $N_A^Y(x)$ is the set

$$N_A^Y(x) = \{y^* \in Y^* : \langle y^*, z - x \rangle \leq 0 \quad \text{for all } z \in A\}. \tag{2.17}$$

Corollary 2.1. *Let $Z \subseteq X$ be a closed subspace and $C \subseteq X$ a polyhedron. Let $x_0 \in C \cap Z$ and let $N_{C\cap Z}^Z(x_0)$ denote the normal cone of $C \cap Z$ at x_0 taken in Z. Then, for each $x_0^* \in N_{C\cap Z}^Z(x_0)$, there exists $x^* \in N_C(x_0)$ such that x^* is an extension of x_0^* (an extension of x_0^* obtained from Lemma* 2.1 *will be referred to as a natural extension of x_0^*).*

Theorem 2.2. *Let I be a finite index set and $\{C, C_i : i \in I\}$ be a finite family of nonempty convex sets in a normed linear space X. Let $x_0 \in C \cap (\cap_{i\in I} C_i)$. Then the family $\{C, C_i : i \in I\}$ has the strong CHIP at x_0 provided that at least one of conditions* (a)*,* (b)*, and* (c) *in Theorem DLW is satisfied.*

Proof. In the case (a), if X is a Banach space and $\{C, C_i : i \in I\}$ is a family of nonempty closed convex sets, the proof is the same as that given in [13], except that here we apply [2, Theorem 2.6, p. 189] instead of [2, Corollary 2.5, p. 113]. Note further that the result is valid for any normed linear space X and any family $\{C, C_i : i \in I\}$ of nonempty convex sets. However, this observation does not constitute a genuine extension. Indeed, let U denote the open unit ball of $\overline{X}$. Let $\overline{C}$ and $\overline{C}_i$, respectively, denote the closures of C and C_i in $\overline{X}$. By (a), there exist $c \in C$ and $\epsilon > 0$ such that

$$(c + \epsilon U) \cap X \subseteq C_i \quad \text{for each } i \in I. \tag{2.18}$$

We claim that

$$c + \epsilon U \subseteq \overline{C}_i \quad \text{for each } i \in I. \tag{2.19}$$

Indeed, let $u \in U$. Then there exists a sequence $\{x_n\} \subseteq U \cap X$ convergent to u. Then by (2.18) $\{c + \epsilon x_n\} \subseteq C_i$, which implies that $c + \epsilon u \in \overline{C}_i$. Thus (2.19) is clear. Let $x_0 \in C \bigcap (\bigcap_{i\in I} C_i)$. Then one can apply the Banach space version of (a) in Theorem 2.2 to conclude that

$$N_{\overline{C}\cap(\cap_{i\in I}\overline{C}_i)}(x_0) \subseteq N_{\overline{C}}(x_0) + \sum_{i\in I} N_{\overline{C}_i}(x_0)$$

and hence

$$N_{C\cap(\cap_{i\in I} C_i)}(x_0) = N_{\overline{C\cap(\cap_{i\in I} C_i)}}(x_0) \subseteq N_C(x_0) + \sum_{i\in I} N_{C_i}(x_0)$$

because $\overline{C \cap (\cap_{i\in I} C_i)} = \overline{C} \cap (\cap_{i\in I} \overline{C}_i)$ thanks to (2.19). This completes the proof of (a).

Now let us verify the conclusion in the case (b). Let $x_0 \in C \bigcap (\bigcap_{i\in I} C_i)$ and let Z denote the subspace spanned by $C - x_0$. Since $\mathrm{ri}\, C \bigcap (\bigcap_{i\in I} C_i) \neq \emptyset$, the intersection of the interior of $C - x_0$ in the subspace Z and the set $\bigcap_{i\in I}(C_i - x_0)$ is nonempty. By the case (a) and Lemma 2.1 (applied to Z in place of X), we obtain that

$$
\begin{aligned}
N^Z_{(C-x_0)\cap(\cap_{i\in I}(C_i-x_0))}(0) &= N^Z_{C-x_0}(0) + N^Z_{Z\cap(\cap_{i\in I}(C_i-x_0))}(0) \\
&= N^Z_{C-x_0}(0) + \sum_{i\in I} N^Z_{Z\cap(C_i-x_0)}(0).
\end{aligned} \tag{2.20}
$$

Let $x^* \in N_{C\cap(\cap C_i)}(x_0)$. Then $x^*|_Z \in N^Z_{(C-x_0)\cap(\cap_{i\in I}(C_i-x_0))}(0)$; consequently, by (2.20), there exist $\widetilde{x}^*_0 \in N^Z_{C-x_0}(0)$ and $\widetilde{x}^*_i \in N^Z_{Z\cap(C_i-x_0)}(0)$ for each $i \in I$ such that

$$
x^*|_Z = \widetilde{x}^*_0 + \sum_{i\in I} \widetilde{x}^*_i \quad \text{on } Z. \tag{2.21}
$$

Let $x^*_0 \in X^*$ be an extension of $\widetilde{x}^*_0$. Then, as $C - x_0 \subseteq Z$, $x^*_0 \in N_{C-x_0}(0) = N_C(x_0)$. Also, for each $i \in I$, as $C_i - x_0$ is a polyhedron in X, there exists a natural extension $x^*_i \in N_{C_i-x_0}(0)$ of $\widetilde{x}^*_i$ by Corollary 2.1, and hence $x^*_i \in N_{C_i}(x_0)$ with $x^*_i|_Z = \widetilde{x}^*_i$ for each $i \in I$. Do this for each $i \in I$ and let $y^* = x^* - x^*_0 - \sum_{i\in I} x^*_i$. Then $y^* \in N_C(x_0)$ by (2.21). Hence, $x^* = y^* + x^*_0 + \sum_{i\in I} x^*_i \in N_C(x_0) + \sum_{i\in I} N_{C_i}(x_0)$ and the conclusion in the case (b) is proved. Therefore the proof of Theorem 2.2 is complete, as (c) follows from (a) and (b). ∎

For a closed convex subset W of X, let P_W denote the projection operator defined by

$$
P_W(x) = \{y \in W : \|x - y\| = d_W(x)\},
$$

where $d_W(x)$ denotes the distance from x to W. Recall that the duality map J from X to 2^{X^*} is defined by

$$
J(x) := \{x^* \in X^* : \ \langle x^*, x\rangle = \|x\|^2, \ \|x^*\| = \|x\|\}. \tag{2.22}
$$

In fact, $J(x) = \partial\phi(x)$, where $\phi(x) := \frac{1}{2}\|x\|^2$. Thus a Banach space X is smooth if and only if for each $x \in X$ the duality map is single-valued. We also need the following proposition, which was established independently by Deutsch [9] and Rubenstein [28] (see also [4]).

PROPOSITION 2.1. *Let W be a convex set in X. Then for any $x \in X$, $z_0 \in P_W(x)$ if and only if $z_0 \in W$ and there exists $x^* \in J(x - z_0)$ such that* $\mathrm{Re}\,\langle x^*, z - z_0\rangle \le 0$ *for any $z \in W$, that is, $J(x - z_0) \cap N_W(z_0) \neq \emptyset$. In particular, when X is smooth, $z_0 \in P_W(x)$ if and only if $z_0 \in W$ and $J(x - z_0) \in N_W(z_0)$.*

3. Extended Minkowski functional, interior-point condition, and continuity condition. Recall that I denotes an index set which is assumed to be a compact metric space. For convenience, a family $\{C, C_i : \ i \in I\}$ is called a closed convex set system with base-set C (CCS-system with base-set C) if C and C_i are nonempty closed convex subsets of X for each $i \in I$.

DEFINITION 3.1. *A CCS-system $\{C, C_i : \ i \in I\}$ with base-set C is said to satisfy*

(i) *the C-interior-point condition if*

$$
C \bigcap \left(\bigcap_{i\in I} \mathrm{rint}_C\, C_i\right) \neq \emptyset; \tag{3.1}
$$

(ii) *the strong C-interior-point condition if*

$$C \bigcap \left(\operatorname{rint}_C \bigcap_{i \in I} C_i \right) \neq \emptyset; \tag{3.2}$$

(iii) *the weak-strong C-interior-point condition with the pair (I_1, I_2) if there exist two disjoint finite subsets I_1 and I_2 of I such that each C_i $(i \in I_2)$ is a polyhedron and*

$$\operatorname{ri} C \bigcap \left(\operatorname{rint}_C \bigcap_{i \in I \setminus (I_1 \cup I_2)} C_i \right) \bigcap \left(\bigcap_{i \in I_1} \operatorname{ri} C_i \right) \bigcap_{i \in I_2} C_i \neq \emptyset; \tag{3.3}$$

(iv) *the interior-point condition (resp., the strong interior-point condition, the weak-strong interior-point condition with the pair (I_1, I_2)) if the operation "rint_C" in* (3.1) *(resp.,* (3.2), (3.3)*) is replaced with "*int*".*

Any point $\bar{x}$ belonging to the set on the left-hand side of (3.1) *(resp.,* (3.2), (3.3)*) is called a C-interior point (resp., a strong C-interior point, a weak-strong C-interior point with the pair (I_1, I_2)) of the CCS-system $\{C, C_i : i \in I\}$. Similarly, the notion of an interior point (resp., a strong interior point, a weak-strong interior point with the pair (I_1, I_2)) of the CCS-system $\{C, C_i : i \in I\}$ is defined.*

It is trivial that (3.2) $\Longrightarrow$ (3.1). The converse also holds in some cases, one of which will be described in terms of continuity of some set-valued functions. For set-valued functions there are many different notions of continuity. In Definitions 3.2 and 3.3 below, we recall two frequently used ones. We assume that Q is a compact metric space.

Definition 3.2. *Let Y be a normed linear space. Then the set-valued function $F : Q \to 2^Y \setminus \{\emptyset\}$ is said to be*

(i) *lower semicontinuous at $t_0 \in Q$ if, for any $y_0 \in F(t_0)$ and any $\epsilon > 0$, there exists an open neighborhood $U(t_0)$ of t_0 such that for each $t \in U(t_0)$, $\mathbf{B}(y_0, \epsilon) \cap F(t) \neq \emptyset$;*

(ii) *locally uniform lower semicontinuous at $t_0 \in Q$ if, for any $y_0 \in F(t_0)$, there exists an open neighborhood $V(y_0)$ of y_0 such that for any $\epsilon > 0$, there exists an open neighborhood $U(t_0)$ of t_0 such that $\mathbf{B}(y, \epsilon) \cap F(t) \neq \emptyset$ holds for each $t \in U(t_0)$ and each $y \in V(y_0) \cap F(t_0)$;*

(iii) *upper semicontinuous at $t_0 \in Q$ if, for any open neighborhood V of $F(t_0)$, there exists an open neighborhood $U(t_0)$ of t_0 such that $F(t) \subseteq V$ for each $t \in U(t_0)$;*

(iv) *lower semicontinuous (resp., locally uniform lower semicontinuous, upper semicontinuous) on Q if it is lower semicontinuous (resp., locally uniform lower semicontinuous, upper semicontinuous) at each $t \in Q$.*

Definition 3.3 (cf. [29, p. 55]). *Let $F : Q \to 2^Y$ be a set-valued function defined on Q and let $t_0 \in Q$. Then F is said to be*

(i) *upper Kuratowski semicontinuous at t_0 if, for any sequence $\{t_k\} \subseteq Q$, the relations $\lim_{k\to\infty} t_k = t_0$, $\lim_{k\to\infty} x_{t_k} = x_{t_0}$, $x_{t_k} \in F(t_k)$, $k = 1, 2, \ldots$, imply $x_{t_0} \in F(t_0)$;*

(ii) *lower Kuratowski semicontinuous at t_0 if, for any sequence $\{t_k\} \subseteq Q$, the relations $\lim_{k\to\infty} t_k = t_0$, $y_0 \in F(t_0)$ imply $\lim_{k\to\infty} d_{F(t_k)}(y_0) = 0$;*

(iii) *Kuratowski continuous at t_0 if F is both upper Kuratowski semicontinuous and lower Kuratowski semicontinuous at t_0;*

(iv) *Kuratowski continuous on Q if it is Kuratowski continuous at each point of Q.*

Remark 3.1. Clearly,

(i) F is upper semicontinuous $\Longrightarrow$ F is upper Kuratowski semicontinuous,

(ii) F is lower semicontinuous $\Longleftrightarrow$ F is lower Kuratowski semicontinuous.

Moreover, the converse of (i) holds provided that the union set $\cup_{t\in Q} F(t)$ is compact.

The following two propositions provide some useful reformulations regarding various lower semicontinuities. Since the proofs are similar, we shall only prove the first proposition.

PROPOSITION 3.1. *Let $F : Q \to 2^Y \setminus \{\emptyset\}$ be a set-valued function. Let $t_0 \in Q$. Then the following statements are equivalent.*

(i) *F is lower semicontinuous at t_0.*

(ii) *For any $y_0 \in F(t_0)$, there exists $y_t \in F(t)$ for each $t \in Q$ such that $\lim_{t\to t_0} \|y_t - y_0\| = 0$.*

(iii) *For any $y_0 \in F(t_0)$, $\lim_{t\to t_0} d_{F(t)}(y_0) = 0$.*

Proof. (i) $\Longrightarrow$ (ii). Let $y_0 \in F(t_0)$. Then, by (i), for each positive k there exists an open neighborhood $U_k(t_0)$ of t_0 such that

$$\mathbf{B}\left(y_0, \frac{1}{k}\right) \cap F(t) \neq \emptyset \quad \text{for each } t \in U_k(t_0). \tag{3.4}$$

Without loss of generality, we may assume that $U_{k+1}(t_0) \subseteq U_k(t_0)$ for each k and $\bigcap_{k\geq 1} U_k(t_0) = \{t_0\}$ because Q is a metric space. Now we construct $y_t \in F(t)$ for each $t \in Q$ as follows:

$$\begin{array}{ll} y_t \in F(t) & \text{if } t \in Q \setminus U_1(t_0), \\ y_t \in \mathbf{B}\left(y_0, \frac{1}{k}\right) \cap F(t) & \text{if } t \in U_k(t_0) \setminus U_{k+1}(t_0), \quad k = 1, 2, \ldots, \\ y_0 & \text{if } t = t_0. \end{array} \tag{3.5}$$

Then $\lim_{t\to t_0} y_t = y_0$.

(ii) $\Longrightarrow$ (iii). It is trivial.

(iii) $\Longrightarrow$ (i). Let $y_0 \in F(x_0)$ and $\epsilon > 0$. By (iii), there exists an open neighborhood $U(t_0)$ of t_0 such that for each $t \in U(t_0)$, one has that $d_{F(t)}(y_0) < \epsilon$ and thus $\mathbf{B}(y_0, \epsilon) \cap F(t) \neq \emptyset$. Therefore (i) holds. The proof is complete. □

The following proposition can be proved similarly.

PROPOSITION 3.2. *Let $F : Q \to 2^Y \setminus \{\emptyset\}$ be a set-valued function. Let $t_0 \in Q$. Then the following statements are equivalent.*

(i) *F is locally uniform lower semicontinuous at t_0.*

(ii) *For any $y_0 \in F(t_0)$, there exists an open neighborhood $V(y_0)$ of y_0 such that for any $y \in V(y_0) \cap F(t_0)$, there exists $z_t(y) \in F(t)$ for each $t \in Q$ such that $\lim_{t\to t_0} \|z_t(y) - y\| = 0$ holds uniformly on $V(y_0) \cap F(t_0)$.*

(iii) *For any $y_0 \in F(t_0)$, there exists an open neighborhood $V(y_0)$ of y_0 such that*

$$\lim_{t\to t_0} \sup_{y\in V(y_0)\cap F(t_0)} d_{F(t)}(y) = 0.$$

(iv) *For any $y_0 \in F(t_0)$, there exists a neighborhood $V(y_0)$ of y_0 such that for any $\epsilon > 0$, there exists a neighborhood $U(t_0)$ of t_0 satisfying*

$$V(y_0) \cap F(t_0) \subseteq \bigcap_{t\in U(t_0)} F(t)^{\varepsilon},$$

where A^{ε} is defined by

$$A^{\varepsilon} = \{y \in Y : \ d_A(y) < \varepsilon\}.$$

Recalling our blanket assumption (1.2), and the definition of CCS-systems made at the beginning of this section, we state our first main result of this section.

THEOREM 3.1. *Let $\{C, C_i : i \in I\}$ be a CCS-system with base-set C, and let $\bar{x} \in C$. Suppose that the set-valued function $i \mapsto (\operatorname{aff} C) \cap C_i$ is locally uniform lower semicontinuous on I. Then $\bar{x}$ is a C-interior point of the system if and only if it is a strong C-interior point of the system.*

Proof. We need to prove only the necessity part. Assume without loss of generality that

$$0 = \bar{x} \in C \bigcap \left(\bigcap_{i \in I} \operatorname{rint}_C C_i \right). \tag{3.6}$$

Then $Z := \operatorname{aff} C$ is a vector subspace of X. It suffices to show that

$$0 \in C \bigcap \left(\operatorname{rint}_C \bigcap_{i \in I} C_i \right). \tag{3.7}$$

Clearly, we need only to show that

$$\inf_{i \in I} d_{\operatorname{bd}_Z C_i}(0) > 0. \tag{3.8}$$

(Indeed, if $d_{\operatorname{bd}_Z C_i}(0) > \gamma$, then $Z \cap \mathbf{B}(0, \gamma) \subseteq C_i$.) Suppose on the contrary that (3.8) does not hold. Then, by the compactness of I, there exist a convergent sequence $(i_n) \subseteq I$ (say with limit $i_0 \in I$) and a sequence (y_{i_n}) with $y_{i_n} \in \operatorname{bd}_Z C_{i_n}$ for each n such that $\lim_n \|y_{i_n}\| = 0$. Write

$$\widehat{C}_i = Z \cap C_i, \quad i \in I. \tag{3.9}$$

By assumptions, $i \mapsto \widehat{C}_i$ is locally uniform lower semicontinuous at i_0. By (iv) of Proposition 3.2 (applied to $0, i_0$ in place of y_0, t_0), there exists a $\delta \in (0, 1)$ such that for any $\varepsilon > 0$ there exists a neighborhood $U_\varepsilon(i_0)$ of i_0 such that

$$\mathbf{B}(0, \delta) \cap \widehat{C}_{i_0} \subseteq \bigcap_{i \in U_\varepsilon(i_0)} \widehat{C}_i^\varepsilon. \tag{3.10}$$

In view of (3.6), we may assume in addition that

$$\mathbf{B}(0, \delta) \cap Z \subseteq \widehat{C}_{i_0} \tag{3.11}$$

(take a smaller $\delta > 0$ if necessary). Combining the above two inclusions, we have

$$\mathbf{B}(0, \delta) \cap Z \subseteq \bigcap_{i \in U_\varepsilon(i_0)} \widehat{C}_i^\varepsilon. \tag{3.12}$$

Let us fix an $\varepsilon \in (0, \frac{\delta}{3})$ and take $\alpha > 0$ such that $\frac{3}{2}\varepsilon < \alpha < \frac{\delta}{2}$ (hence $\alpha < \frac{1}{2}$ as $\delta < 1$). We fix a natural number n which is large enough so that

$$\|y_{i_n}\| < \frac{\delta}{2} \quad \text{and} \quad i_n \in U_\varepsilon(i_0). \tag{3.13}$$

For simplicity of notations, we henceforth write i for the i_n with the above n. Since y_i is a (relative) boundary point of $C_i \cap Z = \widehat{C}_i$ in the vector subspace Z of X and since

$\widehat{C}_i$ has a nonempty relative interior (containing the origin) by (3.6), the separation theorem implies that $N_{\widehat{C}_i}(y_i)|_Z \neq \{0\}$. Hence, by the Hahn–Banach theorem, there exists $x_i^* \in N_{\widehat{C}_i}(y_i)$ such that $\|x_i^*|_Z\|$ is of norm 1. Take $x_i^\varepsilon \in Z$ such that $\|x_i^\varepsilon\| = 1$ and $\langle x_i^*, x_i^\varepsilon \rangle \geq 1 - \frac{\varepsilon}{2}$. Define $z_i := y_i + \alpha x_i^\varepsilon$. Then $z_i \in Z$,

$$\|(y_i + x_i^\varepsilon) - z_i\| = \|(1-\alpha)x_i^\varepsilon\| = 1 - \alpha,$$

and it follows from the triangle inequality that, for any $y \in \widehat{C}_i$,

$$\begin{aligned}\|z_i - y\| &\geq \|(y_i + x_i^\varepsilon) - y\| - (1-\alpha)\\ &\geq \langle x_i^*, y_i - y\rangle + \langle x_i^*, x_i^\varepsilon\rangle - (1-\alpha)\\ &\geq \langle x_i^*, x_i^\varepsilon\rangle - (1-\alpha)\\ &\geq (1 - \tfrac{\varepsilon}{2}) - (1-\alpha)\\ &= \alpha - \tfrac{\varepsilon}{2} > \varepsilon.\end{aligned}$$

Therefore $z_i \notin \widehat{C}_i^\varepsilon$. This contradicts (3.12) and (3.13) because

$$\|z_i\| \leq \|z_i - y_i\| + \|y_i\| = \|\alpha x_i^\varepsilon\| + \|y_i\| < \alpha + \frac{\delta}{2} \leq \delta.$$

Thus (3.8) must hold and the proof is complete. ◻

Let A and $\widehat{C}$ be two closed convex subsets of X with $0 \in \operatorname{rint}_{\widehat{C}} A$. In the following we will show that A admits a "$\widehat{C}$-extended Minkowski functional" p_A in the sense that p_A is a continuous sublinear functional on X such that its restriction $p_A|_{\operatorname{aff}\widehat{C}}$ equals the Minkowski functional of $A \cap \operatorname{aff}\widehat{C}$ in the vector subspace $\operatorname{aff}\widehat{C}$ of X. Note that in this case one has, for each $z \in \operatorname{aff}\widehat{C}$,

$$p_A(z) \leq 1 \iff z \in A, \tag{3.14}$$

$$p_A(z) = 1 \iff z \in \operatorname{bd}_{\widehat{C}} A. \tag{3.15}$$

Lemma 3.1. *Suppose that* $0 \in \operatorname{rint}_{\widehat{C}} A$*; that is,*

$$0 \in \mathbf{B}(0,\alpha) \cap \operatorname{aff}\widehat{C} \subseteq A \tag{3.16}$$

for some $\alpha > 0$*, where* A *and* $\widehat{C}$ *are closed convex subsets of* X*. Denote the closure of* $\operatorname{aff}\widehat{C}$ $(= \operatorname{span}\widehat{C})$ *by* Z *and let* $\widetilde{A}$ *denote the closed convex hull of the set* $(A \cap Z) \cup (\mathbf{B}(0,\alpha))$*. Then*

(i) $\widetilde{A}$ *is a closed convex set in* X *with nonempty interior such that*

$$\widetilde{A} \cap Z = A \cap Z \quad \text{and} \quad 0 \in \operatorname{int}\widetilde{A}. \tag{3.17}$$

(ii) *The corresponding Minkowski functional* $q_{\widetilde{A}}$ *(in the usual sense) on* X *has the properties*

$$q_{\widetilde{A}}(x) \leq \frac{1}{\alpha}\|x\| \quad \text{for each } x \in X \tag{3.18}$$

and

$$q_{\widetilde{A}}(x) = \inf\{\lambda \geq 0 : \ x \in \lambda(A \cap Z)\} \quad \text{for each } x \in Z \tag{3.19}$$

(that is, $q_{\widetilde{A}}$ *is a* $\widehat{C}$*-extended Minkowski functional of* A*).*

Proof. Let D denote the convex hull of the set $(A \cap Z) \cup (\mathbf{B}(0,\alpha))$. Then $\widetilde{A} = \overline{D}$ and $\mathbf{B}(0,\alpha) \subseteq D$. Hence, by elementary functional analysis, the Minkowski functional of $\widetilde{A}$ coincides with that of D and (3.18) holds. Hence, to prove (3.19) it suffices to show that

$$D \cap Z = A \cap Z. \tag{3.20}$$

Let $x = \lambda_1 a + \lambda_2 b \in D \cap Z$ with $a \in A \cap Z$, $b \in \mathbf{B}(0,\alpha)$, and $\lambda_1, \lambda_2 \in (0,1)$ such that $\lambda_1 + \lambda_2 = 1$. Then $b \in Z$. We claim that $b \in A$. In fact, since $b \in Z$, there exists a sequence $\{b_k\} \subset \operatorname{aff} \widehat{C}$ such that $b_k \to b$. If $b \in \operatorname{int} \mathbf{B}(0,\alpha)$, then $b_k \in \mathbf{B}(0,\alpha)$ for all k large enough. This implies that $b_k \in \mathbf{B}(0,\alpha) \cap \operatorname{aff} \widehat{C}$ for all such k. Therefore, by (3.16), $b_k \in A$ and hence $b \in A$. Thus assume that $b \in \operatorname{bd} \mathbf{B}(0,\alpha)$. Define $\tilde{b}_k = \frac{\alpha b_k}{\|b_k\|}$ for each k. Then $\tilde{b}_k \in \mathbf{B}(0,\alpha) \cap \operatorname{aff} \widehat{C}$ and $\tilde{b}_k \to b$ for each k. Hence $b \in A$ by (3.16). Therefore our claim stands and $x \in A$ as A is convex. This shows that (3.20), and hence (3.19), are true. To verify (3.17), let $z \in \widetilde{A} \cap Z$. Then $q_{\widetilde{A}}(z) \leq 1$ and one can apply (3.19) to conclude that $z \in A$ because A is closed and $z \in Z$. Thus (3.17) is seen to be true and the proof is complete. □

Note. The set $\widetilde{A}$ will be referred to as a $\widehat{C}$-Minkowski extension of A (though it also depends on α in (3.16)).

For the remainder of this paper, $\{C, C_i : \ i \in I\}$ denotes a CCS-system with base-set C as defined at the beginning of this section. Now we state the second main result of this section.

THEOREM 3.2. *Let $\bar{x} \in C \cap (\cap_{i\in I} C_i)$ and let $\widehat{C} := C - \bar{x}$, $\widehat{C}_i := C_i - \bar{x}$ for each $i \in I$. Then the following statements are equivalent.*

(i) *$\bar{x}$ is a strong C-interior point of the CCS-system $\{C, C_i : \ i \in I\}$, namely,*

$$\bar{x} \in C \cap \operatorname{rint}_C \left(\cap_{i\in I} C_i\right). \tag{3.21}$$

(ii) *For each $i \in I$, there exists a $\widehat{C}$-extended Minkowski functional $p_{\widehat{C}_i}$ of the set $\widehat{C}_i$ such that the sup-function $P(\cdot)$ of $\{p_{\widehat{C}_i}(\cdot)\}$ defined by*

$$P(x) := \sup_{i\in I} p_{\widehat{C}_i}(x), \quad x \in X \tag{3.22}$$

is continuous on X.

Moreover, if we add an additional assumption that the set-valued map $i \mapsto (\operatorname{aff} C) \cap C_i$ is lower semicontinuous, then (ii) *above can be replaced by a stronger one, as follows:*

$\widetilde{\text{(ii)}}$ (ii) *holds and $i \mapsto p_{\widehat{C}_i}(x)$ is upper semicontinuous for each $x \in X$.*

Proof. (ii) $\Longrightarrow$ (i). Let $Z := \operatorname{aff} \widehat{C} = \operatorname{span} \widehat{C}$. By (3.14), $Z \cap \widehat{C}_i = \{z \in Z : p_{\widehat{C}_i}(z) \leq 1\}$ and hence

$$Z \bigcap \left(\bigcap_{i\in I} \widehat{C}_i \right) = \{z \in Z : \ P(z) \leq 1\}.$$

By the continuity assumption on P, it follows that $0 \in \operatorname{rint}_Z(\cap_{i\in I} \widehat{C}_i)$ and hence $\bar{x} \in C \cap \operatorname{rint}_C(\cap_{i\in I} C_i)$. Thus (3.21) is seen to hold.

(i) $\Longrightarrow$ (ii). By (3.21), there exists $\alpha > 0$ such that

$$\mathbf{B}(0,\alpha) \cap Z \subseteq \widehat{C}_i \quad \text{for each } i \in I. \tag{3.23}$$

Then, by Lemma 3.1, there exists a $\widehat{C}$-extended Minkowski functional $p_{\widehat{C}_i}$ of $\widehat{C}_i$ such that $p_{\widehat{C}_i}$ is the Minkowski functional of the closed convex hull of $(\widehat{C}_i \cap Z) \cup (\mathbf{B}(0,\alpha))$ and, in particular,

$$p_{\widehat{C}_i}(x) \le \frac{1}{\alpha}\|x\| \quad \text{for each } x \in X. \tag{3.24}$$

Hence, by definition of P,

$$P(x) \le \frac{1}{\alpha}\|x\| \quad \text{for each } x \in X, \tag{3.25}$$

and thus P is continuous by an elementary argument. This establishes the implication (i) $\Longrightarrow$ (ii).

For the remainder of the proof we assume, in addition, that the set-valued map $i \mapsto (\operatorname{aff} C) \cap C_i$ is lower semicontinuous and hence the set-valued map $i \mapsto Z \cap \widehat{C}_i$ is lower semicontinuous. Then, to prove (i) $\Longrightarrow$ $\widetilde{\text{(ii)}}$ it remains to show for any $i_0 \in I$ and any $x \in X$ that

$$\limsup_{i \to i_0} p_{\widehat{C}_i}(x) \le p_{\widehat{C}_{i_0}}(x). \tag{3.26}$$

Suppose not. Then there exist $i_0 \in I$ and $x \in X$ such that

$$\limsup_{i \to i_0} p_{\widehat{C}_i}(x) > 1 > p_{\widehat{C}_{i_0}}(x). \tag{3.27}$$

Then $x \in \operatorname{co}((\widehat{C}_{i_0} \cap Z) \cup (\mathbf{B}(0,\alpha)))$ and so $x = \lambda_1 b + \lambda_2 z$ for some $b \in \mathbf{B}(0,\alpha)$, $z \in \widehat{C}_{i_0} \cap Z$, and some $\lambda_1, \lambda_2 \in [0,1]$ with $\lambda_1 + \lambda_2 = 1$. Since the set-valued function $i \mapsto \widehat{C}_i \cap Z$ is lower semicontinuous at i_0, there exists $z_i \in \widehat{C}_i \cap Z$ for each $i \in I$ such that $z_i \to z$ as $i \to i_0$. Define $x_i = \lambda_1 b + \lambda_2 z_i$ for each $i \in I$. Then $x_i \in \operatorname{co}((\widehat{C}_i \cap Z) \cup (\mathbf{B}(0,\alpha)))$ and thus $p_{\widehat{C}_i}(x_i) \le 1$. Consequently, it follows from (3.25) that

$$p_{\widehat{C}_i}(x) \le p_{\widehat{C}_i}(x - x_i) + p_{\widehat{C}_i}(x_i) \le \frac{1}{\alpha}\|x - x_i\| + 1$$

and hence that $\limsup_{i \to i_0} p_{\widehat{C}_i}(x) \le 1$. This contradicts (3.27). Therefore (3.26) must hold for each $i_0 \in I$ and $x \in X$. □

The following proposition deals with a special case in Theorem 3.2 by deleting the words "relative" and "$\widehat{C}$-extended," respectively, from (i) and (ii). We will omit the proof as it is similar to that of Theorem 3.2.

THEOREM 3.3. *Let $\bar{x} \in C \cap (\cap_{i \in I} C_i)$. Then the following statements are equivalent:*

(i) *$\bar{x}$ is a strong interior point of the CCS-system $\{C, C_i : i \in I\}$; namely,*

$$\bar{x} \in C \cap \operatorname{int}(\cap_{i \in I} C_i).$$

(ii) *$\bar{x} \in \operatorname{int} C_i$ for each $i \in I$ and the sup-function $P(\cdot)$ of $\{p_{\widehat{C}_i}(\cdot)\}$ is continuous on X, where $p_{\widehat{C}_i}$ is the Minkowski functional of the set $\widehat{C}_i := C - \bar{x}$.*

Moreover, if we add an additional assumption that the set-valued map $i \mapsto C_i$ is lower semicontinuous, then the above (ii) *can be replaced by a stronger one, as follows:*

$\widetilde{\text{(ii)}}$ (ii) *holds and $i \mapsto p_{\widehat{C}_i}(x)$ is upper semicontinuous for each $x \in X$.*

4. Interior-point condition and the strong CHIP. For the remainder of this paper, we assume that I is a compact metric space and that $\{C, C_i:\ i \in I\}$ is a CCS-system with base-set C as in the beginning of the preceding section. Our main results are to provide sufficient conditions for ensuring the strong CHIP. For $x_0 \in C \cap (\cap_{i\in I} C_i)$, let $I_C^{\mathrm{rb}}(x_0) = \{i \in I:\ x_0 \in \mathrm{bd}_C\, C_i\}$. Since $\mathrm{bd}_C\, C_i = \mathrm{bd}\, C_i \setminus \mathrm{int}_C\, C_i$,

$$I_C^{\mathrm{rb}}(x_0) \subseteq \{i \in I:\ x_0 \in \mathrm{bd}\, C_i\}. \tag{4.1}$$

THEOREM 4.1. *Let $x_0 \in C \cap (\cap_{i\in I} C_i)$. Then the CCS-system $\{C, C_i:\ i \in I\}$ has the strong CHIP at x_0 if the following conditions are satisfied.*

(a) *The system $\{C, C_i:\ i \in I\}$ satisfies the strong C-interior-point condition.*

(b) *The set-valued mapping $i \mapsto (\mathrm{aff}\, C) \cap C_i$ is lower semicontinuous on I.*

(c) *The pair $\{\mathrm{aff}\, C, C_i\}$ has the strong CHIP at x_0 for each $i \in I$.*

(d) *Either C is finite dimensional or $I_C^{\mathrm{rb}}(x_0)$ is a finite set.*

Moreover, the same conclusion also holds if (a), (b), *and* (c) *are replaced simultaneously by* (a*) *and* (b*).

(a*) *The same as* (a) *but delete the word "relative."*

(b*) *The set-valued mapping $i \mapsto C_i$ is lower semicontinuous on I.*

Remark 4.1. In view of Theorem 2.2, (a*) $\Longrightarrow$ (c) because (a*) implies $\mathrm{int}\, C_i \supseteq \mathrm{int}(\cap_{i\in I} C_i) \cap C \neq \emptyset$.

Proof of Theorem 4.1. By the assumptions (a) and (b), let $\bar{x}$, $\widehat{C}$, $\widehat{C}_i$, $p_{\widehat{C}_i}$, and P be as in parts (i), $\widetilde{\text{(ii)}}$ of Theorem 3.2. In particular, P is continuous, the function $i \mapsto p_{\widehat{C}_i}(x)$ is upper semicontinuous for each $x \in X$, and for each $x \in \mathrm{aff}\, C$ and $i \in I$ it holds that

$$p_{\widehat{C}_i}(x - \bar{x}) \le 1 \Longleftrightarrow x \in C_i, \tag{4.2}$$

$$p_{\widehat{C}_i}(x - \bar{x}) = 1 \Longleftrightarrow x \in \mathrm{bd}_C\, C_i, \tag{4.3}$$

where (4.3) holds by (3.15). [*Note*. The above considerations are valid by Lemma 3.1 if one assumes (a*) + (b*) instead of (a) + (b); in fact, in this case, (4.2) and (4.3) hold for all $x \in X$ (not only those x in $\mathrm{aff}\, C$).]

Define, for each $i \in I$,

$$g_i(x) = p_{\widehat{C}_i}(x - \bar{x}) - 1 \quad \text{for each } x \in X$$

and let $G : X \to \mathbb{R}$ be defined by $G(x) := \sup_{i\in I} g_i(x)$ for each $x \in X$. Then G is continuous and the function $i \mapsto g_i(x)$ is upper semicontinuous for each $x \in X$. Thus, one can apply Theorem 2.1 to conclude that

$$N_C(x_0) + \partial G(x_0) = N_C(x_0) + \mathrm{cone} \sum_{i\in I(x_0)} \partial\, g_i(x_0), \tag{4.4}$$

provided that the following condition (d$'$) is satisfied:

(d$'$) C is finite dimensional or $I(x_0)$ is finite.

To prove the theorem, we need only to prove the inclusion

$$N_{C\cap(\cap_{i\in I} C_i)}(x_0) \subseteq N_C(x_0) + \sum_{i\in I} N_{C_i}(x_0), \tag{4.5}$$

as the reverse inclusion is evident. Note that, by (4.2),

$$\begin{aligned} N_{C\bigcap(\bigcap_{i\in I} C_i)}(x_0) &= N_{C\bigcap(\bigcap_{i\in I} g_i^{-1}(\mathbb{R}_-))}(x_0) \\ &= N_{C\bigcap(G^{-1}(\mathbb{R}_-))}(x_0) \\ &= N_C(x_0) + N_{G^{-1}(\mathbb{R}_-)}(x_0) \end{aligned} \tag{4.6}$$

thanks to Theorem DLW because $\bar{x} \in C \cap \operatorname{int} G^{-1}(\mathbb{R}_-)$ as $G(\bar{x}) = -1$. Thus (4.5) is seen to hold if $N_{G^{-1}(\mathbb{R}_-)}(x_0) = 0$. Therefore we may henceforth assume that $G(x_0) = 0$. Then, referring to the corresponding definition stated in (2.11), $I(x_0) = \{i \in I:\ g_i(x_0) = 0\}$. Since $x_0 \in C$ it follows from (4.3) that

$$I(x_0) = I_C^{\mathrm{rb}}(x_0). \tag{4.7}$$

Thus, by assumption (d), the condition (d′) holds. Recall from [7, Corollary 1, p. 56] that

$$N_{g_i^{-1}(\mathbb{R}_-)}(x_0) = \begin{cases} \operatorname{cone} \partial g_i(x_0), & i \in I(x_0), \\ 0, & i \notin I(x_0) \end{cases} \tag{4.8}$$

and, similarly,

$$N_{G^{-1}(\mathbb{R}_-)}(x_0) = \operatorname{cone} \partial G(x_0). \tag{4.9}$$

Hence, by (4.6) and (4.4), we have

$$\begin{aligned} N_{C\bigcap(\bigcap_{i\in I} C_i)}(x_0) &= N_C(x_0) + \operatorname{cone} \partial G(x_0) \\ &= N_C(x_0) + \operatorname{cone} \sum_{i\in I(x_0)} \partial\, g_i(x_0) \\ &= N_C(x_0) + \sum_{i\in I(x_0)} N_{g_i^{-1}(\mathbb{R}_-)}(x_0) \\ &\subseteq N_C(x_0) + \sum_{i\in I(x_0)} N_{C_i\cap \operatorname{aff} C}(x_0) \end{aligned} \tag{4.10}$$

because $g_i^{-1}(\mathbb{R}_-) \supseteq C_i \cap \operatorname{aff} C$. This implies that (4.5) holds because for each $i \in I(x_0) = I_C^{\mathrm{rb}}(x_0)$, one has from assumption (c) that

$$N_{C_i\cap \operatorname{aff} C}(x_0) = N_{C_i}(x_0) + N_{\operatorname{aff} C}(x_0)$$

(note also that $N_C(x_0) = N_C(x_0) + N_{\operatorname{aff} C}(x_0)$ as $C \subseteq \operatorname{aff} C$). Thus the first part of the theorem is proved. The proof of the second part is almost the same because if the assumptions (a) + (b) + (c) are replaced by (a*) + (b*), then (4.7) remains true (noting that $\bar{x} \in C \cap \operatorname{int} C_i$ and $x_0 \in C \cap C_i$ for each $i \in I$ and that $N_{C_i\cap \operatorname{aff} C}(x_0)$ in (4.10) is to be replaced by $N_{C_i}(x_0)$). ∎

Remark 4.2. As assumption (c) is used only at the end of the proof, (4.10) is valid even if (c) is not assumed in Theorem 4.1.

Remark 4.3. The result in the first part of Theorem 4.1 is not true if the assumptions are replaced by (a), (c), (d), and (b*) (instead of (b)), as shown by the following example.

Example 4.1. Let $X = \mathbb{R}^2$ and let $I = \{0, 1, \frac{1}{2}, \ldots\}$. For each $i \in I$, let C_i be the closed convex subset of $\mathbb{R}^2$ that is bounded by four line segments l_1, l_2, l_3, l_4 and the curve defined by

$$t_2 = i(t_1 + 1 + i)^2 \quad \text{for all } t_1 \in [-2, -1 - i],$$

where l_1, l_2, l_3, and l_4 are defined as follows:

$$\begin{aligned} l_1: &\quad t_1 = -2 \quad \text{for all } t_2 \in [i(1-i)^2, 1],\\ l_2: &\quad t_1 = 1 \quad \text{for all } t_2 \in [0, 1],\\ l_3: &\quad t_2 = 1 \quad \text{for all } t_1 \in [-2, 1],\\ l_4: &\quad t_2 = 0 \quad \text{for all } t_1 \in [-1 - i, 1]. \end{aligned}$$

Let $C = \{(t_1, 0) : t_1 \in [-2, 1]\}$. Clearly, $\bar{x} = (0, 0)$ is a strong C-interior point of the system $\{C, C_i : i \in I\}$. Note that $i_0 = 0$ is the only limit point of I and that C_0 is the rectangle $[-2, 1] \times [0, 1]$. Hence it is easy to see that the set-valued function $i \mapsto C_i$ is lower semicontinuous on I. However, since $(\operatorname{aff} C) \cap C_i$ is the segment $[-1-i, 1] \times \{0\}$ for each $i \in I \setminus \{0\}$ and $(\operatorname{aff} C) \cap C_0 = [-2, 1] \times \{0\}$, the set-valued function $i \mapsto (\operatorname{aff} C) \cap C_i$ is not lower semicontinuous on I. Let $x_0 = (-1, 0)$. It is clear that $\{\operatorname{aff} C, C_i\}$ has the strong CHIP at x_0 for each $i \in I$. But the system $\{C, C_i : i \in I\}$ does not have the strong CHIP at x_0 because $C \cap (\cap_{i \in I} C_i) = \{(t_1, 0) : t_1 \in [-1, 1]\}$, $N_{C \cap (\cap_{i \in I} C_i)}(x_0) = \{(t_1, t_2) : t_1 \le 0\}$, and $N_C(x_0) + \sum_{i \in I} N_{C_i}(x_0) = \{(t_1, t_2) : t_1 = 0\}$. □

Remark 4.4. In general, $\{\operatorname{aff} C, C_1\}$ may not have the strong CHIP even if the C-interior-point condition is satisfied by the system $\{C, C_1\}$. For example, let $I = \{1\}$, $X = \mathbb{R}^2$, and C be the line $t_2 = 0$ while C_1 is bounded by the lines $t_2 = 0$, $t_2 = 2$, $t_1 = 1$, and the half-circle $t_1^2 + (t_2 - 1)^2 = 1$, $t_1 \in [-1, 0]$. Clearly, $\{C, C_1\}$ satisfies the C-interior-point condition but $\{\operatorname{aff} C, C_1\}$ does not have the strong CHIP at $x_0 = (0, 0)$. This example also shows that (c) cannot be dropped in the first part of Theorem 4.1.

Our next two theorems address the case when some C_i might have an empty interior. We will use the notion that a subset in X is finite codimensional.

DEFINITION 4.1. *Let A and B be two nonempty convex subsets of X. We say that A is*

(i) *finite codimensional in B if the closed subspace $\overline{\operatorname{span} B} \cap \overline{\operatorname{span} A}$ is a finite codimensional subspace of $\overline{\operatorname{span} B}$;*

(ii) *finite codimensional if $\overline{\operatorname{span} A}$ is finite codimensional in X.*

Obviously, if B is finite dimensional, any nonempty convex subset A of X is finite codimensional in B.

THEOREM 4.2. *Let $x_0 \in C \cap (\cap_{i \in I} C_i)$. The system $\{C, C_i : i \in I\}$ has the strong CHIP at x_0 if the following conditions are satisfied.*

(a) *The system $\{C, C_i : i \in I\}$ satisfies the weak-strong C-interior-point condition with (I_1, I_2).*

(b) *The set-valued mapping $i \mapsto (\operatorname{aff} C) \cap C_i$ is lower semicontinuous on I.*

(c) *The pair $\{\operatorname{aff} C, C_i\}$ has the strong CHIP at x_0 for each $i \in I \setminus (I_1 \cup I_2)$.*

(d) *C is finite dimensional or $I_C^{\mathrm{rb}}(x_0)$ is finite.*

(e) *C_i is finite codimensional for each $i \in I_1$.*

Moreover, the same conclusion also holds if (a), (b), *and* (c) *in the above assumptions are replaced simultaneously by* (a*) *and* (b*).

(a*) *The same as* (a) *but delete the word "relative."*

(b*) *The set-valued mapping $i \mapsto C_i$ is lower semicontinuous on I.*

Proof. By (a), there exist $\bar{x} \in X$ and two disjoint finite subsets I_1, I_2 of I such that $\bar{x}$ is a weak-strong C-interior point with (I_1, I_2) of the CCS-system $\{C, C_i : i \in I\}$; that is, C_i is a polyhedron for each $i \in I_2$ and

$$\bar{x} \in \operatorname{ri} C \bigcap \left(\operatorname{rint}_C \bigcap_{i \in I_0} C_i\right) \bigcap \left(\bigcap_{i \in I_1} \operatorname{ri} C_i\right) \bigcap \left(\bigcap_{i \in I_2} C_i\right), \tag{4.11}$$

where $I_0 = I \setminus (I_1 \cup I_2)$. Let J denote the closure of I_0; namely, J equals the union of I_0 with I^l, where the subset I^l of $I_1 \cup I_2$ is defined by

$$I^l := \{i \in I_1 \cup I_2 : \ i \text{ is a limit point of } I\}. \tag{4.12}$$

For each $i \in I^l$, we define $\vec{C}_i$ by

$$\begin{aligned} \vec{C}_i : \quad &= \{x \in X : \ \exists x_j \in (\operatorname{aff} C) \cap C_j \text{ for all } j \in I \setminus \{i\} \text{ such that } \lim_{j \to i} x_j = x\} \\ &= \{x \in X : \ \exists x_j \in (\operatorname{aff} C) \cap C_j \text{ for all } j \in I_0 \text{ such that } \lim_{j \to i} x_j = x\}, \end{aligned} \tag{4.13}$$

thanks to the fact that $I_1 \cup I_2$ is a finite set. By assumption (b) and Proposition 3.1, we have

$$(\operatorname{aff} C) \cap C_i \subseteq \vec{C}_i \quad \text{for each } i \in I^l. \tag{4.14}$$

Moreover,

$$\bar{x} \in \operatorname{ri} C \bigcap \left(\operatorname{rint}_C \bigcap_{i \in I_0} C_i\right) \bigcap \left(\operatorname{rint}_C \bigcap_{i \in I^l} \vec{C}_i\right). \tag{4.15}$$

In fact, since $\bar{x} \in \operatorname{rint}_C \bigcap_{i \in I_0} C_i$, there exists $\delta > 0$ such that $(\operatorname{aff} C) \cap \mathbf{B}(\bar{x}, \delta) \subseteq C_j$ for each $j \in I_0$. From (4.13) it follows that $(\operatorname{aff} C) \cap \mathbf{B}(\bar{x}, \delta) \subseteq \vec{C}_i$ for each $i \in I^l$. Therefore $\bar{x} \in \operatorname{rint}_C(\bigcap_{i \in I^l} \vec{C}_i)$ and (4.15) holds. Note further that each $\vec{C}_i$ is convex and closed. In fact, let $i \in I^l$ and let $\{x_n\} \subseteq \vec{C}_i$ be such that $x_n \to z$. Then $\lim_{j \to i} d_{C_j \cap \operatorname{aff} C}(x_n) = 0$ for each $n = 1, 2, \ldots$. Since

$$d_{C_j \cap \operatorname{aff} C}(z) \le \|x_n - z\| + d_{C_j \cap \operatorname{aff} C}(x_n),$$

we have that $\lim_{j \to i} d_{C_j \cap \operatorname{aff} C}(z) = 0$. This implies that $z \in \vec{C}_i$ and so $\vec{C}_i$ is closed. The convexity of $\vec{C}_i$ follows from the convexity of the sets $(\operatorname{aff} C) \cap C_j$ $(j \in I)$ and the definition of $\vec{C}_i$. Recall that $J = I_0 \cup I^l$ and define $\vec{C}_i = C_i$ for each $i \in I_0$. Then J is compact and $\{C, \vec{C}_i : i \in J\}$ is a CCS-system with the following properties:

(i) $\{C, \vec{C}_i : i \in J\}$ satisfies the strong C-interior-point condition; in fact,

$$\bar{x} \in \operatorname{ri} C \bigcap \operatorname{rint}_C \left(\bigcap_{j \in J} \vec{C}_j\right) \tag{4.16}$$

(see (4.15)).

(ii) The set-valued function $j \mapsto (\operatorname{aff} C) \cap \vec{C}_j$ is lower semicontinuous on J.

(iii) Either C is finite dimensional or $\vec{I}_C^{\rm rb}(x_0)$ is a finite set, where

$$\vec{I}_C^{\rm rb}(x_0) := \{i \in J : \ x_0 \in {\rm bd}_C\, \vec{C}_i\}. \tag{4.17}$$

In fact, (ii) follows from assumption (b) and the definition of $\vec{C}_j$. Moreover, (iii) follows from (d) and the fact that $\vec{I}_C^{\rm rb}(x_0) \subseteq I_C^{\rm rb}(x_0)$ (because of (4.14) and $x_0 \in (\operatorname{aff} C) \cap C_i$). Thus, by Remark 4.2, we have that

$$N_{C\cap(\cap_{j\in J}\vec{C}_j)}(x_0) \subseteq N_C(x_0) + \sum_{j\in J} N_{(\operatorname{aff} C)\cap\vec{C}_j}(x_0). \tag{4.18}$$

We will show that

$$N_{(\operatorname{aff} C)\cap\vec{C}_j}(x_0) \subseteq N_C(x_0) + N_{C_j}(x_0) \quad \text{for each } j \in J. \tag{4.19}$$

This inclusion is simply from assumption (c) if $j \in I_0$. Next consider the case when $j \in J \cap I_2$. In this case one has $j \in I^l \cap I_2$ and it follows from (4.14) that $(\operatorname{aff} C) \cap C_j \subseteq \vec{C}_j$. Consequently, one has

$$N_{(\operatorname{aff} C)\cap\vec{C}_j}(x_0) \subseteq N_{(\operatorname{aff} C)\cap C_j}(x_0) \subseteq N_{C\cap C_j}(x_0) = N_C(x_0) + N_{C_j}(x_0),$$

where the last equality holds by Theorem 2.2 (which is applicable here because C_j is a polyhedron and $\bar{x} \in \operatorname{ri} C \cap C_j$). It remains to consider the case when $j \in I_1 \cap I^l$ for (4.19). To do this and for a later application, let us consider a general $j \in I_1$ (for the time being regardless of whether $j \in I^l$ or not). Then by (4.11) and definitions, $\bar{x} \in \operatorname{ri} C_j = \operatorname{rint}_{C_j} C_j$. Hence, by Lemma 3.1, $C_j - \bar{x}$ admits a (C_j)-Minkowski extension $\widetilde{C}_j - \bar{x}$: $\widetilde{C}_j$ is a closed convex set such that

$$\bar{x} \in \operatorname{int} \widetilde{C}_j \text{ and } \overline{\operatorname{aff} C_j} \cap \widetilde{C}_j = \overline{\operatorname{aff} C_j} \cap C_j = C_j = (\operatorname{aff} C_j) \cap \widetilde{C}_j \quad \text{for all } j \in I_1. \tag{4.20}$$

Combining this with (4.14),

$$(\operatorname{aff} C) \cap \overline{\operatorname{aff} C_j} \cap \widetilde{C}_j \subseteq (\operatorname{aff} C) \cap \vec{C}_j \quad \text{for each } j \in I_1 \cap I^l. \tag{4.21}$$

Thus if $j \in I_1 \cap I^l$, then it follows from (4.21) and Theorem 2.2 that

$$\begin{aligned} N_{(\operatorname{aff} C)\cap\vec{C}_j}(x_0) &\subseteq N_{\operatorname{aff} C\cap\overline{\operatorname{aff} C_j}\cap\widetilde{C}_j}(x_0) \\ &\subseteq N_{(\operatorname{aff} C)\cap\overline{\operatorname{aff} C_j}}(x_0) + N_{\widetilde{C}_j}(x_0) \\ &\subseteq N_{\operatorname{aff} C}(x_0) + N_{\overline{\operatorname{aff} C_j}}(x_0) + N_{\widetilde{C}_j}(x_0) \\ &\subseteq N_{\operatorname{aff} C}(x_0) + N_{\overline{\operatorname{aff} C_j}\cap\widetilde{C}_j}(x_0) \\ &\subseteq N_C(x_0) + N_{C_j}(x_0) \end{aligned}$$

thanks to (4.20), and thus (4.19) is verified. Here we have used Theorem 2.2 (twice) which is applicable as $\bar{x} \in \operatorname{int} \widetilde{C}_j \cap \overline{\operatorname{aff} C_j} \cap \operatorname{ri}(\operatorname{aff} C)$ and $\overline{\operatorname{aff} C_j}$ is a polyhedron (being an affine subspace of finite codimension) by assumption (e). Therefore (4.19) is established in all possible cases. Combining (4.19) and (4.18), we have

$$N_{C\cap(\cap_{j\in J}\vec{C}_i)}(x_0) \subseteq N_C(x_0) + \sum_{j\in J} N_{C_j}(x_0). \tag{4.22}$$

Recalling $\vec{C}_i = C_i$ for each $i \in I_0$, $J = I_0 \cup I^l$ and $I = J \cup I_1 \cup I_2$, it is a routine matter to verify from (4.14) and (4.20) that

(4.23)
$$C \bigcap \left(\bigcap_{i \in I} C_i \right) = \left\{ C \bigcap \left(\bigcap_{i \in J} \vec{C}_i \right) \bigcap \left(\bigcap_{i \in I_1} \overline{\operatorname{aff} C_i} \right) \bigcap \left(\bigcap_{i \in I_2} C_i \right) \right\} \bigcap \left(\bigcap_{i \in I_1} \widetilde{C}_i \right).$$

(For example, if x is a member of the set on the right-hand side of (4.23) and if $i \in I_1$, then $x \in C_i$ by (4.20) and so it is not difficult to verify that x belongs to the set on the left-hand side of (4.23).) Moreover, by virtue of the general inclusion property, $\operatorname{ri} C \bigcap (\operatorname{rint}_C \bigcap_{i \in J} \vec{C}_i) \subseteq \operatorname{ri}(C \bigcap (\bigcap_{i \in J} \vec{C}_i))$ (which can be verified by definition). This with (4.16) implies that $\bar{x} \in \operatorname{ri}(C \bigcap (\bigcap_{i \in J} \vec{C}_i))$ and hence it follows from (4.11) and (4.20) that

$$\bar{x} \in \operatorname{ri} \left(C \bigcap \left(\bigcap_{i \in J} \vec{C}_i \right) \right) \bigcap \left(\bigcap_{i \in I_1} \overline{\operatorname{aff} C_i} \right) \bigcap \left(\bigcap_{i \in I_2} C_i \right) \bigcap \left(\operatorname{int} \bigcap_{i \in I_1} \widetilde{C}_i \right). \tag{4.24}$$

Thus Theorem 2.2 is applicable to computing the normal cone of the set on the left-hand side of (4.23) at x_0 (noting that each C_i with $i \in I_2$ is a polyhedron and that each $\overline{\operatorname{aff} C_i}$ with $i \in I_1$ is also a polyhedron as noted before):

$$\begin{aligned} &N(x_0; C \cap (\cap_{i \in I} C_i)) \\ &\quad = N(x_0; (C \cap (\cap_{i \in J} \vec{C}_i)) \cap (\cap_{i \in I_1} \overline{\operatorname{aff} C_i}) \cap (\cap_{i \in I_2} C_i)) + \sum_{i \in I_1} N(x_0; \widetilde{C}_i) \\ &\quad = N(x_0; C \cap (\cap_{i \in J} \vec{C}_i)) + \sum_{i \in I_1} N(x_0; \overline{\operatorname{aff} C_i}) + \sum_{i \in I_2} N(x_0; C_i) + \sum_{i \in I_1} N(x_0; \widetilde{C}_i) \\ &\quad \subseteq N(x_0; C) + \sum_{i \in J} N(x_0; C_i) + \sum_{i \in I_1} N(x_0; \overline{\operatorname{aff} C_i} \cap \widetilde{C}_i) + \sum_{i \in I_2} N(x_0; C_i), \end{aligned}$$

thanks to (4.22). Since $I = J \cup I_1 \cup I_2$ and in view of (4.20), this implies that $\{C, C_i : i \in I\}$ has the strong CHIP at x_0. This completes the proof for the first part of Theorem 4.2.

For the proof of the second part, by (a*) there exist $\bar{x} \in X$ and two disjoint finite subsets I_1, I_2 of I such that

$$\bar{x} \in \operatorname{ri} C \bigcap \left(\operatorname{int} \bigcap_{i \in I_0} C_i \right) \bigcap \left(\bigcap_{i \in I_1} \operatorname{ri} C_i \right) \bigcap \left(\bigcap_{i \in I_2} C_i \right). \tag{4.25}$$

Now the proof is completed almost the same as for the first part, with the only modifications as follows. We use (4.25) in place of (4.11). We use "int" to replace "rint_C" in (4.15); in (4.13) and (4.14) we use "C_i" to replace "$(\operatorname{aff} C) \cap C_i$." ∎

Below we provide a sufficient condition ensuring the strong CHIP for a CCS-system with a closed subspace as a base-set.

LEMMA 4.1. *Let $\{C_i : i \in I\}$ be a family of nonempty closed convex subsets of X, $Y \subseteq X$ be a vector subspace, and $x_0 \in Y \bigcap (\bigcap_{i \in I} C_i)$. Then the system $\{Y, C_i : i \in I\}$ has the strong CHIP at x_0 provided that*

(a) *for each $i \in I$, either $C_i \subseteq Y$ or C_i is a polyhedron;*

(b) *$\{Y \cap C_i : i \in I\}$ has the strong CHIP at x_0 in Y.*

Proof. By (b) and recalling the notation of N^Y given in (2.17), we have

$$N^Y_{Y\cap(\cap_{i\in I} C_i)}(x_0) = \sum_{i\in I} N^Y_{Y\cap C_i}(x_0).$$

Now let $x^* \in N_{Y\cap(\cap_{i\in I} C_i)}(x_0)$. Then $x^*|_Y \in N^Y_{Y\cap(\cap_{i\in I} C_i)}(x_0)$ and, hence, there exist a finite subset I_0 of I and $\widetilde{x}_i^* \in N^Y_{Y\cap C_i}(x_0)$, $i \in I_0$, such that

$$x^*|_Y = \sum_{i\in I_0} \widetilde{x}_i^* \quad \text{on } Y. \tag{4.26}$$

For each $i \in I_0$, let $x_i^* \in X^*$ be an extension of $\widetilde{x}_i^*$. Then $x_i^* \in N_{Y\cap C_i}(x_0)$. Note that in the case when $C_i \subseteq Y$, $N_{Y\cap C_i}(x_0) = N_{C_i}(x_0)$, while in the case when C_i is a polyhedron, $N_{Y\cap C_i}(x_0) = N_Y(x_0) + N_{C_i}(x_0)$ by Theorem 2.2. Thus (a) implies that $\sum_{i\in I_0} x_i^* \in N_Y(x_0) + \sum_{i\in I} N_{C_i}(x_0)$. Denote $y^* := x^* - \sum_{i\in I_0} x_i^*$. Then $y^* \in N_Y(x_0)$ and so $x^* = y^* + \sum_{i\in I_0} x_i^* \in N_Y(x_0) + \sum_{i\in I} N_{C_i}(x_0)$. This shows that $N_{Y\cap(\cap_{i\in I} C_i)}(x_0) \subseteq N_Y(x_0) + \sum_{i\in I} N_{C_i}(x_0)$ and the proof is complete. □

The following consequence of Lemma 4.1 and Theorem 4.2 will be further extended in Corollary 4.4.

COROLLARY 4.1. *Let C, C_1 be a pair of closed convex subsets of X such that $\operatorname{ri} C \cap \operatorname{ri} C_1 \neq \emptyset$. Suppose that (at least) one of the sets is finite dimensional or finite codimensional. Then $\{C, C_1\}$ has the strong CHIP.*

Proof. By symmetry, we need only to consider two cases: (i) C_1 is finite codimensional; (ii) C is finite dimensional. The case (i) is clear by Theorem 4.2 (with $I = I_1 = \{1\}$ and $I_2 = \emptyset$). For the case (ii), let $x_0 \in C \cap C_1$. Let $\overline{\operatorname{span} C} + \overline{\operatorname{span} C_1}$ be denoted by Y. Then C_1 is finite codimensional in Y and hence, by what we just proved, the system $\{C, C_1\}$ in Y has the strong CHIP in Y. Since C and C_1 are subsets of Y, it follows from Lemma 4.1 that the system $\{Y, C, C_1\}$ has the strong CHIP. This implies that $\{C, C_1\}$ has the strong CHIP in X since $N_Y(x_0) \subseteq N_C(x_0)$ for each $x_0 \in C \cap C_1$. □

If assumption (d) in Theorem 4.2 is replaced by the stronger assumption that C is finite dimensional, then (e) can be dropped. This will be proved in Theorem 4.3 below. For preparing its proof and also for a later use, we first give a lemma.

LEMMA 4.2. *Let $x_0 \in C \cap (\cap_{i\in I} C_i)$. The CCS-system $\{C, C_i : i \in I\}$ has the strong CHIP at x_0 if it satisfies* (a*) *and* (b*) *of Theorem* 4.2 *as well as the following conditions.*

($\bar{\text{c}}$) *$\{C, C_i : i \in I_1 \cup I_2\}$ has the strong CHIP at x_0.*

(d) *The same as* (d) *in Theorem* 4.2.

Proof. As in the beginning of the proof of Theorem 4.2, let I^l be defined by (4.12) and let $\vec{C}_i$, for each $i \in I^l$, be defined by (4.13) with X in place of C. Let $J := I_0 \cup I^l$. Define $\vec{C}_i = C_i$ for each $i \in I_0$. Then J is compact and $\{C, \vec{C}_i : i \in J\}$ is a CCS-system with the following properties:

(i) $\{C, \vec{C}_i : i \in J\}$ satisfies the strong interior-point condition; in fact,

$$\bar{x} \in \operatorname{ri} C \bigcap \left(\operatorname{int} \bigcap_{j\in J} \vec{C}_j \right). \tag{4.27}$$

(ii) The set-valued function $j \mapsto \vec{C}_j$ is lower semicontinuous on J.

(iii) $C_j \subseteq \vec{C}_j$ for each $j \in J$.

(iv) C is finite dimensional or $\vec{J}(x_0)$ is finite, where

$$\vec{J}(x_0) = \{i \in J : \ x_0 \in \operatorname{bd} \vec{C}_i\}.$$

Thanks to (i), (ii), and (iv), it is easy to see that the system $\{C, \vec{C}_i : \ i \in J\}$ satisfies the conditions (a*), (b*), and (d) of Theorem 4.1. Hence, by Theorems 2.2 and 4.1 and the above (iii), we obtain that

$$\begin{aligned} N(x_0; C) + N(x_0; (\cap_{i\in J} \vec{C}_i)) &= N(x_0; C \cap (\cap_{i\in J} \vec{C}_i)) \\ &= N(x_0; C) + \textstyle\sum_{i\in J} N(x_0; \vec{C}_i) \\ &\subseteq N(x_0; C) + \textstyle\sum_{i\in J} N(x_0; C_i). \end{aligned} \tag{4.28}$$

Noting by (iii) that

$$C \bigcap \left(\bigcap_{i\in I} C_i \right) = C \bigcap \left(\bigcap_{i\in J} \vec{C}_i \right) \bigcap \left(\bigcap_{i\in I_1\cup I_2} C_i \right) \tag{4.29}$$

and that

$$\bar{x} \in \operatorname{ri} C \bigcap \left(\operatorname{int} \bigcap_{j\in J} \vec{C}_j \right) \bigcap \left(\bigcap_{i\in I_1\cup I_2} C_i \right), \tag{4.30}$$

(a) of Theorem 2.2 can be applied to conclude that

$$\begin{aligned} N(x_0; C \cap (\cap_{i\in I} C_i)) &= N(x_0; \cap_{i\in J} \vec{C}_i) + N(x_0; C \cap (\cap_{i\in I_1\cup I_2} C_i)) \\ &= N(x_0; C) + N(x_0; (\cap_{i\in J} \vec{C}_i)) \\ &\quad + \textstyle\sum_{i\in I_1} N(x_0; C_i) + \sum_{i\in I_2} N(x_0; C_i), \end{aligned} \tag{4.31}$$

thanks to assumption ($\bar{\text{c}}$). Combining (4.28) and (4.31) gives the desired conclusion and the proof is complete. ∎

THEOREM 4.3. *Let $x_0 \in C \cap (\cap_{i\in I} C_i)$. The system $\{C, C_i : \ i \in I\}$ has the strong CHIP at x_0 if it satisfies* (a), (b), *and* (c) *of Theorem* 4.2 *and the following condition.*

($\bar{\text{d}}$) *C is finite dimensional.*

Moreover, the same conclusion also holds if (a) + (b) + (c) + ($\bar{\text{d}}$) *in the above assumptions is replaced by* (a*) + (b*) + ($\bar{\text{d}}$), *where* (a*) *and* (b*) *are as in Theorem* 4.2.

Proof. Denote

$$\widehat{C} = C - x_0, \quad Z := \operatorname{span} \widehat{C}, \quad \widehat{C}_i = C_i - x_0 \quad \text{for each } i \in I. \tag{4.32}$$

Then, by assumptions, Z is finite dimensional and

$$\operatorname{ri} \widehat{C} \bigcap \left(\operatorname{rint}_{\widehat{C}} \bigcap_{i\in I_0} \widehat{C}_i \right) \bigcap \left(\bigcap_{i\in I_1} \operatorname{ri} \widehat{C}_i \right) \bigcap \left(\bigcap_{i\in I_2} \widehat{C}_i \right) \neq \emptyset. \tag{4.33}$$

Letting $C_i^\sharp$ denote the intersection $Z \cap \widehat{C}_i$, it follows that

$$\operatorname{ri} \widehat{C} \bigcap \left(\operatorname{rint}_{\widehat{C}} \bigcap_{i\in I_0} C_i^\sharp \right) \bigcap \left(\bigcap_{i\in I_1} \operatorname{ri} C_i^\sharp \right) \bigcap \left(\bigcap_{i\in I_2} C_i^\sharp \right) \neq \emptyset. \tag{4.34}$$

Noting aff $\widehat{C} = Z$, this implies that, as a system in the Banach space Z, $\{\widehat{C}, C_i^\sharp :\ i \in I\}$ satisfies the weak-strong interior-point condition. In fact, with respect to the relative topology in Z, one has

$$\operatorname{rint}_Z \widehat{C} \bigcap \left(\operatorname{rint}_Z \bigcap_{i\in I_0} C_i^\sharp \right) \bigcap \left(\bigcap_{i\in I_1} \operatorname{ri} C_i^\sharp \right) \bigcap \left(\bigcap_{i\in I_2} C_i^\sharp \right) \neq \emptyset. \tag{4.35}$$

By Theorem DLW (applied to the system $\{\widehat{C}, \bigcap_{i\in I} C_i^\sharp\}$ in Z), we have

$$N^Z_{\widehat{C}\cap(\cap_{i\in I} C_i^\sharp)}(0) = N^Z_{\widehat{C}}(0) + N^Z_{\cap_{i\in I} C_i^\sharp}(0);$$

namely, in the normed linear space X,

$$N_{\widehat{C}\cap(\cap_{i\in I} \widehat{C}_i)}(0) = N_{\widehat{C}}(0) + N_{\cap_{i\in I} C_i^\sharp}(0), \tag{4.36}$$

because $\widehat{C} \cap (\cap_{i\in I} C_i^\sharp) = \widehat{C} \cap (\cap_{i\in I} \widehat{C}_i)$, and $\widehat{C}$ as well as $C_i^\sharp$ are subsets of Z. We will show that

$$N_{\cap_{i\in I} C_i^\sharp}(0) \subseteq N_{\widehat{C}}(0) + \sum_{i\in I} N_{\widehat{C}_i}(0). \tag{4.37}$$

Granting this, (4.36) and an easy translation argument imply that

$$N_{C\cap(\cap_{i\in I} C_i)}(x_0) \subseteq N_C(x_0) + \sum_{i\in I} N_{C_i}(x_0),$$

which shows that $\{C, C_i :\ i \in I\}$ has the strong CHIP at x_0. It remains to prove (4.37). To do this, we shall apply Lemma 4.1 to $Y := Z$ and the system $\{D_i\}$, where

$$D_i = \begin{cases} C_i^\sharp = \widehat{C}_i \cap Z, & i \in I \setminus I_2, \\ \widehat{C}_i, & i \in I_2. \end{cases}$$

We suppose, without loss of generality, that $I \setminus I_2 \neq \emptyset$ (otherwise (4.37) holds by Theorem 2.2). Then $\cap_{i\in I} C_i^\sharp = \cap_{i\in I} D_i$. Moreover, assumption (c) (for $i \in I\setminus(I_1 \cup I_2)$) and Corollary 4.1 (for $i \in I_1$) imply that for each $i \in I \setminus I_2$, $\{Z, \widehat{C}_i\}$ has the strong CHIP at 0 and hence that

$$N_{D_i}(0) = N_{Z\cap\widehat{C}_i}(0) = N_Z(0) + N_{\widehat{C}_i}(0) \subseteq N_{\widehat{C}}(0) + N_{\widehat{C}_i}(0). \tag{4.38}$$

We claim that $\{D_i :\ i \in I\}$ has the strong CHIP at 0. Granting this, it follows from (4.38) that

$$\begin{aligned} N_{\cap_{i\in I} C_i^\sharp}(0) = N_{\cap_{i\in I} D_i}(0) &= \sum_{i\in I} N_{D_i}(0) \\ &= \sum_{i\in I\setminus I_2} N_{D_i}(0) + \sum_{i\in I_2} N_{\widehat{C}_i}(0) \subseteq N_{\widehat{C}} + \sum_{i\in I} N_{\widehat{C}_i}(0), \end{aligned}$$

which shows (4.37). Thus it remains to prove the above claim. In view of Lemma 4.1, it suffices to show that, as a system in the subspace Z, $\{Z \cap D_i :\ i \in I\}$ has the strong CHIP at 0 (note that for each $i \in I_2$, $\widehat{C}_i$ is a polyhedron because C_i is a polyhedron). By (4.35), this system in Z satisfies the weak-strong interior-point condition on Z

with (I_1, I_2). Assumption (b) tells us that $i \mapsto D_i \cap Z$ is lower semicontinuous on I. It is trivial that for each $i \in I_1$, $D_i \cap Z$ is finite codimensional in Z as Z is finite dimensional. Hence it is easy to see that the second part of Theorem 4.2 is applicable to the system $\{Z, Z \cap D_i : i \in I\}$ in place of $\{C, C_i : i \in I\}$ if assumptions (a), (b), (c), and $(\bar{\text{d}})$ are assumed. Thus, this system (in Z) does have the strong CHIP at 0. Our claim is therefore established and this completes the proof of the first part of the theorem.

For the second part, suppose that the system $\{C, C_i : i \in I\}$ satisfies $(\text{a}^*) + (\text{b}^*) + (\bar{\text{d}})$ (in place of (a) + (b) + (c) + $(\bar{\text{d}})$). Then, by $(\bar{\text{d}})$ and by the result of the first part (applied to the finite subsystem $\{C, C_i : i \in I_1 \cup I_2\}$), one concludes that $\{C, C_i : i \in I_1 \cup I_2\}$ has the strong CHIP at x_0. Hence, by Lemma 4.2, $\{C, C_i : i \in I\}$ also has the strong CHIP at x_0. ∎

COROLLARY 4.2. *Let $x_0 \in C \cap (\cap_{i\in I} C_i)$. The system $\{C, C_i : i \in I\}$ has the strong CHIP at x_0 if it satisfies* (a^*) *and* (b^*) *of Theorem* 4.2 *as well as the following condition.*

(d^*) *At least one of $\{C, C_i : i \in I_1\}$ is finite dimensional.*

Proof. By assumption (a^*), take $\bar{x}$ satisfying (4.25). By assumption (d^*) and in view of the second part of Theorem 4.3, it suffices to consider the case when C_{i_0} is finite dimensional for some $i_0 \in I_1$. Let

$$I_1' = I_1 \cup \{i_\infty\},$$

where i_∞ is a new index such that $i_\infty \notin I$. Let $I_0 = I \setminus (I_1 \cup I_2)$ and define $D := C_{i_0} \cap (\cap_{i\in I_0} C_i)$. Then $\operatorname{ri} C_{i_0} \cap \operatorname{int}(\cap_{i\in I_0} C_i) \subseteq \operatorname{ri} D$, and thus $\bar{x} \in \operatorname{ri} D$ by (4.25). Thus

$$\bar{x} \in \operatorname{ri} D \bigcap \left(\bigcap_{i\in I_1} \operatorname{ri} C_i \right) \bigcap \operatorname{ri} C \bigcap \left(\bigcap_{i\in I_2} C_i \right). \tag{4.39}$$

Letting $J = I_1' \cup I_2$, $D_{i_\infty} = C$, and $D_i = C_i$ for each $i \in I_1 \cup I_2$, (4.39) implies that the new CCS-system $\{D, D_j : j \in J\}$ satisfies the weak-strong interior-point condition with (I_1', I_2), that is, the condition (a^*) of Theorem 4.3 stated for $\{C, C_i : i \in I\}$. It also satisfies (b^*) of Theorem 4.3 as J is finite. Therefore by applying Theorem 4.3 to the new system we have that

$$\begin{aligned} N_{D\cap(\cap_{j\in J} D_i)}(x_0) &= N_D(x_0) + \textstyle\sum_{j\in J} N_{D_j}(x_0) \\ &= N_{C_{i_0}\cap(\cap_{i\in I_0} C_i)}(x_0) + N_C(x_0) + \textstyle\sum_{i\in I_1\cup I_2} N_{C_i}(x_0) \\ &\subseteq N_C(x_0) + N_{C_{i_0}\cap(\cap_{i\in I} C_i)}(x_0). \end{aligned} \tag{4.40}$$

Applying Theorem 4.3 to the system $\{C_{i_0}, C_i : i \in I\}$ and noting that $D \cap (\cap_{j\in J} D_i) = C \cap (\cap_{i\in I} C_i)$, it follows from (4.40) that

$$N_{C\cap(\cap_{i\in I} C_i)}(x_0) = N_{D\cap(\cap_{j\in J} D_i)}(x_0) = N_C(x_0) + \sum_{i\in I} N_{C_i}(x_0).$$

The proof is complete. ∎

COROLLARY 4.3. *Let $x_0 \in C \cap (\cap_{i\in I} C_i)$. The system $\{C, C_i : i \in I\}$ has the strong CHIP at x_0 if it satisfies* (a^*) *and* (b^*) *of Theorem* 4.2 *as well as the following conditions.*

(c^*) *For any $i, j \in I_1$, C_i is finite codimensional in C as well as in C_j.*

$(\widetilde{\text{d}})$ *$I(x_0) = \{i \in I : x_0 \in \operatorname{bd} C_i\}$ is finite.*

Proof. By Lemma 4.2, it suffices to show that ($\bar{\text{c}}$) of Lemma 4.2 holds; that is, the system $\{C, C_i : i \in I_1 \cup I_2\}$ has the strong CHIP at x_0. For this purpose, take a new index $0 \notin I$ and, for each $i \in I_1$, let Y_0 and Y_i denote the closed subspaces spanned by C and C_i, respectively, and let

$$Y = Y_0 + \sum_{i \in I_1} Y_i. \tag{4.41}$$

We claim that each C_i for $i \in I_1$ is finite codimensional in Y. To verify this claim, let $i \in I_1$ and set $J = \{0\} \cup I_1 \setminus \{i\}$. By assumption (c*), C_i is finite codimensional in Y_j, for each $j \in J$, and it follows from Definition 4.1 that there exists a finite dimensional subspace Y_j' of Y_j such that Y_j is the direct sum of Y_j' and $Y_j \cap Y_i$, that is,

$$Y_j = Y_j' + (Y_j \cap Y_i) \quad \text{and} \quad Y_j' \cap (Y_j \cap Y_i) = \{0\}.$$

Hence, for each $j \in J$,

$$Y_j + Y_i = Y_j' + Y_i \quad \text{and} \quad Y_j' \cap Y_i = \{0\}. \tag{4.42}$$

Then, by (4.41) and (4.42),

$$Y = \sum_{j \in J} (Y_j + Y_i) = \sum_{j \in J} Y_j' + Y_i \quad \text{and} \quad \left(\sum_{j \in J} Y_j' \right) \cap Y_i = \{0\}. \tag{4.43}$$

This implies that Y_i is finite codimensional in Y since $\sum_{j \in J} Y_j'$ is finite dimensional. The claim is proved. Noting that $C_i \cap Y = C_i$ for each $i \in I_1$, this implies that, as a CCS-system in Y, $\{C, C_i \cap Y : i \in I_1 \cup I_2\}$ has property (e) of Theorem 4.2 stated for $\{C, C_i : i \in I\}$. Moreover, it also satisfies (d) thanks to assumption ($\widetilde{\text{d}}$) and (4.1). Therefore one can apply the second part of Theorem 4.2 (with $I_0 = \emptyset$) to conclude that this finite system in the subspace Y has the strong CHIP at x_0. Noting that $C, C_i \subseteq Y$ for each $i \in I_1$ and C_j is a polyhedron for each $j \in I_2$, it follows from Lemma 4.1 that $\{Y, C, C_i : i \in I_1 \cup I_2\}$ has the strong CHIP at x_0 in X. Therefore $\{C, C_i : i \in I_1 \cup I_2\}$ has the strong CHIP at x_0 (because $C \subseteq Y$). ∎

We obtain below an extension of Rockafellar's result [27, Corollary 23.8.1, p. 223] in the setting of general normed linear spaces.

COROLLARY 4.4. *Let $I = J \cup K$ be finite with J, K disjoint such that C_k is a polyhedron for each $k \in K$, and suppose that*

$$\operatorname{ri} C \bigcap \left(\bigcap_{j \in J} \operatorname{ri} C_j \right) \bigcap \left(\bigcap_{k \in K} C_i \right) \neq \emptyset. \tag{4.44}$$

Then the system $\{C, C_i : i \in I\}$ has the strong CHIP if at least one of the following conditions is satisfied.

(a) *At least one of $\{C, C_j : j \in J\}$ is finite dimensional.*

(b) *For each $j \in J$, C_j is finite codimensional in C and C_i, respectively, for each $i \in J$ (e.g., C_j is finite codimensional for each $j \in J$).*

Proof. Since I is finite and thanks to (4.44), the system $\{C, C_i : i \in I\}$ satisfies (a*) and (b*) of Theorem 4.2 (with $(I_1, I_2) = (J, K)$). Now apply Corollary 4.2 and Corollary 4.3 to conclude the proof. ∎

5. Subsystems and the strong CHIP. Recall that I and $\{C, C_i : i \in I\}$ are as explained in the beginning of the preceding section. Our main result of this section is the following.

THEOREM 5.1. *Suppose that the finite subsystem $\{C, C_i : i \in J\}$ has the strong CHIP for each finite subset J of I. Then the system $\{C, C_i : i \in I\}$ has the strong CHIP provided the following conditions are satisfied.*

(a) *C is finite dimensional (say, $\dim C := l < +\infty$).*

(b) *The set-valued function $i \mapsto (\operatorname{aff} C) \cap C_i$ is Kuratowski continuous on I.*

(c) *For any finite subset J of I (with the number of elements $|J| \le l$ if X is real and $|J| \le 2l$ if X is complex), the subsystem $\{C, C_i : i \in J\}$ satisfies the following C-interior-point condition:*

$$C \bigcap \left(\bigcap_{i \in J} \operatorname{rint}_C C_i \right) \neq \emptyset. \tag{5.1}$$

Proof. Since I is compact and metrizable, there exists a sequence $\{I_k\}$ of subsets of I such that

(i) each I_k is finite;

(ii) $I_k \subseteq I_{k+1}$ for $k = 1, 2, \ldots$;

(iii) I equals the closure of $\bigcup_{k=1}^{\infty} I_k$.

Let $S = \bigcap_{i \in I} C_i$, $K = C \cap S$, and $S_k = \bigcap_{i \in I_k} C_i$ for each k. Then

$$K = C \bigcap \left(\bigcap_{i \in I} C_i \right) \subseteq C \cap S_k. \tag{5.2}$$

Let $x_0 \in C \bigcap (\bigcap_{i \in I} C_i)$ and let $x^* \in N_K(x_0)$. We have to show that $x^* \in N_C(x_0) + \sum_{i \in I} N_{C_i}(x_0)$. We will first show that there exist $\{x_k\} \subseteq C$ with $x_k \to x_0$ and $x_k^* \in N_{C \cap S_k}(x_k)$ such that $\{x_k^*\}$ is bounded and

$$\lim_{k \to \infty} \langle x_k^*, y \rangle = \langle x^*, y \rangle \quad \text{for each } y \in \operatorname{span}(C - x_0). \tag{5.3}$$

In fact, since $Z := \operatorname{span}(C - x_0)$ is finite dimensional, we may assume, without loss of generality, that the norm restricted to Z is both strictly convex and smooth. Clearly, we may assume that $x^*|_Z \neq 0$. Take $z_0 \in Z$ such that $\langle x^*, z_0 \rangle = \|x^*|_Z\| \cdot \|z_0\| = \|z_0\|^2$. Write $x := x_0 + z_0$. Then, $x^*|_Z = J(x - x_0)|_Z$. By the Hahn–Banach theorem, let $\bar{x}^* \in X^*$ be a norm-preserving extension of $x^*|_Z$. Then $\bar{x}^* \in N_K(x_0) \cap J(x - x_0)$. Hence by Proposition 2.1, $x_0 = P_K(x)$. Let $x_k = P_{C \cap S_k}(x)$. Then $x - x_k \in Z$ because $x_0 - x_k$, $x - x_0 \in Z$. Moreover,

$$\|x_k\| \le \|x_k - x\| + \|x\| \le \|x - x_0\| + \|x\|;$$

hence $\{x_k\} \subset C$ is bounded. Without loss of generality, assume that $x_k \to \bar{x}$ for some $\bar{x} \in C$. Let $i \in I$. By (ii) and (iii), there exists $\{i_k\} \subseteq I$ with $i_k \in I_k$ for each k such that $i_k \to i$. Noting that $x_k \in (\operatorname{aff} C) \cap C_{i_k}$ and that $x_k \to \bar{x}$, we have that $\bar{x} \in (\operatorname{aff} C) \cap C_i$ by the upper Kuratowski semicontinuity assumed in (b). This shows that $\bar{x} \in K$. Because

$$\|x - \bar{x}\| = \lim_{k \to \infty} \|x - x_k\| \le \|x - y\| \quad \text{for each } y \in K,$$

$\bar{x} = P_K(x) = x_0$ and hence $x_k \to x_0$. On the other hand, by Proposition 2.1, there exists $x_k^* \in N_{C \cap S_k}(x_k) \cap J(x - x_k)$. Consequently, $\{x_k^*\}$ is bounded since

$\|x_k^*\| = \|x - x_k\|$. Moreover, by the smoothness of the norm in Z, the mapping $z \mapsto J(z)|_Z$ is norm-weak* continuous. Hence $x_k^*|_Z \to x^*|_Z$ as $x - x_k \to x - x_0$ in Z. Thus (5.3) holds.

By assumption, the finite subsystem $\{C, C_i : i \in I_k\}$ has the strong CHIP at x_k, and hence

$$x_k^* \in N_{C \cap S_k}(x_k) = N_C(x_k) + \sum_{i \in I_k} N_{C_i}(x_k). \tag{5.4}$$

If there exists a subsequence $\{k_j\}$ of $\{k\}$ such that $x_{k_j}^* \in N_C(x_{k_j})$ for each j, then $x^* \in N_C(x_0)$ by (5.3) (and so $x^* \in N_C(x_0) + \sum_{i \in I} N_{C_i}(x_0)$). Therefore we may assume that for each k, $x_k^* \notin N_C(x_k)$. Recalling that the dimension of Z is l, it follows from (5.4) and [27, Corollary 17.1.2] that there exist $z_k^* \in N_C(x_0)$, $i_j^k \in I_k$, and $\widehat{y}_{i_j^k}^* \in N_{C_{i_j^k}}(x_0)$ such that

$$x_k^* = z_k^* + \sum_{j=1}^{s} \widehat{y}_{i_j^k}^* \quad \text{on } Z \quad \text{for all } k = 1, 2, \ldots, \tag{5.5}$$

where $s \le l$ if X is real and $s \le 2l$ if X is complex. Without loss of generality, assume that $\widehat{y}_{i_j^k}^*|_Z \neq 0$ for each $j = 1, 2, \ldots, s$. Set $\lambda_j^k = \|\widehat{y}_{i_j^k}^*|_Z\|$, $y_{i_j^k}^* = \frac{\widehat{y}_{i_j^k}^*}{\lambda_j^k}$. Then $\lambda_j^k > 0$ for $j = 1, 2, \ldots, s$ and

$$x_k^* = z_k^* + \sum_{j=1}^{s} \lambda_j^k y_{i_j^k}^* \quad \text{on } Z \quad \text{for all } k = 1, 2, \ldots. \tag{5.6}$$

Let $\lambda^k := \sum_{j=1}^{s} \lambda_j^k$. Then $\{\lambda^k\}$ is bounded. Indeed, if not, by considering a subsequence if necessary, we have that $\lim_{k\to\infty} \lambda^k = +\infty$. Thus $\frac{x_k^*}{\lambda^k} \to 0$ as $k \to \infty$. Furthermore, without loss of generality, we may assume that as $k \to \infty$,

$$i_j^k \to i_j \quad \text{and} \quad \frac{\lambda_j^k}{\lambda^k} \to \mu_j, \quad j = 1, 2, \ldots, s. \tag{5.7}$$

Then $\sum_{j=1}^{s} \mu_j = 1$. Since $\{y_{i_j^k}^*\}$ is bounded, by (5.6), $\{\frac{z_k^*}{\lambda^k}|_Z\}$ is bounded too. Thus, we may also assume that there exist $\widetilde{z}_0^*, \widetilde{y}_{i_j}^* \in Z^*$ such that

$$\frac{z_k^*}{\lambda^k} \to \widetilde{z}_0^* \quad \text{and} \quad y_{i_j^k}^* \to \widetilde{y}_{i_j}^* \quad \text{on } Z \tag{5.8}$$

as $k \to \infty$. Then, since $z_k^* \in N_C(x_k)$, by (5.8) and the fact that $x_k \to x_0$,

$$\langle \widetilde{z}_0^*, z - x_0 \rangle \le 0 \quad \text{for each } z \in C. \tag{5.9}$$

Let $z \in (Z + x_0) \cap C_{i_j}$. Since $i_j^k \to i_j$ and thanks to assumption (b), there exists $\{z_k\}$ with each $z_k \in (\operatorname{aff} C) \cap C_{i_j^k}$ such that $z_k \to z$ as $k \to \infty$. Then $\langle y_{i_j^k}^*, z_k - x_k \rangle \le 0$. Since $x_k \to x_0$, it follows from (5.8) that for each j,

$$\langle \widetilde{y}_{i_j}^*, z - x_0 \rangle \le 0 \quad \text{for each } z \in (Z + x_0) \cap C_{i_j}. \tag{5.10}$$

Consequently, by the Hahn–Banach theorem, $\widetilde{z}_0^*$ and $\widetilde{y}_{i_j}^*$ can be extended to $z_0^* \in N_C(x_0)$ and $y_{i_j}^* \in N_{C_{i_j} \cap (Z+x_0)}(x_0)$. Clearly, by (5.6), (5.7), and (5.8),

$$0 = z_0^* + \sum_{j=1}^{s} \mu_j y_{i_j}^* \quad \text{on } Z. \tag{5.11}$$

By assumption (c), there exists $\bar{y} \in C \cap \operatorname{rint}_C C_{i_j}$ for each $j = 1, 2, \ldots, s$. Then, by (5.10), $\langle y_{i_j}^*, \bar{y} - x_0 \rangle \le 0$ for each $j = 1, 2, \ldots, s$. We claim that each of the above inequalities must be strict. Indeed, suppose otherwise that $\langle y_{i_j}^*, \bar{y} - x_0 \rangle = 0$ for some j. Let $z \in Z$ and let $z_t := t(z + x_0) + (1-t)\bar{y}$. Then for any t with $|t|$ small enough, $z_t \in (\operatorname{aff} C) \cap C_{i_j}$ and thus, by (5.10),

$$t\langle y_{i_j}^*, z \rangle = \langle y_{i_j}^*, z_t - x_0 \rangle - (1-t)\langle y_{i_j}^*, \bar{y} - x_0 \rangle = \langle y_{i_j}^*, z_t - x_0 \rangle \le 0.$$

This implies that $\langle y_{i_j}^*, z \rangle = 0$, that is, $y_{i_j}^*|_Z = 0$, which contradicts that $\|y_{i_j}^*|_Z\| = \|\widetilde{y}_{i_j}^*\| = 1$. Hence,

$$\left\langle z_0^* + \sum_{j=1}^{s} \mu_j y_{i_j}^*, \bar{y} - x_0 \right\rangle < 0,$$

which contradicts (5.11). This shows that $\{\lambda^k\}$ is bounded. Note that $N_{(Z+x_0) \cap C_{i_j}}(x_0) \subseteq N_C(x_0) + N_{C_{i_j}}(x_0)$. Thus, taking the limits on the two sides of (5.6) and using the similar arguments as above (if necessary, using subsequences), we get that

$$x^* = z_0^* + \sum_{j=1}^{s} \lambda_j y_{i_j}^* \quad \text{on } Z \tag{5.12}$$

for some $\lambda_j \ge 0$, $z_0^* \in N_C(x_0)$, and $y_{i_j}^* \in N_{C_{i_j}}(x_0)$ $(j = 1, 2, \ldots, s)$. Let $y^* = x^* - z_0^* - \sum_{j=1}^{s} \lambda_j y_{i_j}^*$. Then $y^* \in N_C(x_0)$ by (5.12) and thus $x^* \in N_C(x_0) + \sum_{i \in I} N_{C_i}(x_0)$. The proof is complete. □

COROLLARY 5.1. *Suppose that the CCS-system* $\{C, C_i : i \in I\}$ *satisfies the interior-point condition,* $\dim C < +\infty$*, and the set-valued function* $i \mapsto (\operatorname{aff} C) \cap C_i$ *is Kuratowski continuous. Then the system* $\{C, C_i : i \in I\}$ *has the strong CHIP.*

Proof. The assumed interior-point condition clearly implies (c) of Theorem 5.1; it also implies that each of the finite subsystems of $\{C, C_i : i \in I\}$ has the strong CHIP by Theorem DLW. Hence the conclusion holds by Theorem 5.1. □

Remark 5.1. Examples 5.1, 5.2, and 5.3 will show that none of the conditions (a), (b), and (c) in Theorem 5.1 can be dropped. Each of these examples will be a CCS-system without the strong CHIP, but each of the finite subsystems of each of these CCS-systems does have the strong CHIP (each C_i being a polyhedron, and the base-set being the whole space). In each of these examples, I is the compact subset of $\mathbb{R}$ defined by $I = \{0, 1, \frac{1}{2}, \ldots, \frac{1}{i}, \ldots\}$.

Example 5.1. Let $C = X = \{x = (x_1, x_2, \ldots, x_k, \ldots) : x_k \in \mathbb{R}, \lim_k x_k \text{ exists}\}$ with the norm defined by

$$\|x\| = \sup_k |x_k|, \quad x = (x_k) \in X.$$

Define

$$C_i = \begin{cases} \{x = (x_k) \in X : \lim_k x_k \le 0\}, & i = 0, \\ \{x = (x_k) \in X : x_{\frac{1}{i}} \le 0\}, & i \in I \setminus \{0\}. \end{cases}$$

Then the set-valued function $i \mapsto C_i$ is Kuratowski continuous on I. In fact, let $\{i_n\} \subseteq I$ be a sequence satisfying $i_n \to 0$. To show the Kuratowski continuity at 0, let $\{x^{i_n}\}$ be a sequence satisfying $x^{i_n} \in C_{i_n}$ and $x^{i_n} \to x^0$. Noting

$$x^0_{\frac{1}{i_n}} = x^0_{\frac{1}{i_n}} - x^{i_n}_{\frac{1}{i_n}} + x^{i_n}_{\frac{1}{i_n}} \le x^0_{\frac{1}{i_n}} - x^{i_n}_{\frac{1}{i_n}} \le \|x^0 - x^{i_n}\| \to 0,$$

one has that $\lim_k x^0_k \le 0$ and thus $x^0 \in C_0$. This proves that the set-valued function $i \mapsto C_i$ is upper Kuratowski continuous at 0. To show the lower Kuratowski continuity at 0, let $x^0 \in C_0$. Then $a = \lim_k x^0_k \le 0$. Define $x^{i_n} = (x^{i_n}_k)$ with $x^{i_n}_k = x^0_k$ if $k \ne \frac{1}{i_n}$ and $x^{i_n}_k = a$ if $k = \frac{1}{i_n}$. Then $x^{i_n} \in C_{i_n}$ and $\lim_n x^{i_n} = x^0$. This shows that the set-valued function $i \mapsto C_i$ is lower Kuratowski continuous at 0. Note that $\bar{x} \in \operatorname{int}(\bigcap_{i\in I} C_i)$ for $\bar{x} = (-1, -1, \ldots) \in X$. Hence the conditions (b) and (c) in Theorem 5.1 are satisfied. Let $x_0 = 0$. Then $x_0 \in \bigcap_{i\in I} C_i$. It is easy to see that $\sum_{i\in I} N_{C_i}(x_0)$ is not closed; hence this system does not have the strong CHIP at x_0. □

Example 5.2. Let $C = X = \mathbb{R}^2$. Define

$$C_i = \begin{cases} \{x = (x_1, x_2) \in X : \; x_1 + x_2 \le 0\}, & i = 0, \\ \{x = (x_1, x_2) \in X : \; x_1 + i x_2 \le 0\}, & i \in I \setminus \{0\}. \end{cases}$$

Then $\bigcap_{i\in I} C_i = \{(x_1, x_2) : \; x_1 \le 0, \, x_1 + x_2 \le 0\}$. Let $x_0 = 0$. Then $x_0 \in \operatorname{bd} \bigcap_{i\in I} C_i$. Clearly, (a) in Theorem 5.1 is satisfied. Since $\bar{x} = (-1, \frac{1}{2}) \in \operatorname{int}(\bigcap_{i\in I} C_i)$, condition (c) in Theorem 5.1 is satisfied too. However, $N_{\bigcap_{i\in I} C_i}(x_0) = \{(t_1, t_2) : \; 0 \le t_2 \le t_1\}$ and $\sum_{i\in I} N_{C_i}(x_0) = \{(t_1, t_2) : \; 0 < t_2 \le t_1\} \cup \{(0, 0)\}$. Therefore this system does not have the strong CHIP at x_0. Note that condition (b) is not satisfied. □

Example 5.3. Let $C = X = \mathbb{R}^2$ and define

$$C_i = \begin{cases} \{x = (x_1, x_2) : \; x_2 \le 0\}, & i = 1, \\ \{x = (x_1, x_2) : \; i x_1 - x_2 - i^2 \le 0\}, & i \in I \setminus \{1\}. \end{cases}$$

Then $\bigcap_{i\in I} C_i = \{(x_1, 0) \in \mathbb{R}^2 : \; x_1 \le 0\}$. Let $x_0 = 0$. Hence

$$N_{\bigcap_{i\in I} C_i}(x_0) = \{(t_1, t_2) \in \mathbb{R}^2 : \; t_1 \ge 0\}$$

and

$$\sum_{i\in I} N_{C_i}(x_0) = \operatorname{cone}\{(0, -1), \, (0, 1)\} = \{(t_1, t_2) \in \mathbb{R}^2 : \; t_1 = 0\}.$$

Consequently, this system does not have the strong CHIP at x_0. Note that conditions (a) and (b) in Theorem 5.1 are satisfied but condition (c) is not. □

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