

On relationship between convergence ball of Euler iteration in Banach spaces and its dynamical behavior on Riemann spheres

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Abstract The relationship between the convergence ball of the Euler iteration in Banach Spaces and its exclusive fixed points on Riemann spheres is investigated. By using an exclusive fixed point of the Euler iteration, the convergence ball is determined accurately for a class of operators whose derivatives satisfy some generalized Lipschitz condition on Banach spaces.

Keywords: solution of operator equation, the Euler iteration, convergence ball, exclusive fixed point, complex analytic dynamical system.

In ref. [1], we discovered that, for complex polynomials, the Euler iteration has non-exclusive extraneous fixed points of every type, and, in particular, adjoint to these fixed points is the Fatou domain, that is, the super-attractive domain, attractive domain or the parabolic domain in the Sullivan basin. Hence the phenomenon of “numerical extraneous root” appears and this causes much inconvenience to numerical computation. Only from this point, it seems that exclusive extraneous fixed points would have no influence on numerical computation. But now we have found that, under the quite extensive condition, the convergence domain of the Euler iteration in Banach spaces is closely relative to its exclusive extraneous fixed points on Riemann spheres. In this paper, using an exclusive fixed point of the Euler iteration, we will determine accurately its convergence balls for a class of operators whose derivatives satisfy some generalized Lipschitz condition on Banach spaces.

Let us look at the convergence ball for the Newton iteration in this viewpoint^[2-6]. We will see that it has remarkable difference from the Euler iteration. In fact, the convergence ball for the Newton iteration is dependent upon its periodic orbit of period 2 for a man-made real function on the reals. Because the man-made function can generally not be extended into an analytic function on the Riemann sphere, it has no relation with complex analytic dynamical system. In ref. [7], the determining of convergence balls was considered as the local behavior of iterations. The distinction of radiuses r_U , r_N , r_E of several balls shows that the local behavior of iteration presented in ref. [7] is short of unity under the quite extensive condition (This was stated for the Euler iteration in ref. [7] according to the result of ref. [8] under the special condition given in ref. [9]). This is in strong contrast to the unity of the semi-local behavior showed in refs. [9-12]

under the corresponding condition.

In the remainder of the paper, we always assume that f is a nonlinear operator from a region D in a Banach space X , real or complex, to another Banach space of the same type and that the zero x^* of f belongs to D . For convenience, with no loss of generality, we assume that D is some ball $B(x^*, r)$ with center x^* and radius r . Furthermore, we still assume that $f'(x^*)^{-1}$ exists and that $f'(x^*)^{-1}f'$ or $f'(x^*)^{-1}f''$ satisfies various generalized, weaker or stronger, Lipschitz conditions in the ball $B(x^*, r)$, see for example ref. [2]. But it should be noted that the stronger condition implies the weaker condition completely in some proper way. In addition, we always write $\rho(x) = \|x - x^*\|$.

The Euler iteration E_f is defined as follows:

$$E_f(x) = x - (I - P_f(x))f'(x)^{-1}f(x),$$

where

$$P_f(x) = -\frac{1}{2}f'(x)^{-1}f''(x)f'(x)^{-1}f(x).$$

1 One periodic orbit of period 2 in \mathbb{R} determining the convergence ball of the Newton iteration

Let us recall simply some related results of ref. [2]. Let L be a positive integrable function on the interval $[0, T]$, where T is a sufficient large positive number. For $t \in [0, T]$, set

$$h(t) = -t + \int_0^t (t - u)L(u)du.$$

Obviously, the derivative of $h(t)$

$$h'(t) = -1 + \int_0^t L(u)du$$

has a unique zero in $[0, T]$, say r_0 , which satisfies that

$$\int_0^{r_0} L(u)du = 1.$$

In addition, when t increases from 0 to r_0 , $h(t)$ decreases from 0 to the negative number $h(r_0)$ while when t increases from r_0 to $+\infty$, $h(t)$ does from $h(r_0)$ to $+\infty$. Hence, $h(T) > 0$ if T is large enough. This implies that $h(t)$ has a unique zero in $(0, T)$, say r_U , which satisfies that

$$\int_0^{r_U} (r_U - u)L(u)du = r_U.$$

Theorem 1.1. Suppose that f satisfies that

$$\|f'(x^*)^{-1}(f'(x) - f'(x^*))\| \leq \int_0^{\rho(x)} L(u)du, \quad \forall x \in B(x^*, r_U).$$

Then x^* is the unique zero of f in $B(x^*, r_U)$.

Theorem 1.2. Under the assumption of Theorem 1.1, for all $x \in B(x^*, r_U)$, $f'(x)^{-1}$ exists and satisfies

$$\|f'(x)^{-1}f'(x^*)\| \leq \left(1 - \int_0^{\rho(x)} L(u)du\right)^{-1} = -h'(t)^{-1},$$

where $t = \rho(x)$.

Furthermore, assume that L is non-decreasing in $[0, r_0)$. We extend h into an odd function \bar{h} on $(-r_0, r_0)$. Then the Newton iteration $N_{\bar{h}}(t)$ for \bar{h} has a unique periodic point $r_N \in (0, r_0)$ of period 2. Clearly, r_N satisfies

$$\frac{\int_0^{r_N} uL(u)du}{r_N \left(1 - \int_0^{r_N} L(u)du\right)} = 1.$$

It is just the periodic orbit of period 2 in $\mathbb{R} \left\{N_{\bar{h}}^n(r_N)\right\}$ that determine the convergence ball of the Newton iteration for the corresponding family of operators.

Theorem 1.3. Suppose that f satisfies that

$$\|f'(x^*)^{-1}(f'(x) - f'(x^\tau))\| \leq \int_{\tau\rho(x)}^{\rho(x)} L(u)du, \quad \forall x \in B(x^*, r_N),$$

where $x^\tau = x^* + \tau(x - x^*)$, $0 \leq \tau \leq 1$. Then for any $x_0 \in B(x^*, r_U)$, the sequence of the Newton iteration $\{N_f^n(x_0)\}$ converges to x^* and satisfies that

$$\|N_f^n(x_0) - x^*\| \leq q^{2^n - 1} \|x_0 - x^*\|, \quad n = 1, 2, \dots,$$

where

$$q = \frac{\int_0^{t_0} uL(u)du}{t_0 \left(1 - \int_0^{t_0} L(u)du\right)} < 1, \quad t_0 = \rho(x_0).$$

2 One extraneous fixed point on \bar{C} determining the convergence ball of the Euler iteration

Furthermore, we assume that L has both non-decreasing and positive derivative L' in the interval $[0, r_0)$ and $L(0) > 0$. In particular, L may be taken to be polynomials of degree more than or equal to 1 with positive coefficients or to be analytic functions with positive Maclaurin coefficients.

Consider the function

$$P_h(t) = -\frac{1}{2} \frac{h''(t)h(t)}{h'(t)^2}.$$

Clearly, when t increases monotonically from 0 to r_0 , $P_h(t)$ does from 0 to $+\infty$ so that $1 - P_h(t)$ has a unique zero r_E in $(0, r_0)$. It is easy to see that $t = r_E$ is the unique fixed point of the Euler iteration $E_h(t)$ in $(0, r_0)$. Since $h(r_E) \neq 0$, it is an extraneous fixed point (The fixed point $p \in X$ of the Euler iteration $E_f(x)$ is called an extraneous fixed point, if p is not a zero of f). Using the fact that $P_h(t) = 1$ at $t = r_E$, we obtain

$$\frac{dE_h(t)}{dt} = \frac{h(t)^2}{2h'(t)^4} (3h''(t)^2 - h'''(t)h'(t)).$$

Observe that $h'(t) < 0$, $h'''(t) = L'(t) > 0$. It follows that its characteristic value

$$\lambda = \frac{dE_h(t)}{dt} > 6 \left(\frac{h(t)h''(t)}{2h'(t)^2} \right)^2 = 6P_h(t)^2 = 6.$$

This means that the extraneous fixed point $t = r_E$ is exclusive.

Theorem 2.1. The Euler iteration for h has a unique fixed point r_E in the interval $(0, r_0)$, which is an exclusive extraneous fixed point. In addition, $r_E < r_N$.

Proof. It suffices to verify that $r_E < r_N$. Note that, for $t = r_N$,

$$\frac{h(t)}{h'(t)} = 2t, \quad h'(t) = -\frac{1}{t} \int_0^t uL(u)du.$$

Substituting this into $P_h(t)$, we have that

$$P_h(t) = -\frac{1}{2} \frac{h''(t)}{h'(t)} \frac{h(t)}{h'(t)} = \frac{t^2 L(t)}{\int_0^t uL(u)du} > 2.$$

The proof is complete.

In ref. [1], we studied the influence of non-exclusive extraneous fixed points upon the solving of an algebra equation. In the present paper, we will investigate influence of exclusive extraneous fixed points upon the solving of an operator equation in a Banach space. We have the following theorem.

Theorem 2.2. Suppose that f satisfies that

$$\|f'(x^*)^{-1}f''(x^*)\| \leq L(0)$$

and

$$\|f'(x^*)^{-1}(f''(x) - f''(x^\tau))\| \leq \int_{\tau\rho(x)}^{\rho(x)} L'(u)du, \quad \forall x \in B(x^*, r_E),$$

where $x^\tau = x^* + \tau(x - x^*)$, $0 \leq \tau \leq 1$. Then, for all $x_0 \in B(x^*, r_E)$, the Euler iteration $\{E_f^n(x_0)\}$ converges to x^* and satisfies that

$$\|E_f^n(x_0) - x^*\| \leq q^{3^n - 1} \|x_0 - x^*\|, \quad n = 1, 2, \dots,$$

where

$$q = \sqrt{\frac{E_h(t_0)}{t_0}} < 1, \quad t_0 = \rho(x_0).$$

Furthermore, the radius r_E of the convergence ball is optimal as a constant that only depends upon L but not f .

In order to verify the theorem we need the following lemma.

Lemma 2.1. Suppose that $L'(t) > 0$ is increasing monotonically. Then so is the function

$$\frac{1}{t} \int_0^1 \tau d\tau \int_{\tau t}^t L'(u)du.$$

Proof. It suffices to show that the function $\frac{1}{t} \int_{\tau t}^t L'(u)du$ is increasing monotonically in $[0, 1]$. For the end, take $t_1, t_2 \in [0, 1]$, $0 < t_1 < t_2$. If $0 < t_1 < \tau t_2$, then

$$\left(\frac{1}{t_2} \int_{\tau t_2}^{t_2} - \frac{1}{t_1} \int_{\tau t_1}^{t_1}\right) L'(u)du \geq L'(t_1) \left(\frac{1}{t_2} \int_{\tau t_2}^{t_2} - \frac{1}{t_1} \int_{\tau t_1}^{t_1}\right) du = 0.$$

Otherwise,

$$\begin{aligned} \left(\frac{1}{t_2} \int_{\tau t_2}^{t_2} - \frac{1}{t_1} \int_{\tau t_1}^{t_1}\right) L'(u)du &= \left(\frac{1}{t_2} \int_{t_1}^{t_2} - \left(\frac{1}{t_1} - \frac{1}{t_2}\right) \int_{\tau t_2}^{t_1} - \frac{1}{t_1} \int_{\tau t_1}^{\tau t_2}\right) L'(u)du \\ &\geq L'(t_1) \left(\frac{1}{t_2} \int_{t_1}^{t_2} - \left(\frac{1}{t_1} - \frac{1}{t_2}\right) \int_{\tau t_2}^{t_1} - \frac{1}{t_1} \int_{\tau t_1}^{\tau t_2}\right) du \\ &= 0. \end{aligned}$$

This completes the proof of the lemma.

Now let us prove Theorem 2.2. It is easy to see that, under the assumption of the theorem, f satisfies that

$$\|f'(x^*)^{-1}f''(x)\| \leq L(t) = h''(t),$$

where $t = \rho(x)$. This implies that f satisfies the assumption of Theorem 1.3 on $B(x^*, r_E)$. On the other hand, for any $x \in B(x^*, r_E)$,

$$\begin{aligned} E_f(x) - x^* &= x - x^* - f'(x)^{-1}f(x) - \frac{1}{2}f'(x)^{-1}f''(x)(x - x^*)^2 \\ &\quad + \frac{1}{2}f'(x)^{-1}f''(x)(x - x^* - f'(x)^{-1}f(x))(x - x^*) \\ &\quad + \frac{1}{2}f'(x)^{-1}f''(x)f'(x)^{-1}f(x)(x - x^* - f'(x)^{-1}f(x)) \\ &= -f'(x)^{-1} \int_0^1 \tau(f''(x) - f''(x^\tau))d\tau(x - x^*)^2 \\ &\quad + \frac{1}{2}f'(x)^{-1}f''(x)f'(x)^{-1} \int_0^1 \tau f''(x^\tau)d\tau(x - x^*)^3 \\ &\quad + \frac{1}{2}f'(x)^{-1}f''(x) \left(1 - f'(x)^{-1} \int_0^1 \tau f''(x^\tau)d\tau(x - x^*)\right) (x - x^*) \\ &\quad \cdot f'(x)^{-1} \int_0^1 \tau f''(x^\tau)d\tau(x - x^*)^2. \end{aligned}$$

Hence, we have that, for any $x \in B(x^*, r_E)$,

$$\begin{aligned} \|E_f(x) - x^*\| &\leq -h'(t)^{-1} \int_0^1 \tau d\tau \int_{\tau t}^t L'(u)du \cdot t^2 \\ &\quad + \frac{1}{2}(-h'(t))^{-2}L(t) \int_0^1 \tau L(\tau t)d\tau \cdot t^3 \\ &\quad + \frac{1}{2}(-h'(t))^{-1}L(t) \left(1 - h'(x)^{-1}d \int_0^1 \tau L(\tau t)d\tau \cdot t\right) \cdot t \\ &\quad \cdot (-h'(x))^{-1}d \int_0^1 \tau L(\tau t)d\tau \cdot t^2 \\ &= \frac{E_h(t)}{t^3} \cdot t^3. \end{aligned}$$

By Lemma 2.1, $\frac{E_h(t)}{t^3}$ is increasing monotonically. Thus, if $x_0 \in B(x^*, r_E)$ and $t = \rho(x) \leq t_0 = \rho(x_0)$,

$$\|E_f(x) - x^*\| \leq \left(\frac{q}{t_0}\right)^2 t^3.$$

From this, we can easily establish the inequality in the theorem by mathematical induction.

Finally, note that the positive non-decreasing function L' on $[0, r_0)$ can be extended into a function on $(-r_0, r_0)$ in any way such that the function is non-decreasing on $(-r_0, r_0)$ and takes the same values as the original function on $(-r_0, 0]$. Thus, $f = h$ is a function defined on $(-r_0, r_0)$ and satisfies the condition of the theorem with $X = \mathbb{R}$, $x^* = 0$. Since, $E_h^n(r_E) = r_E$, it does not converge to $x^* = 0$. This shows that the radius of the convergence ball r_E is optimal. The proof of Theorem 2.2 is complete.

Remark 2.1. The function $L : [0, r_0) \rightarrow (0, +\infty)$ in this section is required to have positive non-decreasing derivative L' . This property, as shown in the proof of Theorem 2.2, makes it possible that the function h is extended into a function f defined on $(-r_0, r_0)$ such that the condition of Theorem 2.2 is satisfied with $X = \mathbb{R}$, $x^* = 0$. In fact, in many common cases, h can be naturally extended into an analytic function on \mathbb{C} , for example, when L is polynomials of degree more than or equal to 1 with positive coefficients or analytic functions with positive Maclaurin coefficients. Hence, r_E is an exclusive fixed point of a rational or analytic iteration, which belongs to its Julia set. However, it should be pointed out that the point r_N in the previous section is only a periodic point of period 2 of the Newton iteration for the man-made function \bar{h} on \mathbb{R} and, in general, \bar{h} cannot be extended into an analytic function on \mathbb{C} . In fact, if the odd function \bar{h} can, L must be odd too so that $L(0) = 0$. Consequently, the operator family majorized by L (i.e., the family of operators satisfying the condition of Theorem 1.3) is a quite poor one, which has no practical meaning.

Remark 2.2. If the constant $L(0)$ and the function L' in the condition of Theorem 2.2 are, respectively, replaced by a general constant γ and a general positive non-decreasing function L_1 , then the condition of Theorem 2.2 implies the one of Theorem 1.3 on $B(x^*, r)$ ($0 < r \leq r_0$) if and only if $\gamma \leq L(0)$ and $L_1(t) \leq L'(t), \forall t \in [0, r)$. This just means that the stronger condition implies completely the weaker one in some proper way stated in the introduction. According to the difference of r_U, r_N, r_E , we claim that the local behaviors of iterations are short of unity.

3 Some special majorizing operators and pictures of the Euler iteration

The assumption of Theorem 2.2 is quite general. For the weak Kantorovich's condition

$$\begin{cases} \|f'(x^*)^{-1}f''(x^*)\| \leq \gamma \\ \|f'(x^*)^{-1}(f''(x) - f''(x^\tau))\| \leq L' \cdot (1 - \tau)\|x - x^*\|, \end{cases}$$

where L' is a positive constant, we can take $L(u) = \gamma + L'u$ in Theorem 2.2 and then,

$$h(t) = -t + \frac{\gamma}{2}t^2 + \frac{L'}{6}t^3,$$

while for the Smale's condition^[5,6]

$$\|f'(x^*)^{-1}f^{(k)}(x^*)\| \leq k!\gamma^{k-1}, \quad k = 1, 2, \dots,$$

or the weak Smale's condition^[9], we can take

$$L(u) = \frac{2\gamma}{(1 - \gamma u)^3},$$

and then

$$h(t) = -t + \frac{\gamma t^2}{(1 - \gamma t)}.$$

Both of the two majorizing operators h can be extended naturally into rational functions from $\bar{\mathbb{C}}$ to $\bar{\mathbb{C}}$. In particular, we investigate Fatou sets and Julia sets of the Euler iteration for the later majorizing operator.

Without loss of generality, assume that $\gamma = 1$. Then we have that

$$E_h(z) = z - \frac{h(z)}{h'(z)^3} = \frac{4z^4 - 16z^3 + 22z^2 - 9z + 1}{(1 - z)^4}.$$

Hence, the Euler iteration has four extraneous fixed points, which are four zeroes of the polynomial $4z^4 - 16z^3 + 22z^2 - 9z + 1$: $0.181462390 \dots, 0.400820747 \dots, 1.708858431 \dots \pm i \cdot 0.719020217 \dots$. Substituting them into the following equality:

$$\frac{d}{dz} E_h(z) = \frac{h(z)^2}{h'(z)^4} \frac{3}{(1-z)^6} (3 - 4z + 2z^2),$$

we can easily obtain their characteristic values. Clearly, the four extraneous fixed points are all exclusive. From the above equality, we can also get the free critical points $1 \pm i\sqrt{\frac{1}{2}}$ of the Euler iteration (Each zero of f is of course free critical points of $E_f(x)$ and it belongs to its attractive domain. Any other critical point is called a free critical point). Because they all belong to the attractive domain of the zero $z^* = 0$ of h , the Julia set, by refs. [13—15], is non-dense, what's more, the Fatou set consists of the two attractive domains of the zero $z^* = 0$ and another zero $z^{**} = \frac{1}{2}$ of h . Under the Euler iteration $E_h(z)$, the two attractive domains are completely invariant, respectively, while the Julia set is just their common boundary. Note that $E_h(\infty) = -\frac{1}{2}$ belongs to the attractive domain of $z^* = 0$ so that the Julia set is compact in Gauss plane \mathbb{C} .

Of course, the four extraneous fixed points belong to the Julia set but the value of the smaller positive fixed point determines the radius of the convergence ball $B(x^*, r_E)$, see plate I. The size of the window of this figure is $[-1, 2] \times [-1.5, 1.5]$. The green region is the attractive domain of $z^* = 0$, the red region the one of $z^{**} = \frac{1}{2}$ while the yellow disk the convergence ball determined by r_E and r_E is on the boundary of the two regions, which is showed better by the small figure in the corner. From the curvature of the yellow boundary, we can directly see how large part of the big figure is enlarged into the small figure, the size of the window of which is $[0.180, 0.183] \times [-0.0015, 0.0015]$.

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