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Newton's method for sections on Riemannian manifolds: Generalized covariant α-theory☆

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Abstract

One kind of the L-average Lipschitz condition is introduced to covariant derivatives of sections on Riemannian manifolds. A convergence criterion of Newton's method and the radii of the uniqueness balls of the singular points for sections on Riemannian manifolds, which is independent of the curvatures, are established under the assumption that the covariant derivatives of the sections satisfy this kind of the L-average Lipschitz condition. Some applications to special cases including Kantorovich's condition and the γ-condition as well as Smale's α-theory are provided. In particular, the result due to Ferreira and Svaiter [Kantorovich's Theorem on Newton's method in Riemannian manifolds, J. Complexity 18 (2002) 304–329] is extended while the results due to Dedieu Priouret, Malajovich [Newton's method on Riemannian manifolds: covariant alpha theory, IMA J. Numer. Anal. 23 (2003) 395–419] are improved significantly. Moreover, the corresponding results due to Alvarez, Bolter, Munier [A unifying local convergence result for Newton's method in Riemannian manifolds, Found. Comput. Math. to appear] for vector fields and mappings on Riemannian manifolds are also extended.

Keywords: Newton's method; Riemannian manifold; Section; Convergence criterion; Approximate singular point

1. Introduction

Recently, there has been an increased interest in studying numerical algorithms on manifolds for there are a lot of numerical problems posed in manifolds arising in many natural contexts.

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Classical examples are given by eigenvalue problems, symmetric eigenvalue problems, invariant subspace computations, optimization problems with equality constraints, etc., see for example [1,8,12,23–25]. For such problems, one often has to compute solutions of a system of equations or to find singular points of a vector field on a Riemannian manifold.

In a vector space framework, the most famous method to approximately solve a nonlinear differentiable equation $F(x) = 0$ is Newton's method, where F is a differentiable mapping from a Banach space E to another Y. As is well-known, one of the most important results on Newton's method is Kantorovich's theorem (cf. [16,17]). Under the mild condition that the second Frechet derivative of F is bounded (or more general, the first derivative is Lipschitz continuous) on a proper open metric ball of the initial point x_0, Kantorovich's theorem provides a simple and clear criterion, based on the knowledge of the first derivative around the initial point, ensuring the existence, uniqueness of the solution of the equation and the quadratic convergence of Newton's method. Another important result on Newton's method is Smale's point estimate theory (i.e., α-theory and γ-theory) in [21], where the notion of an approximate zero was introduced and the rules to judge an initial point x_0 to be an approximate zero were established, depending on the information of the analytic nonlinear operator at this initial point and a solution x^*, respectively. There are a lot of works on the weakness and/or extension of the Lipschitz continuity made on the mapping F, see for example, [9,10,14,15,28,33] and references therein. In particular, Zabrejko–Nguen parametrized in [33] the classical Lipschitz continuity. Wang introduced in [28] the notion of Lipschitz conditions with L-average to unify both Kantorovich's and Smale's criteria.

In a Riemannian manifold framework, an analogue of the well-known Kantorovich's theorem was given in [11] for Newton's method for vector fields on Riemannian manifolds while the extensions of the famous Smale's α-theory and γ-theory in [21] to analytic vector fields and analytic mappings on Riemannian manifolds were done in [6], where the convergence criteria depending on the injective radius of the exponential map were presented. In the recent paper [19], the convergence criteria in [6] were improved by using the notion of the γ-condition for the vector fields and mappings on Riemannian manifolds. The radii of uniqueness balls of singular points of vector fields satisfying the γ-conditions were estimated in [26], while the local behavior of Newton's method on Riemannian manifolds was studied in [18]. These results are still dependent on the injective radius of the exponential map. Recently, inspired by previous work of Zabrejko and Nguen in [33] on Kantorovich's majorant method, Alvarez et al. introduced in [2] a Lipschitz-type radial function for the covariant derivative of vector fields and mappings on Riemannian manifolds, and established a unified convergence criterion of Newton's method on Riemannian manifolds, applications of which to analytic vector fields and mappings give first a curvature-free generalization of Smale's α-theory in Euclidean space setting, which improves significantly the corresponding results in [6,19].

The purpose of the present paper is to provide a unified frame which includes vector fields and mappings on Riemannian manifolds as special cases, and a unified convergence criterion which includes Kantorovich's convergence theorem and Smale's α-theory for Newton's method on Riemannian manifolds as special cases. For this purpose, we extend Newton's method and modify the notion of the Lipschitz conditions with L-average in vector spaces to suit sections on Riemannian manifolds. In the spirit of the previous work of Wang [28], our approach is based on the construction of a real-valued function, namely the majorizing function, using an adequate L-average Lipschitz condition for the covariant derivatives of sections on Riemannian manifolds. Similar techniques were also used in [2,11,18]. The key techniques used in the study of the uniqueness result are taken from [11]. Our results are completely independent of the injective radius of the exponential map, and valid for not only the Levi–Civita connection but also any affine connection on sections (in particular, the underlying bundle does not necessarily have a metric structure). In particular, when the results are applied to the Lipschitz continuous sections on Riemannian manifolds, the corresponding results in [11] are extended; while applied to analytic sections on Riemannian manifolds, Smale's α-theory in [6] is extended and improved significantly, and Smale's theory of the approximate zero is developed, which seems new even in the case when the section is a vector field and the connection is Levi–Civita connection. Furthermore, even in the case of vector fields and mappings on Riemannian manifold, the corresponding results due to Alvarez et al. in [2] are seen to be extended because the connection used here is not necessarily the Levi–Civita connection.

We end this section by describing simply the organization of the present paper. The useful definitions and preliminaries about sections on Riemannian manifolds are given in the next section. Some auxiliary results for preparations for the proofs of the main theorems are developed in Section 3. The main theorems are given in Section 4. Section 5 is devoted to the applications to the case of Kantorovich's condition and the case of the γ-condition. Applications to Smale's α-theory and Smale's theory of the approximate singular points are given in the last section.

2. Notions and preliminaries

Let $\kappa \in \mathbb{N} \cup \{\infty, \omega\}$ and let M be a complete m-dimensional C^κ-Riemannian manifold with countable bases, where C^κ means smooth or analytic in the case when $\kappa = \infty$ or ω. Let $p \in M$ and let T_pM denote the tangent space at p to M. We denote by $\langle \cdot, \cdot \rangle_p$ the scalar product on T_pM with the associated norm $\| \cdot \|_p$, where the subscript p is sometimes omitted. The tangent bundle TM of M is defined by

$$TM := \bigcup_{p \in M} T_pM.$$

Thus, a vector field X on M is a mapping from M to TM satisfying that $X(p) \in T_pM$ for each $p \in M$. For $p, q \in M$, let c: [0, 1] $\to M$ be a piecewise smooth curve connecting p and q. Then the arc-length of c is defined by $l(c) := \int_0^1 \| c'(t) \| \, dt$, while the Riemannian distance from p to q is defined by $d(p, q) := \inf_c l(c)$, where the infimum is taken over all piecewise smooth curves c: [0, 1] $\to M$ connecting p and q. Thus (M, d) is a complete metric space by the Hopf–Rinow Theorem (*cf.* [7]). Noting that M is complete, the exponential map $\exp_p : T_pM \to M$ at p is well-defined on T_pM. Recall that a geodesic c in M connecting p and q is called a minimizing geodesic if its arc-length equals its Riemannian distance between p and q. Clearly, a curve c: [0, 1] $\to M$ is a minimizing geodesic connecting p and q if and only if there exists a vector $v \in T_pM$ such that $\|v\| = d(p, q)$ and $c(t) = \exp_p(tv)$ for each $t \in [0, 1]$.

Let ∇ denote the Levi–Civita connection on M and let $c: \mathbb{R} \to M$ be a C^κ-curve. Then we use $P_{c,\cdot,\cdot}$ to denote the parallel transport on tangent bundle TM along c with respect to ∇.

In the remainder of this section, we shall describe simply the notions of sections, connections and parallel transports as well as some relative facts. For the details, the readers are referred to some text books, for example, [5,32]. Recall that $\kappa \in \mathbb{N} \cup \{\infty, \omega\}$. Throughout the whole paper, we shall always assume that E and M are C^κ-manifolds.

Definition 2.1. Let $\pi: E \to M$ be a C^κ-morphism. Then $\pi: E \to M$ is called a C^κ-vector bundle of rank $\hat{m}$ if the following conditions are satisfied.

(1) For each $p \in M$, $E_p := \pi^{-1}(p)$ is a real vector space of dimension $\hat{m}$.

(2) For each $p \in M$, there exist a neighborhood U of p and a C^κ-diffeomorphism $h: \pi^{-1}(U) \to U \times \mathbb{R}^{\hat{m}}$ such that, for each $q \in U$, $h(E_q) \subset \{q\} \times \mathbb{R}^{\hat{m}}$ and the mapping $h^q: E_p \to \mathbb{R}^{\hat{m}}$ defined by

$$h^q(x) = \mathbf{proj} \circ h(x) \quad \text{for each } x \in E_q \tag{2.1}$$

is a linear isomorphism, where $\mathbf{proj}: \{q\} \times \mathbb{R}^{\hat{m}} \to \mathbb{R}^{\hat{m}}$ is the natural projection on $\mathbb{R}^{\hat{m}}$.

Definition 2.2. Let $\pi: E \to M$ be a C^κ-vector bundle of rank $\hat{m}$ and $\xi: M \to E$ a C^κ-morphism. Then $\xi: M \to E$ is called a C^κ-section of the C^κ-vector bundle $\pi: E \to M$ if $\pi \circ \xi = \mathbf{I}_M$, where $\mathbf{I}_M$ denotes the identity on M.

The set of all C^κ-sections of the C^κ-vector bundle $\pi: E \to M$ is denoted by $C^\kappa(M, E)$. In the particular cases when $\kappa = \infty$, or ω, a C^κ-section ξ is called a smooth section or an analytic section, respectively. Let $C^\kappa(TM)$ denote the set of all the C^κ-vector fields on M and $C^\kappa(M)$ the set of all C^κ-mappings from M to $\mathbb{R}$, respectively.

Definition 2.3. Let $\pi: E \to M$ be a C^κ-vector bundle of rank $\hat{m}$. Then a mapping $\mathrm{D}: C^\kappa(M, E) \times C^\kappa(TM) \to C^{\kappa-1}(M, E)$ is called a connection on this vector bundle if, for every $X, Y \in C^\kappa(TM)$, $\xi, \eta \in C^\kappa(M, E)$, $f \in C^\kappa(M)$ and $\lambda \in \mathbb{R}$, the following conditions are satisfied:

$$\begin{aligned} &\mathrm{D}_{X+fY}\xi = \mathrm{D}_X\xi + f\mathrm{D}_Y\xi, \quad \mathrm{D}_X(\xi + \lambda\eta) = \mathrm{D}_X\xi + \lambda\mathrm{D}_X\eta \quad \text{and} \\ &\mathrm{D}_X(f\xi) = X(f)\xi + f\mathrm{D}_X\xi. \end{aligned} \tag{2.2}$$

Note that connections on the vector bundle $\pi: E \to M$ exist because M is a C^κ-Riemannian manifold with countable bases (cf. [32] for the case when $\kappa = \infty$ and its proof for the general case is similar). For any $(\xi, X) \in C^\kappa(M, E) \times C^\kappa(TM)$, $\mathrm{D}_X\xi$ is called the covariant derivative of ξ with respect to X. Since D is tensorial in X, the value of $\mathrm{D}_X\xi$ at $p \in M$ only depends on the tangent vector $v = X(p) \in T_pM$. Hence, the mapping $\mathrm{D}\xi(p): T_pM \to \pi^{-1}(p)$ given by

$$\mathrm{D}\xi(p)v := \mathrm{D}_X\xi(p) \quad \text{for each } v \in T_pM \tag{2.3}$$

is well-defined and is a linear map from T_pM to $\pi^{-1}(p)$.

Definition 2.4. Let $c: \mathbb{R} \to M$ be a C^κ-curve. For any $a, b \in \mathbb{R}$, define the mapping $\mathcal{P}_{c,c(b),c(a)}: \pi^{-1}(c(a)) \to \pi^{-1}(c(b))$ by $\mathcal{P}_{c,c(b),c(a)}(v) = \eta_v(c(b))$ for each $v \in \pi^{-1}(c(a))$, where η_v is the unique C^κ-section such that $\mathrm{D}_{c'(t)}\eta_v = 0$ and $\eta_v(c(a)) = v$. Then $\mathcal{P}_{c,\cdot,\cdot}$ is called the parallel transport on vector bundle E along c.

In particular, we write $\mathcal{P}_{q,p}$ for $\mathcal{P}_{c,q,p}$ in the case when c is a minimizing geodesic connecting p and q.

The notion of the higher order covariant derivatives for tensor fields was known, see for example, [7,32]. Below we shall define the higher order covariant derivative for sections. Let $k \leqslant \kappa$ be a positive integer and let ξ be a C^κ-section. Recall that D is a connection on the vector bundle $\pi: E \to M$ and ∇ is the Levi–Civita connection on M. Then the covariant derivative of order k can be inductively defined as follows.

Define the map $\mathcal{D}^1\xi = \mathcal{D}\xi: (C^\kappa(TM))^1 \to C^{\kappa-1}(M, E)$ by

$$\mathcal{D}\xi(X) = \mathrm{D}_X\xi \quad \text{for each } X \in C^\kappa(TM), \tag{2.4}$$

and define the map $\mathcal{D}^k\xi\colon (C^\kappa(TM))^k \to C^{\kappa-k}(M, E)$ by

$$\mathcal{D}^k\xi(X_1, \ldots, X_{k-1}, X) = \mathrm{D}_X(\mathcal{D}^{k-1}\xi(X_1, \ldots, X_{k-1})) - \sum_{i=1}^{k-1} \mathcal{D}^{k-1}\xi(X_1, \ldots, \nabla_X X_i, \ldots, X_{k-1}) \tag{2.5}$$

for each $X_1, \ldots, X_{k-1}, X \in C^\kappa(TM)$. Then, in view of the definition and thanks to (2.2), one can use mathematical induction to prove easily that $\mathcal{D}^k\xi(X_1, \ldots, X_k)$ is tensorial with respect to each component X_i, that is, k multi-linear map from $(C^\kappa(TM))^k$ to $C^{\kappa-k}(M, E)$, where the linearity refers to the structure of $C^k(M)$-module. This implies that the value of $\mathcal{D}^k\xi(X_1, \ldots, X_k)$ at $p \in M$ only depends on the k-tuple of tangent vectors $(v_1, \ldots, v_k) = (X_1(p), \ldots, X_k(p)) \in (T_pM)^k$. Consequently, for a given $p \in M$, the map $\mathcal{D}^k\xi(p)\colon (T_pM)^k \to E_p$, defined by

$$\mathcal{D}^k\xi(p)v_1 \ldots v_k := \mathcal{D}^k\xi(X_1, \ldots, X_k)(p) \quad \text{for any } (v_1, \ldots, v_k) \in (T_pM)^k \tag{2.6}$$

is well-defined, where $X_i \in C^\kappa(TM)$ satisfy $X_i(p) = v_i$ for each $i = 1, \ldots, k$. Let $p_0 \in M$ be such that $\mathrm{D}\xi(p_0)^{-1}$ exists. Thus, for any piece-geodesic curve c connecting p_0 and p, $\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathcal{D}^k\xi(p)$ is a k-multilinear map from $(T_pM)^k$ to $T_{p_0}M$. We define the norm of $\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathcal{D}^k\xi(p)$ by

$$\| \mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathcal{D}^k\xi(p) \| = \sup \| \mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathcal{D}^k\xi(p)v_1v_2 \ldots v_k \|_{p_0},$$

where the supremum is taken over all k-tuple of vectors $(v_1, \ldots, v_k) \in (T_pM)^k$ with each $\|v_j\|_p = 1$. Furthermore, for any geodesic $c\colon \mathbb{R} \to M$ on M, since $\nabla_{c'(s)}c'(s) = 0$, it follows from (2.5) that

$$\mathcal{D}^k\zeta(c(s))(c'(s))^k = \mathrm{D}_{c'(s)}(\mathcal{D}^{k-1}\zeta(c(s))(c'(s))^{k-1}) \quad \text{for each } s \in \mathbb{R}. \tag{2.7}$$

For study in the next sections, the following two lemmas will play a key role. Recall that $\pi\colon E \to M$ is a C^κ-vector bundle of rank $\hat{m}$ with a connection D.

Lemma 2.1. *Let $c\colon \mathbb{R} \to M$ be a geodesic and let $\zeta \in C^\kappa(M, E)$. Let $\{e_i\}_{i=1}^{\hat{m}}$ be a basis of $\pi^{-1}(c(0))$. Then, there exist $\hat{m}$ real-valued C^κ-functions $\{\zeta^i\}_{i=1}^{\hat{m}}$ on $\mathbb{R}$ such that*

$$\mathcal{D}^k\zeta(c(s))(c'(s))^k = \sum_{i=1}^{\hat{m}} \frac{\mathrm{d}^k\zeta^i(s)}{\mathrm{d}s^k}\mathcal{P}_{c,c(s),c(0)}e_i \quad \textit{for each } k = 0, 1, \ldots, \kappa. \tag{2.8}$$

Proof. Define

$$\eta_i(c(\cdot)) := \mathcal{P}_{c,c(\cdot),c(0)}e_i \quad \text{for each } i = 1, \ldots, \hat{m}. \tag{2.9}$$

Then $\{\eta_i(c(s))\}_{i=1}^{\hat{m}}$ is a basis of $\pi^{-1}(c(s))$ because $\eta_i(c(0)) = e_i$ for each i and $\mathcal{P}_{c,c(s),c(0)}\colon \pi^{-1}(c(0)) \to \pi^{-1}(c(s))$ is a linear isomorphism. Let $\zeta \in C^\kappa(M, E)$. Then there exist $\hat{m}$ real-valued functions $\{\zeta^i\}_{i=1}^{\hat{m}}$ on $\mathbb{R}$ such that

$$\zeta(c(s)) = \sum_{i=1}^{\hat{m}} \zeta^i(s)\eta_i(c(s)) \quad \text{for any } s \in \mathbb{R}. \tag{2.10}$$

Below we shall show that each ζ^i is a C^κ-function on $\mathbb{R}$ for each $i = 1, 2, \ldots, \hat{m}$. Let $s_0 \in \mathbb{R}$ and $c(s_0) = p$. Then by Definition 2.1, there exist a neighborhood U of p and a C^κ-isomorphism $h: \pi^{-1}(U) \to U \times R^{\hat{m}}$ such that $h(E_q) \subset \{q\} \times \mathbb{R}^{\hat{m}}$ for each $q \in U$, and the mapping h^q defined by (2.1) is a linear isomorphism. Let $\{\delta_i\}$ be the natural basis of $\mathbb{R}^{\hat{m}}$, i.e., $\delta_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ for each $i = 1, 2, \ldots, \hat{m}$. Define $\omega_j(\cdot) := h^{-1}(\cdot, \delta_j)$ for each $1 \leqslant j \leqslant \hat{m}$. Then, for each $q \in U$, $\{\omega_j(q)\}$ is a basis of $\pi^{-1}(q)$. Let I be an open interval such that $s_0 \in I$ and $c(I) \subseteq U$, and let $\eta \in C^\kappa(M, E)$. Then there exist $\hat{m}$ real-valued function η on I such that

$$\eta(c(s)) = \sum_{j=1}^{\hat{m}} \phi^j(s)\omega_j(c(s)) \quad \text{for each } s \in I.$$

We claim that, for each $j = 1, 2, \ldots, \hat{m}$, ϕ^j is a C^κ-function on I. In fact, recall that $h^q = \mathbf{proj} \circ h$ is a linear isomorphism from $E_q \to \mathbb{R}^{\hat{m}}$ and that $\mathbf{proj} \circ h(\omega_j(q)) = h^q(\omega_j(q)) = \delta_j$ for each $j = 1, 2, \ldots, \hat{m}$. Then

$$(\phi^1(s), \ldots, \phi^{\hat{m}}(s)) = \sum_{j=1}^{\hat{m}} \phi^j(s)(\mathbf{proj} \circ h)(\omega_j(c(s))) = ((\mathbf{proj} \circ h) \circ \eta \circ c)(s).$$

This implies that the claim holds because $(\mathbf{proj} \circ h) \circ \eta \circ c: I \to \mathbb{R}^{\hat{m}}$ is a C^κ-function. In particular, for each $i = 1, 2, \ldots, \hat{m}$, there exist $\hat{m}$ C^κ-functions $\eta_{i1}, \ldots, \eta_{i\hat{m}}$ on I such that

$$\eta_i(c(s)) = \sum_{j=1}^{\hat{m}} \eta_{ij}(s)\omega_j(c(s)) \quad \text{for each } s \in I.$$

Consequently, each $\omega_j \circ c$ can be expressed as a linear combination of $\{\eta_1 \circ c, \ldots, \eta_{\hat{m}} \circ c\}$ with the coefficients being C^κ-functions on I. Since $\zeta \in C^\kappa(M, E)$, ζ can be expressed as a linear combination of $\{\omega_1 \circ c, \ldots, \omega_{\hat{m}} \circ c\}$ with the coefficients being C^κ-functions on I by the claim above and hence as a linear combination of $\{\eta_1 \circ c, \ldots, \eta_{\hat{m}} \circ c\}$ with the coefficients being C^κ-functions on I. This implies that each ζ^i is a C^κ-function because such an expression is unique as $\{\eta_1 \circ c, \ldots, \eta_{\hat{m}} \circ c\}$ is a basis on I. Consequently, each ζ^i is a C^κ-function on $\mathbb{R}$ since $s_0 \in \mathbb{R}$ is arbitrary. Note that

$$\mathrm{D}_{c'(s)}\eta_i(c(s)) = 0 \quad \text{for each } i = 1, \ldots, \hat{m} \tag{2.11}$$

thanks to (2.9). It follows from (2.10), (2.4) and Definition 2.3 that

$$\begin{aligned} \mathcal{D}_{c'(s)}\zeta(c(s)) &= \sum_{i=1}^{\hat{m}} (\mathrm{D}_{c'(s)}\zeta^i(s))\eta_i(c(s)) + \sum_{i=1}^{\hat{m}} \zeta^i(s)\mathrm{D}_{c'(s)}\eta_i(c(s)) \\ &= \sum_{i=1}^{\hat{m}} \frac{\mathrm{d}\zeta^i(s)}{\mathrm{d}s}\eta_i(c(s)). \end{aligned} \tag{2.12}$$

Below, we claim that, for each $k = 0, 1, \ldots, \kappa$,

$$\mathcal{D}^k\zeta(c(s))(c'(s))^k = \sum_{i=1}^{\hat{m}} \frac{\mathrm{d}^k\zeta^i(s)}{\mathrm{d}s^k}\eta_i(c(s)). \tag{2.13}$$

Granting this, (2.8) is seen to hold by (2.9) and the proof is complete. We will show (2.13) by mathematical induction. Clearly, the cases when $k = 0, 1$ are trivial thanks to (2.10) and (2.12). Assume that (2.13) is true for $k = l - 1$. Then, by (2.7), we get that

$$\begin{aligned}\mathcal{D}^l\zeta(c(s))(c'(s))^l &= \mathrm{D}_{c'(s)}\left(\sum_{i=1}^{\hat{m}}\frac{\mathrm{d}^{l-1}\zeta^i(s)}{\mathrm{d}s^{l-1}}\eta_i(c(s))\right)\\ &= \sum_{i=1}^{\hat{m}}\left(\mathrm{D}_{c'(s)}(\frac{\mathrm{d}^{l-1}\zeta^i(s)}{\mathrm{d}s^{l-1}})\right)\eta_i(c(s)) + \sum_{i=1}^{\hat{m}}\frac{\mathrm{d}^{l-1}\zeta^i(s)}{\mathrm{d}s^{l-1}}\mathrm{D}_{c'(s)}\eta_i(c(s))\\ &= \sum_{i=1}^{\hat{m}}\frac{\mathrm{d}^{l}\zeta^i(s)}{\mathrm{d}s^{l}}\eta_i(c(s)),\end{aligned}$$

where the last equality holds because of (2.11). Hence, the claim stands. □

Lemma 2.2. *Let $c: \mathbb{R} \to M$ be a geodesic and let $\zeta \in C^\kappa(M, E)$. Then, for each $t \in \mathbb{R}$,*

$$\mathcal{P}_{c,c(0),c(t)}\zeta(c(t)) = \zeta(c(0)) + \int_0^t \mathcal{P}_{c,c(0),c(s)}(\mathrm{D}\zeta(c(s))c'(s))\,\mathrm{d}s. \tag{2.14}$$

Proof. Consider the function $\eta: \mathbb{R} \to \pi^{-1}(c(0))$ defined by $\eta(\cdot) := \mathcal{P}_{c,c(0),c(\cdot)}\zeta(c(\cdot))$. It suffices to verify that

$$\eta'(s) = \mathcal{P}_{c,c(0),c(s)}(\mathrm{D}\zeta(c(s))c'(s)) \quad \text{for each } s \in \mathbb{R}. \tag{2.15}$$

To prove (2.15), let $\{e_i\}$ be a basis of $\pi^{-1}(c(0))$. By Lemma 2.1, there exist $\hat{m}$ real-valued functions $\{\zeta^i\}_{i=1}^{\hat{m}}$ on such that

$$\zeta(c(s)) = \sum_{i=1}^{\hat{m}}\zeta^i(s)\mathcal{P}_{c,c(s),c(0)}e_i \quad \text{and} \quad \mathrm{D}_{c'(s)}\zeta(c(s)) = \sum_{i=1}^{\hat{m}}\frac{\mathrm{d}\zeta^i(s)}{\mathrm{d}s}\mathcal{P}_{c,c(s),c(0)}e_i \ \text{ for each } s \in \mathbb{R}. \tag{2.16}$$

It follows that, for any $s, h \in \mathbb{R}$,

$$\frac{\mathcal{P}_{c,c(s),c(s+h)}\zeta(c(s+h)) - \zeta(c(s))}{h} = \sum_{i=1}^{\hat{m}}\frac{\zeta^i(s+h) - \zeta^i(s)}{h}\mathcal{P}_{c,c(s),c(0)}e_i. \tag{2.17}$$

Letting $h \to 0$ in (2.17) and using (2.16), we get that

$$\mathrm{D}\zeta(c(s))c'(s) = \lim_{h\to 0}\frac{1}{h}(\mathcal{P}_{c,c(s),c(s+h)}\zeta(c(s+h)) - \zeta(c(s))) \quad \text{for each } s \in \mathbb{R}. \tag{2.18}$$

Hence (2.15) holds because $\mathcal{P}_{c,c(0),c(s)}$ is linear. □

We conclude this section by extending Newton's method in [11] to sections on M. Let $\xi \in C^1(M, E)$ and $p_0 \in M$. Then Newton's method with initial point p_0 for ξ is defined as follows.

$$p_{n+1} = \exp_{p_n}(-\mathrm{D}\xi(p_n)^{-1}\xi(p_n)) \quad \text{for each } n = 0, 1, 2, \ldots. \tag{2.19}$$

3. Auxiliary results

For a Banach space or a Riemannian manifold Z, we use $\mathbf{B}_Z(p,r)$ and $\overline{\mathbf{B}_Z(p,r)}$ to denote, respectively, the open metric ball and the closed metric ball at p with radius r, that is,

$$\mathbf{B}_Z(p,r)=\{q\in Z: d(p,q)<r\} \quad \text{and} \quad \overline{\mathbf{B}_Z(p,r)}=\{q\in Z: d(p,q)\leqslant r\}.$$

We often omit the subscript Z if no confusion caused.

Let $C_2(p_0,r)$ denote the set of all piecewise geodesics $c:[0,T]\to M$ with $c(0)=p_0$ and $l(c)<r$ such that $c|_{[0,\tau]}$ is a minimizing geodesic and $c|_{[\tau,T]}$ is a geodesic for some $\tau\in(0,T]$. Inspired by the work of Zabrejko and Nguen in [33] on Kantorovich's majorant method, Alvarez et al. introduced in [2] a Lipschitz–type radial function $L:[0,R]\to[0,+\infty)$ for the covariant derivative of vector fields on Riemannian manifolds which satisfies that for every $r\in[0,R]$ and $c\in C_2(p_0,r)$,

$$\| \mathrm{D}X(p_0)^{-1}[P_{c,c(0),c(b)}\mathrm{D}X(c(b))-P_{c,c(0),c(a)}\mathrm{D}X(c(a))] \| \leqslant L(u)l(c|_{[a,b]})$$
$$\text{for any } 0\leqslant a\leqslant b.$$

where R is a positive real number. Below, we will modify the notion of the Lipschitz condition with L-average for mappings on Banach spaces to suit sections. Let L be a positive nondecreasing integrable function on $[0,R]$, where R is a positive number large enough such that $\int_0^R(R-u)L(u)\,\mathrm{d}u\geqslant R$. The notion of Lipschitz condition in the inscribed sphere with the L average for operators from Banach spaces to Banch spaces was first introduced in [28] by Wang for the study of Smale's point estimate theory. The following definition extends this notion to sections on Riemannian manifold M. Let $\pi: E\to M$ be a C^κ-vector bundle with a connection D and ξ a C^κ-section of this vector bundle.

Definition 3.1. Let $R>r>0$ and let $p_0\in M$ be such that $\mathrm{D}\xi(p_0)^{-1}$ exists. Then $\mathrm{D}\xi(p_0)^{-1}\mathrm{D}\xi$ is said to satisfy the 2-piece L-average Lipschitz condition in $\mathbf{B}(p_0,r)$, if, for any two points $p,q\in\mathbf{B}(p_0,r)$, any geodesic c_2 connecting p,q and minimizing geodesic c_1 connecting p_0, p with $l(c_1)+l(c_2)<r$,

$$\| \mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}\circ(\mathcal{P}_{c_2,p,q}\mathrm{D}\xi(q)P_{c_2,q,p}-\mathrm{D}\xi(p)) \| \leqslant \int_{l(c_1)}^{l(c_1)+l(c_2)} L(u)\,\mathrm{d}u. \tag{3.1}$$

The majorizing function h defined in the following, which was first introduced and studied by Wang (cf. [28]), is a powerful tool in our study. Let $r_0>0$ and $b>0$ be such that

$$\int_0^{r_0} L(u)\,\mathrm{d}u=1 \quad \text{and} \quad b=\int_0^{r_0} L(u)u\,\mathrm{d}u. \tag{3.2}$$

For $\beta>0$, define the majorizing function h by

$$h(t)=\beta-t+\int_0^t L(u)(t-u)\,\mathrm{d}u \quad \text{for each } 0\leqslant t\leqslant R. \tag{3.3}$$

Some useful properties are described in the following proposition, see [28].

Proposition 3.1. *The function h is monotonic decreasing on* $[0,r_0]$ *and monotonic increasing on* $[r_0,R]$. *Moreover, if* $\beta\leqslant b$, *h has a unique zero, respectively, in* $[0,r_0]$ *and* $[r_0,R]$, *which are denoted by* r_1 *and* r_2.

Throughout the remainder of the paper, we always assume that $\hat{m} = m$ and that $p_0 \in M$ is such that $\mathrm{D}\xi(p_0)^{-1}$ exists. Let $\beta := \|\mathrm{D}\xi(p_0)^{-1}\xi(p_0)\| \leqslant b$. The first lemma of this section estimates the norm of the inverse $\mathrm{D}\xi(q)^{-1}$ around the point p_0.

Lemma 3.1. *Let $0 < r \leqslant r_0$ and suppose that $\mathrm{D}\xi(p_0)^{-1}\mathrm{D}\xi$ satisfies the 2-piece L-average Lipschitz condition in $\mathbf{B}(p_0, r)$. Let $p, q \in \mathbf{B}(p_0, r)$ and let c_1 be a minimizing geodesic connecting p_0, p and c_2 a geodesic connecting p, q satisfying $l(c_1) + l(c_2) < r$. Then, $\mathrm{D}\xi(q)^{-1}$ exists and*

$$\|\mathrm{D}\xi(q)^{-1}\mathcal{P}_{c_2,q,p} \circ \mathcal{P}_{c_1,p,p_0}\mathrm{D}\xi(p_0)\| \leqslant \frac{1}{1 - \int_0^{l(c_1)+l(c_2)} L(u)\,\mathrm{d}u} = \frac{-1}{h'(l(c_1) + l(c_2))}. \tag{3.4}$$

Proof. In view of Banach Lemma, it is sufficient to show that

$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p} \circ \mathcal{P}_{c_2,p,q}\mathrm{D}\xi(q)P_{c_2,q,p} \circ P_{c_1,p,p_0} - \mathbf{I}_{T_{p_0}M}\| \leqslant \int_0^{l(c_1)+l(c_2)} L(u)\mathrm{d}u < 1 \tag{3.5}$$

because $P_{c_1,p_0,p}$ is an isometry from T_pM to $T_{p_0}M$ and $P_{c_2,p,q}$ is an isometry from T_qM to T_pM, where $\mathbf{I}_{T_{p_0}M}$ is the identity on $T_{p_0}M$. By (3.1), we have

$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}(\mathcal{P}_{c_2,p,q}\mathrm{D}\xi(q)P_{c_2,q,p} - \mathrm{D}\xi(p))\| \leqslant \int_{l(c_1)}^{l(c_1)+l(c_2)} L(u)\,\mathrm{d}u \tag{3.6}$$

and

$$\|\mathrm{D}\xi(p_0)^{-1}(\mathcal{P}_{c_1,p_0,p}\mathrm{D}\xi(p)P_{c_1,p,p_0} - \mathrm{D}\xi(p_0))\| \leqslant \int_0^{l(c_1)} L(u)\,\mathrm{d}u. \tag{3.7}$$

Since

$$\begin{aligned}
&\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p} \circ \mathcal{P}_{c_2,p,q}\mathrm{D}\xi(q)P_{c_2,q,p} \circ P_{c_1,p,p_0} - \mathbf{I}_{T_{p_0}M} \\
&\quad = \mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}(\mathcal{P}_{c_2,p,q}\mathrm{D}\xi(q)P_{c_2,q,p} - \mathrm{D}\xi(p))P_{c_1,p,p_0} \\
&\qquad + \mathrm{D}\xi(p_0)^{-1}(\mathcal{P}_{c_1,p_0,p}\mathrm{D}\xi(p)P_{c_1,p,p_0} - \mathrm{D}\xi(p_0)),
\end{aligned} \tag{3.8}$$

it follows from (3.6) and (3.7) that

$$\begin{aligned}
&\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p} \circ \mathcal{P}_{c_2,p,q}\mathrm{D}\xi(q)P_{c_2,q,p} \circ P_{c_1,p,p_0} - \mathbf{I}_{T_{p_0}M}\| \\
&\quad \leqslant \int_{l(c_1)}^{l(c_1)+l(c_2)} L(u)\,\mathrm{d}u + \int_0^{l(c_1)} L(u)\,\mathrm{d}u \\
&\quad < \int_0^r L(u)\,\mathrm{d}u < 1.
\end{aligned} \tag{3.9}$$

Hence, (3.5) is seen to hold and the proof is complete. □

In the remainder of this section, we shall assume that $\mathrm{D}\xi(p_0)^{-1}\mathrm{D}\xi$ satisfies the 2-piece L-average Lipschitz condition in $\mathbf{B}(p_0, r_1)$.

Let $\theta \in [0, 1]$ and let the pair $(t, p) \in [0, r_1) \times \mathbf{B}(p_0, r_1)$. Define

$$\hat{t}(\theta) = t - \theta h'(t)^{-1} h(t) \quad \text{and} \quad \hat{p}(\theta) = \exp_p(-\theta \mathrm{D}\xi(p)^{-1}\xi(p)) \tag{3.10}$$

and consider the following condition:

$$d(p_0, p) \leqslant t < r_1 \quad \text{and} \quad \|\mathrm{D}\xi(p)^{-1}\xi(p)\| \leqslant -h'(t)^{-1}h(t). \tag{3.11}$$

For a pair $(\tilde{t}, \tilde{p}) \in [0, R] \times M$, we say that the pair $(\tilde{t}, \tilde{p})$ satisfies (3.11) if (3.11) holds with $(\tilde{t}, \tilde{p})$ in place of (t, p). The following lemma shows that $(\hat{t}(\theta), \hat{p}(\theta))$ retains the condition (3.11).

The following lemma is an extension and refinement of [11, Lemma 3.7], [2, Lemma 4.3].

Lemma 3.2. *Suppose that the pair* $(t, p) \in [0, r_1) \times \mathbf{B}(p_0, r_1)$ *satisfies* (3.11) *and* $\theta \in [0, 1]$. *Then*, $t \leqslant \hat{t}(\theta) < r_1$ *and the pair* $(\hat{t}(\theta), \hat{p}(\theta))$ *satisfies* (3.11). *Moreover, the following assertions hold*:

$$\|\mathrm{D}\xi(\hat{p}(1))^{-1}\xi(\hat{p}(1))\| \leqslant \left(\frac{h'(\hat{t}(1))^{-1}h(\hat{t}(1))}{h'(t)^{-1}h(t)}\right) \|\mathrm{D}\xi(p)^{-1}\xi(p)\|, \tag{3.12}$$

$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,p}\mathcal{P}_{c,p,\hat{p}(1)}\xi(\hat{p}(1))\| \leqslant \left(\frac{h(\hat{t}(1))}{h(t)}\right) \|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,q}\mathcal{P}_{\hat{c},q,p}\xi(p)\|, \tag{3.13}$$

where c is the geodesic of M defined by $c(\lambda) := \exp_p(-\lambda\theta \mathrm{D}\xi(p)^{-1}\xi(p))$ *for each* $\lambda \in [0, 1]$, $q \in \mathbf{B}(p_0, r_1)$ *and $\hat{c}$ is a geodesic connecting q and p such that* $d(p_0, q) + l(\hat{c}) \leqslant t$.

Proof. Noting that $\hat{t}(\cdot)$ is increasing on $[0, 1]$, we have $t \leqslant \hat{t}(\theta) \leqslant \hat{t}(1)$. As the function $t \mapsto t - h'(t)^{-1}h(t)$ is strictly monotonic increasing on $[0, r_1]$ and $h(r_1) = 0$, one has that

$$\hat{t}(1) = t - h'(t)^{-1}h(t) < r_1 - h'(r_1)^{-1}h(r_1) = r_1. \tag{3.14}$$

Suppose that (3.11) holds. Then

$$\theta\|\mathrm{D}\xi(p)^{-1}\xi(p)\| \leqslant -\theta h'(t)^{-1}h(t). \tag{3.15}$$

It follows that

$$d(p_0, \hat{p}(\theta)) \leqslant d(p_0, p) + d(p, \hat{p}(\theta)) \leqslant t - \theta h'(t)^{-1}h(t) = \hat{t}(\theta) < r_1. \tag{3.16}$$

Set

$$s = -\theta h'(t)^{-1}h(t) \quad \text{and} \quad v = -\theta \mathrm{D}\xi(p)^{-1}\xi(p). \tag{3.17}$$

Then, $c(1) = \exp_p(v) = \hat{p}(\theta)$ and

$$d(p_0, p) + l(c) \leqslant t - \theta h'(t)^{-1}h(t) = \hat{t}(\theta) \leqslant \hat{t}(1) < r_1 \tag{3.18}$$

thanks to (3.14) and (3.15). Hence, by Lemma 2.2, we have that

$$\mathcal{P}_{c,p,\hat{p}(\theta)}\xi(\hat{p}(\theta)) - \xi(p) = \int_0^1 \mathcal{P}_{c,p,c(\lambda)}\mathrm{D}\xi(c(\lambda))c'(\lambda)\,\mathrm{d}\lambda. \tag{3.19}$$

Note that $h'' = L$ and $\|v\| \leqslant s$. By (3.18), (3.1) is applicable, and so

$$\begin{aligned}
&\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,p}(\mathcal{P}_{c,p,\hat{p}(\theta)}\xi(\hat{p}(\theta)) - \xi(p) - \mathrm{D}\xi(p)v)\| \\
&\quad \leqslant \int_0^1 \int_{d(p_0,p)}^{d(p_0,p)+\lambda\|v\|} L(u)\,\mathrm{d}u\,\|v\|\,\mathrm{d}\lambda \\
&\quad \leqslant \int_0^1 \int_t^{t+\lambda s} h''(u)\,\mathrm{d}u\,\mathrm{d}\lambda\theta\|\mathrm{D}\xi(p)^{-1}\xi(p)\| \\
&\quad = (h(\hat{t}(\theta)) + (\theta-1)h(t))\left(\frac{\|\mathrm{D}\xi(p)^{-1}\xi(p)\|}{-h'(t)^{-1}h(t)}\right)
\end{aligned} \tag{3.20}$$

thanks to (3.19) and (3.17). Since $l(c) + d(p, p_0) \leqslant \hat{t}(\theta) < r_1$ by (3.18), it follows from Lemma 3.1 that

$$\begin{aligned}
\|\mathrm{D}\xi(\hat{p}(\theta))^{-1}\mathcal{P}_{c,\hat{p}(\theta),p} \circ \mathcal{P}_{p,p_0}\mathrm{D}\xi(p_0)\| &\leqslant -h'(l(c) + d(p, p_0))^{-1} \\
&\leqslant -h'(\hat{t}(\theta))^{-1}.
\end{aligned} \tag{3.21}$$

In particular, taking $\theta = 1$ in (3.20) and (3.21), we have

$$\begin{aligned}
&\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,p}\mathcal{P}_{c,p,\hat{p}(1)}\xi(\hat{p}(1))\| \\
&\quad = \|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,p}(\mathcal{P}_{c,p,\hat{p}(1)}\xi(\hat{p}(1)) - \xi(p) - \mathrm{D}\xi(p)v)\| \\
&\quad \leqslant h(\hat{t}(1))\frac{\|\mathrm{D}\xi(p)^{-1}\xi(p)\|}{-h'(t)^{-1}h(t)}
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
\|\mathrm{D}\xi(\hat{p}(1))^{-1}\mathcal{P}_{c,\hat{p}(1),p} \circ \mathcal{P}_{p,p_0}\mathrm{D}\xi(p_0)\| &\leqslant -h'(l(c) + d(p, p_0))^{-1} \\
&\leqslant -h'(\hat{t}(1))^{-1}.
\end{aligned} \tag{3.23}$$

Thus (3.12) follows from (3.22) and (3.23). Furthermore, by assumptions, $d(p_0, q) + l(\hat{c}) \leqslant t < r_1 \leqslant r_0$. Therefore, one can apply Lemma 3.1 again to get that

$$\|\mathrm{D}\xi(p)^{-1}\mathcal{P}_{\hat{c},p,q}\mathcal{P}_{q,p_0}\mathrm{D}\xi(p_0)\| \leqslant -h'(d(p_0, q) + l(\hat{c}))^{-1} \leqslant -h'(t)^{-1}, \tag{3.24}$$

hence $\|\mathrm{D}\xi(p)^{-1}\xi(p)\| \leqslant -h'(t)^{-1}\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,q}\mathcal{P}_{\hat{c},q,p}\xi(p)\|$. This together with (3.22) yields (3.13).

Thus, in view of (3.16), it remains to verify that

$$\|\mathrm{D}\xi(\hat{p}(\theta))^{-1}\xi(\hat{p}(\theta))\| \leqslant -h'(\hat{t}(\theta))^{-1}h(\hat{t}(\theta)). \tag{3.25}$$

To this purpose, note that (3.20) together with the condition (3.11) implies that

$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,p}(\mathcal{P}_{c,p,\hat{p}(\theta)}\xi(\hat{p}(\theta)) - \xi(p) - \mathrm{D}\xi(p)v)\| \leqslant h(\hat{t}(\theta)) + (\theta-1)h(t). \tag{3.26}$$

Combining this with (3.21) gives that

$$\begin{aligned}
&\|\mathrm{D}\xi(\hat{p}(\theta))^{-1}\mathcal{P}_{c,\hat{p}(\theta),p}(\mathcal{P}_{c,p,\hat{p}(\theta)}\xi(\hat{p}(\theta)) - \xi(p) - \mathrm{D}\xi(p)v)\| \\
&\quad \leqslant \frac{h(\hat{t}(\theta)) + (\theta-1)h(t)}{-h'(\hat{t}(\theta))}.
\end{aligned} \tag{3.27}$$

Taking $\theta = 0$ in (3.21) gives that $\|D\xi(p)^{-1}\mathcal{P}_{p,p_0}D\xi(p_0)\| \leqslant \frac{1}{|h'(t)|}$. Since $h'' = L$ and $\|v\| \leqslant s$, it follows from (3.1) that

$$\begin{aligned}
&\|(D\xi(p)^{-1}\mathcal{P}_{c,p,\hat{p}(\theta)}D\xi(\hat{p}(\theta)P_{c,\hat{p}(\theta),p} - \mathbf{I}_{T_pM})\| \\
&\quad \leqslant \|D\xi(p)^{-1}\mathcal{P}_{p,p_0}D\xi(p_0)\| \|D\xi(p_0)^{-1}\mathcal{P}_{p_0,p}(\mathcal{P}_{c,p,\hat{p}(\theta)}D\xi(\hat{p}(\theta)P_{c,\hat{p}(\theta),p} - D\xi(p))\| \\
&\quad \leqslant \frac{1}{|h'(t)|}\int_t^{t+s} h''(u)\,du \\
&\quad = \frac{h'(\hat{t}(\theta))}{|h'(t)|} + 1 < 1. \qquad (3.28)
\end{aligned}$$

Thus the Banach Lemma is applicable to concluding that $\|D\xi(\hat{p}(\theta))^{-1}\mathcal{P}_{c,\hat{p}(\theta),p}D\xi(p)\| \leqslant h'(\hat{t}(\theta))^{-1}h'(t)$ because $P_{c,p,\hat{p}(\theta)}$ is an isometry; consequently,

$$\begin{aligned}
&\|D\xi(\hat{p}(\theta))^{-1}\mathcal{P}_{c,\hat{p}(\theta),p}\xi(p)\| \leqslant \|D\xi(\hat{p}(\theta))^{-1}\mathcal{P}_{c,\hat{p}(\theta),p}D\xi(p)\| \|D\xi(p)^{-1}\xi(p)\| \\
&\quad \leqslant -h'(\hat{t}(\theta))^{-1}h(t) \qquad (3.29)
\end{aligned}$$

thanks to (3.11). Therefore, combining (3.27) and (3.29), one has that

$$\begin{aligned}
&\|D\xi(\hat{p}(\theta))^{-1}\xi(\hat{p}(\theta))\| \\
&\quad \leqslant \|D\xi(\hat{p}(\theta))^{-1}\mathcal{P}_{c,\hat{p}(\theta),p}(\mathcal{P}_{c,p,\hat{p}(\theta)}\xi(\hat{p}(\theta)) - \xi(p) - D\xi(p)v)\| \\
&\qquad +(1-\theta)\|D\xi(\hat{p}(\theta))^{-1}\mathcal{P}_{c,\hat{p}(\theta),p}\xi(p)\| \\
&\quad \leqslant \frac{h(\hat{t}(\theta)) + (\theta-1)h(t)}{-h'(\hat{t}(\theta))} + \frac{(1-\theta)h(t)}{-h'(\hat{t}(\theta))} \\
&\quad = -h'(\hat{t}(\theta))^{-1}h(\hat{t}(\theta)). \qquad (3.30)
\end{aligned}$$

Therefore (3.25) is seen to hold and the proof is complete. □

Let $\{\hat{t}_n\}$ and $\{\hat{p}_n\}$ denote the sequences generated by Newton's method, respectively, for h with initial point $\hat{t}_0 = t$ and for ξ with initial point $\hat{p}_0 = p$; that is,

$$\hat{t}_0 = t, \quad \hat{t}_{n+1} = \hat{t}_n - h'(\hat{t}_n)^{-1}h(\hat{t}_n) \text{ for each } n = 0, 1, \ldots$$

and

$$\hat{p}_0 = p, \quad \hat{p}_{n+1} = \exp_{\hat{p}_n}(-D\xi(\hat{p}_n)^{-1}\xi(\hat{p}_n)) \text{ for each } n = 0, 1, \ldots .$$

In particular, in the case when $t = 0$ and $p = p_0$, for simplicity, we denote the sequences $\{\hat{t}_n\}$ and $\{\hat{p}_n\}$ by $\{t_n\}$ and $\{p_n\}$, respectively. Hence

$$t_0 = 0, \quad t_{n+1} = t_n - h'(t_n)^{-1}h(t_n) \text{ for each } n = 0, 1, \ldots \qquad (3.31)$$

and

$$p_{n+1} = \exp_{p_n}(-D\xi(p_n)^{-1}\xi(p_n)) \quad \text{for each } n = 0, 1, \ldots . \qquad (3.32)$$

Note that, by Lemma 3.2 and mathematical induction, if the pair $(t, p) \in [0, r_1) \times \mathbf{B}(p_0, r_1)$ satisfies (3.11), then for each $n = 0, 1, \ldots$, the pair $(\hat{t}_n, \hat{p}_n)$ is well-defined and satisfies

$$d(p_0, \hat{p}_n) \leqslant \hat{t}_n < r_1 \quad \text{and} \quad \|D\xi(\hat{p}_n)^{-1}\xi(\hat{p}_n)\| \leqslant -h'(\hat{t}_n)^{-1}h(\hat{t}_n). \qquad (3.33)$$

Furthermore we have the following proposition.

Proposition 3.2. *Suppose that the pair* $(t, p) \in [0, r_1) \times \mathbf{B}(p_0, r_1)$ *satisfies* (3.11). *Then the following assertions hold.*

(i) *The sequence* $\{\hat{t}_n\}$ *is strictly increasing and convergent to* r_1.

(ii) *The sequence* $\{\hat{p}_n\}$ *is well-defined, convergent to a singular point* q^* *of* ξ *in* $\overline{\mathbf{B}(p_0, r_1)}$, *and the following assertions hold*:

$$\|\mathrm{D}\xi(\hat{p}_n)^{-1}\xi(\hat{p}_n)\| \leqslant \left(\frac{h'(\hat{t}_n)^{-1}h(\hat{t}_n)}{h'(\hat{t}_{n-1})^{-1}h(\hat{t}_{n-1})}\right)\|\mathrm{D}\xi(\hat{p}_{n-1})^{-1}\xi(\hat{p}_{n-1})\|; \tag{3.34}$$

$$d(\hat{p}_{n+1}, \hat{p}_n) \leqslant \hat{t}_{n+1} - \hat{t}_n \quad \textit{for each } n = 1, 2, \ldots . \tag{3.35}$$

Proof. (i). Note that the function φ defined by $\varphi(t) := t - h'(t)^{-1}h(t)$ for each $t \in [0, r_1]$ is strictly monotonic increasing on $[0, r_1]$ because $\varphi'(t) = \frac{h''(t)h(t)}{h'(t)^2} > 0$ for each $t \in [0, r_1)$. Thus, it is easy to show by mathematical induction that

$$\hat{t}_n < \hat{t}_{n+1} \quad \text{and} \quad 0 \leqslant \hat{t}_n < r_1 \ \text{ for each } n = 0, 1, \ldots . \tag{3.36}$$

Hence (i) is proved.

(ii) It is clear that the sequence $\{\hat{p}_n\}$ is well-defined and by (3.33), for each $n = 1, 2, \ldots,$

$$\|\mathrm{D}\xi(\hat{p}_n)^{-1}\xi(\hat{p}_n)\| \leqslant -h'(\hat{t}_n)^{-1}h(\hat{t}_n).$$

Hence (3.34) holds by (3.12), and

$$d(\hat{p}_{n+1}, \hat{p}_n) \leqslant \|\mathrm{D}\xi(\hat{p}_n)^{-1}\xi(\hat{p}_n)\| \leqslant -h'(\hat{t}_n)^{-1}h(\hat{t}_n) = \hat{t}_{n+1} - \hat{t}_n \tag{3.37}$$

holds for each $n = 0, 1, \ldots$. By (i), the proof is complete. □

The following result extends [11, Lemma 3.8, Corollary 3.9], [2, Lemmas 4.4, 4.5] to our section framework.

Lemma 3.3. *Suppose that the pair* $(t, p) \in [0, r_1) \times \mathbf{B}(p_0, r_1)$ *satisfies* (3.11). *Let* $q^* \in \overline{\mathbf{B}(p_0, r_1)}$ *be a singular point of* ξ *satisfying* $t + d(p, q^*) = r_1$. *Then, for each* $n = 0, 1, \ldots,$

$$d(p_0, \hat{p}_n) = \hat{t}_n \quad \textit{and} \quad \hat{t}_{n+1} + d(\hat{p}_{n+1}, q^*) = r_1. \tag{3.38}$$

Consequently, $d(p_0, q^*) = r_1$.

Proof. Let $(\tau, q) \in [0, r_1) \times \mathbf{B}(p_0, r_1)$ and let

$$\hat{\tau} = \tau - h'(\tau)^{-1}h(\tau) \quad \text{and} \quad \hat{q} = \exp_q(-\mathrm{D}\xi(q)^{-1}\xi(q)). \tag{3.39}$$

We first verify the following implication:

$$\left.\begin{array}{l}(\tau, q) \text{ satisfies } (3.11)\\ \tau + d(q, q^*) = r_1\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}d(p_0, q) = \tau\\ \hat{\tau} + d(\hat{q}, q^*) = r_1.\end{array}\right. \tag{3.40}$$

To do this, suppose that (τ, q) satisfies (3.11) and $\tau + d(q, q^*) = r_1$. Let $v \in T_qM$ be such that $q^* = \exp_q(v)$ and $\|v\| = d(q, q^*)$. Then the curve c defined by $c(s) := \exp_q(sv)$ for each $s \in [0, 1]$ is a minimizing geodesic connecting q and q^*. Note that

$$\mathcal{P}_{c,q,q^*}\xi(q^*) - \xi(q) - \mathrm{D}\xi(q)v = \int_0^1 (\mathcal{P}_{c,q,c(s)}\mathrm{D}\xi(c(s))P_{c,c(s),q}v - \mathrm{D}\xi(q)v)\,\mathrm{d}s \tag{3.41}$$

thanks to Lemma 2.2. This together with (3.1) and (3.11) yields

$$\begin{aligned}
&\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,q}(\xi(q)+\mathrm{D}\xi(q)v)\| \\
&\quad = \|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,q}(\mathcal{P}_{c,q,q^*}\xi(q^*)-\xi(q)-\mathrm{D}\xi(q)v)\| \\
&\quad \leqslant \int_0^1 \int_{d(p_0,q)}^{d(p_0,q)+s\|v\|} L(u)\,\mathrm{d}u\|v\|\,\mathrm{d}s \\
&\quad \leqslant \int_0^1 \int_{\tau}^{\tau+s\|v\|} h''(u)\,\mathrm{d}u\|v\|\,\mathrm{d}s \\
&\quad = -h(\tau)-h'(\tau)(r_1-\tau),
\end{aligned} \tag{3.42}$$

where the last equality holds because $\|v\| = r_1 - \tau$. Write $\lambda = d(p_0, q)$. Then $0 < \lambda \leqslant \tau < r_1$. It follows from Lemma 3.1 that

$$\frac{1}{\|\mathrm{D}\xi(q)^{-1}\mathcal{P}_{q,p_0}\mathrm{D}\xi(p_0)\|} \geqslant |h'(\lambda)| \geqslant |h'(\tau)| > 0, \tag{3.43}$$

because $|h'|$ is strictly monotonic decreasing on $[0, r_1]$. Since the pair (τ, q) satisfies (3.11) and $\|v\| = d(q, p^*) = r_1 - \tau$, one has that

$$\|v\| - \|\mathrm{D}\xi(q)^{-1}\xi(q)\| \geqslant r_1 - \tau + h'(\tau)^{-1}h(\tau). \tag{3.44}$$

Therefore, from (3.43) and (3.44), we get that

$$\begin{aligned}
&\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,q}(\xi(q)+\mathrm{D}\xi(q)v)\| \\
&\quad = \|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,q}\mathrm{D}\xi(q)(\mathrm{D}\xi(q)^{-1}\xi(q)+v)\| \\
&\quad \geqslant \frac{\|\mathrm{D}\xi(q)^{-1}\xi(q)+v\|}{\|\mathrm{D}\xi(q)^{-1}\mathcal{P}_{q,p_0}\mathrm{D}\xi(p_0)\|} \\
&\quad \geqslant |h'(\lambda)|(\|v\| - \|\mathrm{D}\xi(q)^{-1}\xi(q)\|) \\
&\quad \geqslant |h'(\tau)|(r_1-\tau+h'(\tau)^{-1}h(\tau)) \\
&\quad = -h(\tau)-h'(\tau)(r_1-\tau).
\end{aligned} \tag{3.45}$$

Combining (3.42) and (3.45), it is seen that the inequalities in (3.45) must be equalities; hence

$$\begin{aligned}
&\|\mathrm{D}\xi(q)^{-1}\xi(q)+v\| = \|v\| - \|\mathrm{D}\xi(q)^{-1}\xi(q)\|, \\
&\|\mathrm{D}\xi(q)^{-1}\xi(q)\| = -h'(\tau)^{-1}h(\tau)
\end{aligned} \tag{3.46}$$

and

$$\frac{1}{\|\mathrm{D}\xi(q)^{-1}\mathcal{P}_{q,p_0}\mathrm{D}\xi(p_0)\|} = |h'(\lambda)| = |h'(\tau)|. \tag{3.47}$$

Recall that $|h'|$ is strictly monotonic decreasing on $[0, r_1]$. It follows from (3.47) that $\lambda = \tau$, i.e., $d(p_0, q) = \tau$. Thus, to complete the proof of (3.40), it remains to verify $\hat{\tau} + d(\hat{q}, q^*) = r_1$. Note that

$$\|\mathrm{D}\xi(q)^{-1}\xi(q)+v\|^2 = \|\mathrm{D}\xi(q)^{-1}\xi(q)\|^2 + 2\langle \mathrm{D}\xi(q)^{-1}\xi(q), v\rangle + \|v\|^2.$$

On the other hand, by (3.46), we have that

$$\|\mathrm{D}\xi(q)^{-1}\xi(q)+v\|^2=\|\mathrm{D}\xi(q)^{-1}\xi(q)\|^2-2\|\mathrm{D}\xi(q)^{-1}\xi(q)\|\,\|v\|+\|v\|^2.$$

Therefore, $\|\mathrm{D}\xi(q)^{-1}\xi(q)\|\,\|v\|=\langle \mathrm{D}\xi(q)^{-1}\xi(q),-v\rangle$. This implies that there exists $s_0\geqslant 0$ such that $\mathrm{D}\xi(q)^{-1}\xi(q)=-s_0 v$ as $\|v\|>0$. Moreover, since $\hat{\tau}<r_1$ by Lemma 3.2, one has that

$$\|\mathrm{D}\xi(q)^{-1}\xi(q)\|=-h'(\tau)^{-1}h(\tau)=\hat{\tau}-\tau<r_1-\tau=\|v\|.$$

Then, $s_0<1$. Hence,

$$\hat{q}=\exp_q(-\mathrm{D}\xi(q)^{-1}\xi(q))=\exp_q(s_0 v)=c(s_0). \tag{3.48}$$

Recalling that c is a minimizing geodesic connecting q and q^* and $s_0<1$, one gets that

$$d(q,\hat{q})=\|s_0 v\| \quad \text{and} \quad d(q,q^*)=d(q,\hat{q})+d(\hat{q},q^*). \tag{3.49}$$

Therefore, by (3.46) and (3.48), we have that

$$d(q,\hat{q})=\|s_0 v\|=\|\mathrm{D}\xi(q)^{-1}\xi(q)\|=-h'(\tau)^{-1}h(\tau)=\hat{\tau}-\tau. \tag{3.50}$$

Note that $\tau+d(q,q^*)=r_1$ by assumption. It follows from (3.49) and (3.50) that $\hat{\tau}+d(\hat{q},q^*)=r_1$ and the proof of implication (3.40) is complete.

Now let us to show that (3.38) holds for each n. Clearly, applying (3.40) with $(\tau,q)=(t,p)$ we see that (3.38) holds for $n=0$. Thus, if (3.38) holds for $n=k-1$, then, $\hat{t}_k+d(\hat{p}_k,q^*)=r_1$. Recall from (3.33) that $(\hat{t}_k,\hat{p}_k)$ satisfies (3.11). Hence (3.40) with $(\tau,q)=(\hat{t}_k,\hat{p}_k)$ is applicable and we conclude that

$$d(p_0,\hat{p}_k)=\hat{t}_k, \quad \hat{t}_{k+1}+d(\hat{p}_{k+1},q^*)=r_1.$$

This means that (3.38) holds for $n=k$ and hence (3.38) holds for each n by mathematical induction.

Finally, by Proposition 3.2, the sequence $\{\hat{p}_n\}$ is convergent, say to $\hat{q}^*$ and $\lim_{n\to+\infty}\hat{t}_n=r_1$. Therefore, taking limits in two hand sides of equalities in (3.38), respectively, one has $d(p_0,\hat{q}^*)=r_1$ and $d(\hat{q}^*,q^*)=0$. This implies that $d(p_0,q^*)=r_1$, completing the proof. □

4. Convergence criterion of Newton's method and uniqueness ball of the singular point

Recall that $b=\int_0^{r_0}L(u)u\,\mathrm{d}u$ with r_0 satisfying $\int_0^{r_0}L(u)\,\mathrm{d}u=1$, h is the majorizing function defined by (3.3), $r_1,\ r_2$ are two zeroes of h in $[0,R]$ and that $\{t_n\}$ denotes Newton's sequence of h with initial point $t_0=0$. Recall also that $p_0\in M$ is such that $\mathrm{D}\xi(p_0)^{-1}$ exists.

Theorem 4.1. *Suppose that*

$$\beta:=\|\ \mathrm{D}\xi(p_0)^{-1}\xi(p_0)\ \|\ \leqslant b \tag{4.1}$$

and that $\mathrm{D}\xi(p_0)^{-1}\mathrm{D}\xi$ *satisfies the* 2*-piece L-average Lipschitz condition in* $\mathbf{B}(p_0,r_1)$. *Let* $\{p_n\}$ *be the sequence generated by Newton's method* (2.19) *with initial point* p_0. *Then* $\{p_n\}$ *is well-defined and convergent to a singular point* p^* *of* ξ *in* $\overline{\mathbf{B}(p_0,r_1)}$. *Moreover, there hold*

$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,p_{n-1}}\mathcal{P}_{c_n,p_{n-1},p_n}\xi(p_n)\|\leqslant\left(\frac{t_{n+1}-t_n}{t_n-t_{n-1}}\right)$$
$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,p_{n-2}}\mathcal{P}_{c_{n-1},p_{n-2},p_{n-1}}\xi(p_{n-1})\|, \tag{4.2}$$

for each $n = 2, 3, \ldots$,

$$\|\mathrm{D}\xi(p_n)^{-1}\xi(p_n)\| \leqslant \left(\frac{t_{n+1} - t_n}{t_n - t_{n-1}}\right) \|\mathrm{D}\xi(p_{n-1})^{-1}\xi(p_{n-1})\| \quad \textit{for each } n = 1, 2, \ldots, \tag{4.3}$$

$$d(p_{n+1}, p_n) \leqslant t_{n+1} - t_n \quad \textit{for each } n = 0, 1, \ldots \tag{4.4}$$

and

$$d(p_n, p^*) \leqslant r_1 - t_n \quad \textit{for each } n = 0, 1, \ldots, \tag{4.5}$$

where, for each n, c_n *is the geodesic of* M *defined by*

$$c_n(\lambda) := \exp_{p_{n-1}}(-\lambda \mathrm{D}\xi(p_{n-1})^{-1}\xi(p_{n-1})) \quad \textit{for each } \lambda \in [0, 1]. \tag{4.6}$$

Proof. Note that the pair $(0, p_0)$ satisfies (3.11) as $\|\mathrm{D}\xi(p_0)^{-1}\xi(p_0)\| = \beta = -h'(0)^{-1}h(0)$. By Proposition 4.1, $\{p_n\}$ is well-defined and converges to a singular point p^* of ξ in $\overline{\mathbf{B}(p_0, r_1)}$. Furthermore, applying Proposition 3.2(ii) to the case when $\hat{t}_0 = 0$ and $\hat{p}_0 = p_0$, one sees that (4.4) holds. Since (4.5) is a direct consequence of (4.4), it remains to verify (4.2) and (4.3). To do this, note that, for each $n = 1, 2, \ldots$,

$$-h'(t_{n-1})^{-1} \leqslant -h'(t_n)^{-1} \quad \text{and} \quad -h'(t_n)^{-1}h(t_n) = t_{n+1} - t_n.$$

Hence,

$$\frac{h(t_n)}{h(t_{n-1})} = \left(\frac{t_{n+1} - t_n}{t_n - t_{n-1}}\right)\left(\frac{h'(t_n)}{h'(t_{n-1})}\right) \leqslant \frac{t_{n+1} - t_n}{t_n - t_{n-1}} \quad \text{for each } n = 1, 2, \ldots .$$

Thus, in view of Lemma 3.2, we can easily use mathematical induction to conclude (4.2) and (4.3). □

The technique used in the proof of the following theorem is an adaptation of that for [11, Lemma 3.11] to our section framework. This technique was also adapted in [2,26].

Theorem 4.2. *Suppose that* (4.1) *holds. Let* $r_1 \leqslant r < r_2$ *if* $\beta < b$ *and* $r = r_1$ *if* $\beta = b$. *Suppose that* $\mathrm{D}\xi(p_0)^{-1}\mathrm{D}\xi$ *satisfies the* 2*-piece L-average Lipschitz condition in* $\overline{\mathbf{B}(p_0, r)}$. *Then, there exists a unique singular point* $p^* \in \overline{\mathbf{B}(p_0, r_1)}$ *of* ξ *in* $\overline{\mathbf{B}(p_0, r)}$.

Proof. By Theorem 4.1, there exists at least one singular point of ξ in $\overline{\mathbf{B}(p_0, r_1)}$. Hence we only need to prove the uniqueness. We first prove the following assertion.

$$\text{The singular point of } \xi \text{ in } \overline{\mathbf{B}(p_0, r_1)} \text{ is unique.} \tag{4.7}$$

Let p^* be the limit of the sequence generated by Newton's method (2.19) with initial point p_0. Assume that $q^* \in \overline{\mathbf{B}}(p_0, r_1)$ is another singular point of ξ. We claim that

$$d(p_n, q^*) + t_n \leqslant r_1 \quad \text{for each } n \geqslant 0. \tag{4.8}$$

Granting this, we have $q^* = \lim p_n = p^*$ and hence (4.7) is proved. To verify (4.8), assume first that $d(p_0, q^*) = r_1$. Then $d(p_0, q^*) + t_0 = r_1$ thanks to $t_0 = 0$. Since the pair $(0, p_0)$ satisfies (3.11), Lemma 3.3 is applicable to the pair $(0, p_0)$ to concluding that $d(p_n, q^*) + t_n = r_1$ for each

$n = 0, 1, \ldots$. Therefore (4.8) holds in this case. Now consider the case when $d(p_0, q^*) < r_1$. We shall use mathematical induction to show that, for each $n = 0, 1 \ldots,$

$$d(p_n, q^*) + t_n < r_1. \tag{4.9}$$

Eq. (4.9) is clear for $n = 0$ as $t_0 = 0$. Suppose that (4.9) is true for $n = k$. We have to show that (4.9) is true for $n = k + 1$. To do this, write $\hat{t}_k(\theta) = t_k + \theta(t_{k+1} - t_k)$ and $\hat{p}_k(\theta) = \exp_{p_k}(-\theta \mathrm{D}\xi(p_k)^{-1}\xi(p_k))$. Furthermore, define

$$\varphi(\theta) := d(\hat{p}_k(\theta), q^*) + \hat{t}_k(\theta) \quad \text{for each } \theta \in [0, 1]. \tag{4.10}$$

Then φ is a continuous function on [0, 1] satisfying $\varphi(0) < r_1$. Suppose that on the contrary (4.9) is not true for $n = k + 1$. Then, $\varphi(1) \geqslant r_1$ since $p_{k+1} = \hat{p}_k(1)$. Hence there exists $\overline{\theta} \in (0, 1]$ such that $\varphi(\overline{\theta}) = r_1$, or equivalently,

$$d(\hat{p}_k(\overline{\theta}), q^*) + \hat{t}_k(\overline{\theta}) = r_1. \tag{4.11}$$

It follows from Lemma 3.2 (applied to the pair (t_k, p_k)) that the pair $(\hat{t}_k(\overline{\theta}), \hat{p}_k(\overline{\theta}))$ satisfies (3.11). Then applying Lemma 3.3 to the pair $(\hat{t}_k(\overline{\theta}), \hat{p}_k(\overline{\theta}))$, we conclude that $d(p_0, q^*) = r_1$ which contradicts assumption (**a**). Thus, (4.9) holds and the claim stands.

Next, we shall show that ξ has a unique singular point in $\overline{\mathbf{B}(p_0, r)}$. For this purpose, let q^* be a singular point of ξ in $\overline{\mathbf{B}(p_0, r)}$. Let $v \in T_{p_0}M$ be such that $q^* = \exp_{p_0}(v)$ and $d(p_0, q^*) = \|v\|$. Then the curve c defined by $c(t) := \exp_{p_0}(tv)$, $t \in [0, 1]$ is a minimizing geodesic connecting p_0 and q^*. By Lemma 2.2, we get

$$\mathcal{P}_{c,p_0,q^*}\xi(q^*) - \xi(p_0) = \int_0^1 \mathcal{P}_{c,p_0,c(s)} \mathrm{D}\xi(c(s)) P_{c,c(s),p_0} v \, \mathrm{d}s. \tag{4.12}$$

Hence, it follows from (4.12) and (3.1) that

$$\begin{aligned} \|\mathrm{D}\xi(p_0)^{-1}\xi(p_0) + v\| &= \|\mathrm{D}\xi(p_0)^{-1}(\mathcal{P}_{c,p_0,q^*}\xi(q^*) - \xi(p_0) - \mathrm{D}\xi(p_0)v)\| \\ &\leqslant \int_0^1 \int_0^{s\|v\|} L(u)\, \mathrm{d}u \|v\| \, \mathrm{d}s \\ &= h(\|v\|) - \beta + \|v\|. \end{aligned} \tag{4.13}$$

On the other hand, we observe that

$$\|\mathrm{D}\xi(p_0)^{-1}\xi(p_0) + v\| \geqslant \|v\| - \|\mathrm{D}\xi(p_0)^{-1}\xi(p_0)\| = d(p_0, q^*) - \beta. \tag{4.14}$$

Combining (4.13) and (4.14) gives that $h(d(p_0, q^*)) \geqslant 0$, which implies $d(p_0, q^*) \leqslant r_1$ because $d(p_0, q^*) \leqslant r < r_2$. By (4.7), we complete the proof. □

5. Theorems under the Kantorovich's condition and the γ-condition

Assume that M is a real complete m-dimensional C^1-Riemannian manifold, $\pi: E \to M$ is a C^1-vector bundle of rank m and $\xi \in C^1(M, E)$ is a C^1-section of this vector bundle. Recall from [11] that, in the case when ξ is a vector field and D is the Levi–Civita connection, $\mathrm{D}\xi$ is Lipschitz in $\mathbf{B}(p_0, r)$ with constant $\hat{K} > 0$ if, for any two points $p, q \in \mathbf{B}(p_0, r)$ and any geodesics c with $c(0) = p$, $c(1) = q$ and $c([0, 1]) \subseteq \mathbf{B}(p_0, r)$,

$$\| P_{c,p,q}\mathrm{D}\xi(q) P_{c,q,p} - \mathrm{D}\xi(p) \| \leqslant \hat{K} l(c).$$

Below, we will use the following weaker version of the Lipschitz condition than that above.

Definition 5.1. Let $K > 0$, $r > 0$ be constants and let $p_0 \in M$ be such that $\mathrm{D}\xi(p_0)^{-1}$ exists. Then $\mathrm{D}\xi(p_0)^{-1}\mathrm{D}\xi$ is said to be 2-piece Lipschitz in $\mathbf{B}(p_0, r)$ with constant *K*, if, for any two points $p, q \in \mathbf{B}(p_0, r)$, any geodesic c_2 connecting p, q, and minimizing geodesic c_1 connecting p_0, p with $l(c_1) + l(c_2) < r$,

$$\| \mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p} \circ (\mathcal{P}_{c_2,p,q}\mathrm{D}\xi(q)P_{c_2,q,p} - \mathrm{D}\xi(p)) \| \leqslant Kl(c_2). \tag{5.1}$$

Clearly, that $\mathrm{D}\xi$ is Lipschitz in $\mathbf{B}(p_0, r)$ with constant $C > 0$ implies that $\mathrm{D}\xi(p_0)^{-1}\mathrm{D}\xi$ is 2-piece Lipschitz in $\mathbf{B}(p_0, r)$ with constant $K = \hat{K}\|\mathrm{D}\xi(p_0)^{-1}\|$, which is equivalent that $\mathrm{D}\xi(p_0)^{-1}\mathrm{D}\xi$ satisfies the 2-piece *L*-average Lipschitz condition in $\mathbf{B}(p_0, r)$ with $L(\cdot) \equiv K$ on $[0, R]$. The corresponding quadratic function *h* is $h(t) = \beta - t + \frac{K}{2}t^2$ for each $t \geqslant 0$. Clearly, if $\lambda = K\beta \leqslant \frac{1}{2}$, the zeros of *h* are equal to

$$r_1 = \frac{1 - \sqrt{1-2\lambda}}{K} \quad \text{and} \quad r_2 = \frac{1 + \sqrt{1-2\lambda}}{K}.$$

Thus, the following corollaries, which extend [11, Theorem 3.2], are, respectively trivial applications of Theorems 4.1 and 4.2, where the estimate (5.3) holds because of the estimate for the Newton sequence $\{t_n\}$ for *h* with $t_0 = 0$ (cf. [13,20,27,28]):

$$r_1 - t_n = \frac{(1-\rho)\rho^{2^n-1}}{1-\rho^{2^n}}r_1 \quad \text{for each } n = 0, 1, \ldots, \tag{5.2}$$

where $\rho = \frac{1-\sqrt{1-2\lambda}}{1+\sqrt{1-2\lambda}}$.

Corollary 5.1. *Suppose that* $\lambda = K\beta \leqslant \frac{1}{2}$ *and that* $\mathrm{D}\xi(p_0)^{-1}\mathrm{D}\xi$ *satisfies the* 2*-piece Lipschitz condition with the Lipschitz constant K in* $\mathbf{B}(p_0, r_1)$. *Then Newton's method* (2.19) *with initial point* p_0 *is well-defined and the generated sequence* $\{p_n\}$ *converges to a singular point* p^* *of* ξ *in* $\overline{\mathbf{B}(p_0, r_1)}$. *Moreover*,

$$d(p_n, p^*) \leqslant \frac{1-\rho}{1-\rho^{2^n}}\rho^{2^n-1}r_1 \leqslant \rho^{2^n-1}r_1 \quad \textit{for each } n = 0, 1, \ldots . \tag{5.3}$$

Corollary 5.2. *Suppose that* $\lambda = K\beta \leqslant \frac{1}{2}$. *Let* $r_1 \leqslant r < r_2$ *if* $\lambda < \frac{1}{2}$ *and* $r = r_1$ *if* $\lambda = \frac{1}{2}$. *Suppose that* $\mathrm{D}\xi(p_0)^{-1}\mathrm{D}\xi$ *satisfies the* 2*-piece Lipschitz condition with the Lipschitz constant K in* $\overline{\mathbf{B}(p_0, r)}$. *Then, there exists a unique singular point* $p^* \in \overline{\mathbf{B}(p_0, r_1)}$ *of* ξ *in* $\overline{\mathbf{B}(p_0, r)}$.

The γ-condition for operators in Banach spaces was first introduced by Wang [30] for the study of Smale's point estimate theory and extended to vector fields on Riemannian manifolds in [19]. In the remainder of this section, we shall always assume that ξ is a C^2-section. Let $r > 0$ and $\gamma > 0$ be such that $r\gamma < 1$. Definition 5.2 extends this notion to sections on Riemannian manifolds but with a stronger version than that in [19]. Recall that the norm of a *k* multi-linear operator *T* on a Banach space *E* is defined by

$$\|T\| = \sup\{\|T\ v_1 v_2 \ldots v_k\| : v_i \in E \text{ and } \|v_i\| \leqslant 1 \text{ for each } i = 1, 2, \ldots, k\}.$$

Definition 5.2. ξ is said to satisfy the 2-piece γ-condition at p_0 in $\mathbf{B}(p_0, r)$, if for any two points $p, q \in \mathbf{B}(p_0, r)$, any geodesic c_2 connecting p, q and minimizing geodesic c_1 connecting p_0, p with $l(c_1) + l(c_2) < r$,

$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p} \circ \mathcal{P}_{c_2,p,q}\mathcal{D}^2\xi(q)\| \leqslant \frac{2\gamma}{(1-\gamma(l(c_1)+l(c_2)))^3}. \tag{5.4}$$

Let $\gamma > 0$ and let L be the function defined by

$$L(u) = \frac{2\gamma}{(1-\gamma u)^3} \quad \text{for each } 0 < u < \frac{1}{\gamma}. \tag{5.5}$$

The following proposition shows that the γ-condition implies the 2-piece L-average Lipschitz condition.

Proposition 5.1. *Suppose that ξ satisfies the 2-piece γ-condition at p_0 in $\mathbf{B}(p_0, r)$. Then $\mathrm{D}\xi(p_0)^{-1}$ $\mathrm{D}\xi$ satisfies the 2-piece L-average Lipschitz condition in $\mathbf{B}(p_0, r)$ with L is defined by* (5.5).

Proof. For any $p, q \in \mathbf{B}(p_0, r)$, let c_1 be a minimizing geodesic connecting p_0, p and c_2 a geodesic connecting p, q such that $l(c_1) + l(c_2) < r$. It is sufficient to prove that

$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}(\mathcal{P}_{c_2,p,q}\mathrm{D}\xi(q)P_{c_2,q,p} - \mathrm{D}\xi(p))\| \leqslant \int_{l(c_1)}^{l(c_1)+l(c_2)} \frac{2\gamma}{(1-\gamma u)^3}\,\mathrm{d}u. \tag{5.6}$$

Let $v \in T_pM$ be arbitrary. Then there exists a unique vector field Y such that $Y(c_2(0)) = v$ and $\nabla_{c_2'(t)}Y = 0$. Then $Y(c_2(s)) = P_{c_2,c_2(s),p}v$ for each $s \in [0, 1]$. Thus we apply Lemma 2.2 (to $\zeta = \mathrm{D}_Y\xi$) to conclude that

$$\begin{aligned}\mathcal{P}_{c_2,p,q}\mathrm{D}\xi(q)P_{c_2,q,p}v - \mathrm{D}\xi(p)v &= \mathcal{P}_{c_2,p,q}\mathrm{D}\xi(q)Y(q) - \mathrm{D}\xi(p)Y(p)\\ &= \mathcal{P}_{c_2,p,q}\mathrm{D}_Y\xi(q) - \mathrm{D}_Y\xi(p)\\ &= \int_0^1 \mathcal{P}_{c_2,p,c_2(s)}(\mathrm{D}(\mathrm{D}_Y\xi(c_2(s)))c_2'(s))\,\mathrm{d}s.\end{aligned} \tag{5.7}$$

Since $\nabla_{c_2'(s)}Y(c_2(s)) = 0$, it follows that

$$\begin{aligned}\mathcal{D}^2\xi(c_2(s))Y(c_2(s))c_2'(s) &= \mathrm{D}_{c'(s)}(\mathrm{D}_Y\xi(c_2(s)) - \mathrm{D}\xi(\nabla_{c_2'(s)}Y(c_2(s))))\\ &= \mathrm{D}(\mathrm{D}_Y\xi(c_2(s)))c_2'(s).\end{aligned}$$

Combining this with (5.7), we have that

$$\mathcal{P}_{c_2,p,q}\mathrm{D}\xi(q)P_{c_2,q,p}v - \mathrm{D}\xi(p)v = \int_0^1 \mathcal{P}_{c_2,p,c_2(s)}(\mathcal{D}^2\xi(c_2(s))Y(c_2(s))c_2'(s))\,\mathrm{d}s. \tag{5.8}$$

Since c_2 is a geodesic connecting p and q, there exists $\overline{v} \in T_pM$ such that $q = \exp_p(\overline{v})$ and $l(c_2) = \|\overline{v}\|$. It follows from (5.8) and (5.4) that

$$\begin{aligned}&\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}(\mathcal{P}_{c_2,p,q}\mathrm{D}\xi(q)P_{c_2,q,p} - \mathrm{D}\xi(p))v\|\\ &\quad\leqslant \int_0^1 \frac{2\gamma}{(1-\gamma(l(c_1)+s\|\overline{v}\|))^3}\|\overline{v}\|\|v\|\,\mathrm{d}s\\ &\quad= \int_{l(c_1)}^{l(c_1)+l(c_2)} \frac{2\gamma}{(1-\gamma u)^3}\,\mathrm{d}u\|v\|.\end{aligned}$$

As $v \in T_pM$ is arbitrary, (5.6) is seen to hold. □

Corresponding to the function L defined by (5.5), r_0 and b in (3.2) are $r_0 = \left(1 - \frac{\sqrt{2}}{2}\right)\frac{1}{\gamma}$ and $b = (3 - 2\sqrt{2})\frac{1}{\gamma}$ and the majorizing function given in (3.3) reduces to

$$h(t) = \beta - t + \frac{\gamma t^2}{1 - \gamma t} \quad \text{for each } 0 \leqslant t \leqslant R.$$

Hence the condition $\beta \leqslant b$ is equivalent that $\alpha = \gamma\beta \leqslant 3 - 2\sqrt{2}$. Let $\{t_n\}$ denote the sequence generated by Newton's method with the initial value $t_0 = 0$ for h. Then the following proposition was proved in [29], see also [19,28].

Proposition 5.2. *Assume that* $\alpha = \gamma\beta \leqslant 3 - 2\sqrt{2}$. *Then the zeros of h are*

$$r_1 = \frac{1 + \alpha - \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma}, \quad r_2 = \frac{1 + \alpha + \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma} \tag{5.9}$$

and

$$\beta \leqslant r_1 \leqslant \left(1 + \frac{1}{\sqrt{2}}\right)\beta \leqslant \left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{\gamma} \leqslant r_2 \leqslant \frac{1}{2\gamma}. \tag{5.10}$$

Moreover, the following assertions hold:

$$t_{n+1} - t_n = \frac{(1 - \mu^{2^n})\sqrt{(1+\alpha)^2 - 8\alpha}}{2\alpha(1 - \nu\mu^{2^n-1})(1 - \nu\mu^{2^{n+1}-1})}\nu\mu^{2^n-1}\beta \leqslant \mu^{2^n-1}\beta \quad \textit{for each } n = 0, 1, \ldots \tag{5.11}$$

and

$$\frac{t_{n+1} - t_n}{t_n - t_{n-1}} \leqslant \mu^{2^{n-1}} \quad \textit{for each } n = 1, 2, \ldots,$$

where

$$\mu = \frac{1 - \alpha - \sqrt{(1+\alpha)^2 - 8\alpha}}{1 - \alpha + \sqrt{(1+\alpha)^2 - 8\alpha}} \quad \textit{and} \quad \nu = \frac{1 + \alpha - \sqrt{(1+\alpha)^2 - 8\alpha}}{1 + \alpha + \sqrt{(1+\alpha)^2 - 8\alpha}}. \tag{5.12}$$

In view of Propositions 5.1 and 5.2, Corollaries 5.3 and 5.5 are direct consequence of Theorems 4.1 and 4.2.

Corollary 5.3. *Suppose that* $\alpha := \beta\gamma \leqslant 3 - 2\sqrt{2}$ *and that* ξ *satisfies the* 2-*piece* γ-*condition at* p_0 *in* $\mathbf{B}(p_0, r_1)$. *Then Newton's method* (2.19) *with initial point* p_0 *is well-defined and the generated sequence* $\{p_n\}$ *converges to a singular point* p^* *of* ξ *in* $\overline{\mathbf{B}(p_0, r_1)}$. *Moreover, if* $\alpha = \beta\gamma < 3 - 2\sqrt{2}$, *there hold*

$$\begin{aligned} &\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,p_{n-1}}\mathcal{P}_{c_n,p_{n-1},p_n}\xi(p_n)\| \\ &\quad \leqslant \mu^{2^{n-1}}\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,p_{n-2}}\mathcal{P}_{c_{n-1},p_{n-2},p_{n-1}}\xi(p_{n-1})\| \end{aligned} \tag{5.13}$$

for each $n = 2, 3, \ldots,$

$$\|\mathrm{D}\xi(p_n)^{-1}\xi(p_n)\| \leqslant \mu^{2^{n-1}}\|\mathrm{D}\xi(p_{n-1})^{-1}\xi(p_{n-1})\| \quad \textit{for each } n = 1, 2, \ldots \tag{5.14}$$

and

$$d(p_{n+1}, p_n) \leqslant \mu^{2^n - 1}\beta \quad \text{for each } n = 0, 1, 2, \dots , \tag{5.15}$$

where for each $n = 1, 2, \dots, c_n$ *is the geodesic in* M *defined by* (4.6).

Note that μ increases as α does on $[0, \frac{13-3\sqrt{17}}{4}]$ and the value of μ at $\alpha = \frac{13-3\sqrt{17}}{4}$ is $\frac{1}{2}$. Hence the following corollary is direct.

Corollary 5.4. *Suppose that* $\alpha = \beta\gamma \leqslant \frac{13-3\sqrt{17}}{4} \approx 0.157671$ *and that* ξ *satisfies the* 2*-piece* γ*-condition at* p_0 *in* $\mathbf{B}(p_0, r_1)$. *Then Newton's method* (2.19) *with initial point* p_0 *is well-defined and the generated sequence* $\{p_n\}$ *converges to a singular point* p^* *of* ξ *in* $\overline{\mathbf{B}(p_0, r_1)}$. *Moreover, the estimates* (5.13)–(5.15) *hold for* $\mu = \frac{1}{2}$.

Corollary 5.5. *Suppose that* $\alpha := \beta\gamma \leqslant 3 - 2\sqrt{2}$. *Let* $r_1 \leqslant r < r_2$ *if* $\beta < b$ *and* $r = r_1$ *if* $\beta = b$. *Suppose that* ξ *satisfies the* 2*-piece* γ*-condition at* p_0 *in* $\overline{\mathbf{B}(p_0, r)}$. *Then, there exists a unique singular point* $p^* \in \overline{\mathbf{B}(p_0, r_1)}$ *of* ξ *in* $\overline{\mathbf{B}(p_0, r)}$.

Remark 5.1. In the case when ξ is a vector field and ξ satisfies the 2-piece γ-condition given in [19,26], which requires that (5.4) holds only for minimizing geodesic c_2 connecting p, q. Corollaries 5.3–5.5 (except for the estimates (5.13) and (5.14)) were proved, respectively, in [19,26] but with an additional assumption that $\beta \leqslant (2-\sqrt{2})\mathbf{r}_{p_0}$, where $\mathbf{r}_{p_0}$ is the radius of injectivity of the exponential map at p_0. Their proofs are based on the key Lemma 4.4 in [6], which indeed is not true, for example, M is the 'hat' surface $x^2 + y^2 = 1 + z^2 - \frac{5}{8}z^3$, which has $\mathbf{r}_{(1,0,0)} = \pi$ and $\mathbf{r}_{(0,0,2)} = \infty$. Fortunately, Lemma 4.4 in [6] becomes true if $\mathbf{r}_{p_0}$ is assumed to be the radius of the totally normal ball around p_0 instead of the radius of injectivity of the exponential map at p_0. Therefore, all results in [19,26] are true and their proofs are still valid provided $\mathbf{r}_{p_0}$ is replaced by the radius of the totally normal ball around p_0. Note that the convergence criterions in the theorems obtained in this section are independent of any parameters of the Riemannian manifold M.

6. Applications to analytic sections and generalized Smale's α-theory

Throughout the whole section, we always assume that M is a real complete analytic Riemannian manifold, $\pi: E \to M$ is a real analytic vector bundle of rank m, and ξ is an analytic section of this vector bundle.

Following [6], we define, for a point $p \in M$,

$$\gamma(\xi, p) = \sup_{k \geqslant 2} \| \mathrm{D}\xi(p)^{-1} \frac{\mathcal{D}^k \xi(p)}{k!} \|_p^{1/(k-1)} . \tag{6.1}$$

Also we adopt the convention that $\gamma(\xi, p) = \infty$ if $\mathrm{D}\xi(p)$ is not invertible. Note that this definition is justified and in the case when $\mathrm{D}\xi(p)$ is invertible, by analyticity, $\gamma(\xi, p)$ is finite. Recall that $p_0 \in M$ is such that $\mathrm{D}\xi(p_0)^{-1}$ exists and $\beta = \| \mathrm{D}\xi(p_0)^{-1}\xi(p_0) \|$.

For the study in the remainder, we need the key Taylor formula. For notational simplicity, for $p \in M$ and $v \in T_{p_0}M$, c will denote the geodesic defined by $c(t) := \exp_{p_0}(tv)$ for each $t \in [0, 1]$ in Proposition 6.1 and its proof.

Proposition 6.1. *Let* $r = \frac{1}{\gamma(\xi, p_0)}$. *Let* $p \in M$ *and* $v \in T_{p_0}M$ *be such that* $\|v\| < r$ *and* $p = \exp_{p_0}(v)$. *Then*

$$\mathcal{D}^j \xi(p) = \mathcal{P}_{c,p,p_0} \left(\sum_{k=0}^{\infty} \frac{1}{k!} \mathcal{D}^{k+j} \xi(p_0) v^k \right) P^j_{c,p_0,p} \quad \text{for each } j = 0, 1, 2, \ldots, \tag{6.2}$$

where $P^j_{c,p_0,p}$ *stands for the map from* $(T_pM)^j$ *to* $(T_{p_0}M)^j$ *defined by*

$$P^j_{c,p_0,p}(v_1, \ldots, v_j) = (P_{c,p_0,p}v_1, \ldots, P_{c,p_0,p}v_j) \quad \text{for each } (v_1, \ldots, v_j) \in (T_pM)^j.$$

Proof. We first show the following Taylor formula:

$$\xi(p) = \mathcal{P}_{c,p,p_0} \left(\sum_{k=0}^{\infty} \frac{1}{k!} \mathcal{D}^k \xi(p_0) v^k \right). \tag{6.3}$$

Let $\{e_i\}$ be a basis of $\pi^{-1}(p_0)$ such that $\{\mathrm{D}\xi(p_0)^{-1}e_i\}$ is an orthonormal base of $T_{p_0}M$. By Lemma 2.1, there exist m analytic functions $\xi^i(t)$, $i = 1, 2, \ldots, m$, such that

$$\mathcal{D}^k \xi(c(t))(c'(t))^k = \sum_{i=1}^{m} \frac{\mathrm{d}^k \xi^i(t)}{\mathrm{d}t^k} \mathcal{P}_{c,c(t),p_0} e_i \quad \text{for each } k = 0, 1, \ldots . \tag{6.4}$$

In particular, as $c'(0) = v$, we get

$$\mathcal{D}^k \xi(p_0) v^k = \sum_{i=1}^{m} \frac{\mathrm{d}^k \xi^i(t)}{\mathrm{d}t^k}|_{t=0} e_i \quad \text{for each } k = 0, 1, \ldots . \tag{6.5}$$

Let $j = 1, 2, \ldots, m$. It follows that

$$\langle \mathrm{D}\xi(p_0)^{-1} \mathcal{D}^k \xi(p_0) v^k, \mathrm{D}\xi(p_0)^{-1} e_j \rangle = \frac{\mathrm{d}^k \xi^j(t)}{\mathrm{d}t^k}|_{t=0} \quad \text{for each } k = 0, 1, \ldots \tag{6.6}$$

because $\{\mathrm{D}\xi(p_0)^{-1}e_i\}$ is an orthonormal basis of $T_{p_0}M$. Note that

$$\overline{\lim_{k\to\infty}} \left(\frac{\|\mathrm{D}\xi(p_0)^{-1}\mathcal{D}^k \xi(p_0)\|}{k!} \right)^{1/k} \leqslant \sup_{k \geqslant 2} \left(\frac{\|\mathrm{D}\xi(p_0)^{-1}\mathcal{D}^k \xi(p_0)\|}{k!} \right)^{1/(k-1)} = \gamma(\xi, p_0).$$

This together with (6.6) yields that

$$\overline{\lim_{k\to\infty}} \left(\frac{1}{k!} \left| \frac{\mathrm{d}^k \xi^j(t)}{\mathrm{d}t^k}|_{t=0} \right| \right)^{1/k} \leqslant \overline{\lim_{k\to\infty}} \left(\frac{1}{k!} \|\mathrm{D}\xi(p_0)^{-1}\mathcal{D}^k \xi(p_0)\| \right)^{1/k} \|v\| < 1 \tag{6.7}$$

since $\|v\| < r = \frac{1}{\gamma(\xi, p_0)}$. Hence

$$\xi^j(t) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k \xi^j(t)}{\mathrm{d}t^k}|_{t=0} t^k \quad \text{for each } t \in [0, 1]. \tag{6.8}$$

Combining this with (6.4) and the fact that $p = c(1)$ gives that

$$\xi(p) = \sum_{i=1}^{m} \xi^i(1) \mathcal{P}_{c,p,p_0} e_i = \sum_{i=1}^{m} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k \xi^i(t)}{\mathrm{d}t^k}|_{t=0} \mathcal{P}_{c,p,p_0} e_i. \tag{6.9}$$

Noting that $\mathcal{P}_{c,p,p_0}$ is a linear isomorphism from $\pi^{-1}(p_0)$ to $\pi^{-1}(p)$, one has that

$$\xi(p)=\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i=1}^{m}\frac{\mathrm{d}^k\xi^i(t)}{\mathrm{d}t^k}|_{t=0}\mathcal{P}_{c,p,p_0}e_i=\mathcal{P}_{c,p,p_0}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i=1}^{m}\frac{\mathrm{d}^k\xi^i(t)}{\mathrm{d}t^k}|_{t=0}e_i\right). \tag{6.10}$$

Thus, (6.3) is seen to hold by (6.5).

Below we will show that (6.2) holds. To do this, let $j=1,2,\dots$ and $v_1,\dots,v_j\in T_pM$. It is sufficient to prove that

$$\begin{aligned}&\mathcal{D}^j\xi(p)(v_1,\dots,v_j)\\&=\mathcal{P}_{c,p,p_0}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\mathcal{D}^{k+j}\xi(p_0)(P_{c,p_0,p}v_1,\dots,P_{c,p_0,p}v_j,\underbrace{v,\dots,v}_{k})\right)\end{aligned} \tag{6.11}$$

because, by the analyticity of ξ,

$$\begin{aligned}&\mathcal{D}^{k+j}\xi(p_0)(\underbrace{v,\dots,v}_{k},P_{c,p_0,p}v_1,\dots,P_{c,p_0,p}v_j)\\&=\mathcal{D}^{k+j}\xi(p_0)(P_{c,p_0,p}v_1,\dots,P_{c,p_0,p}v_j,\underbrace{v,\dots,v}_{k})\end{aligned}$$

holds for each k.

To show (6.11), for each $i=1,\dots,j$, let Y_i be the vector field such that $Y_i(p)=v_i$, $\nabla_{c'(s)}Y_i=0$ and $Y_i(p_0)=P_{c,p_0,p}v_i$. Let $\eta=\mathcal{D}^j\xi(Y_1,\dots,Y_j)$. Then η is a section. Thus, applying (6.3) with ξ replaced by η, we have

$$\eta(p)=\mathcal{P}_{c,p,p_0}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\mathcal{D}^k\eta(p_0)v^k\right). \tag{6.12}$$

In view of the definitions of η and $\mathcal{D}^k\eta$, one can use mathematical induction to verify that

$$\begin{aligned}&\mathcal{D}^k\eta(\underbrace{c'(s),\dots,c'(s)}_{k})\\&=\mathcal{D}^{k+j}\xi(Y_1,\dots,Y_j,\underbrace{c'(s),\dots,c'(s)}_{k})\quad\text{for each } k=0,1,\dots.\end{aligned} \tag{6.13}$$

Since $\mathcal{D}^j\xi(p)(v_1,\dots,v_j)=\eta(p)$ and $Y_i(p_0)=P_{c,p_0,p}v_i$ for each $i=1,\dots,j$, it follows that

$$\begin{aligned}&\mathcal{D}^k\eta(p_0)(\underbrace{v,\dots,v}_{k})\\&=\mathcal{D}^{k+j}\xi(p_0)(Y_1(p_0),\dots,Y_j(p_0),\underbrace{v,\dots,v}_{k})\quad\text{for each } k=0,1,\dots.\end{aligned} \tag{6.14}$$

Combining this with (6.12), (6.11) is seen to hold and the proof is complete. □

Throughout the remainder, let $\gamma=\gamma(\xi,p_0)$. We will show that any analytic section satisfies the γ-condition. For this purpose, we need a simple known fact (cf. [3, p. 150]):

$$\sum_{j=0}^{\infty}\frac{(k+j)!}{k!\,j!}t^j=\frac{1}{(1-t)^{k+1}}\quad\text{for each } t\in[-1,1] \text{ and } k=0,1,\dots. \tag{6.15}$$

For simplicity, we use the function ψ defined by

$$\psi(u) := 1 - 4u + 2u^2 \quad \text{for each } u \in \left[0, 1 - \frac{\sqrt{2}}{2}\right). \tag{6.16}$$

Note that ψ is strictly monotonically decreasing on $[0, 1 - \frac{\sqrt{2}}{2})$.

Lemma 6.1. *Let $p \in M$ and let c be a geodesic connecting p_0 and p such that*

$$u := \gamma l(c) < 1 - \tfrac{\sqrt{2}}{2}. \tag{6.17}$$

Then $\mathrm{D}\xi(p)^{-1}$ exists,

$$\|\mathrm{D}\xi(p)^{-1}\mathcal{P}_{c,p,p_0}\mathrm{D}\xi(p_0)\| \leqslant \frac{(1-u)^2}{\psi(u)} \quad \text{and} \quad \gamma(\xi, p) \leqslant \frac{\gamma}{(1-u)\psi(u)}. \tag{6.18}$$

Proof. Assume that c is defined by

$$c(t) = \exp_{p_0}(tv) \quad \text{for each } t \in [0, 1], \tag{6.19}$$

where $v \in T_{p_0}M$. Then $p = \exp_{p_0}(v)$ and $l(c) = \|v\|$; hence, by (6.17),

$$\gamma\|v\| = u < 1 - \tfrac{\sqrt{2}}{2} < 1. \tag{6.20}$$

Thus, Proposition 6.1 (with $j = 1$) is applicable to concluding that

$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathrm{D}\xi(p)P_{c,p,p_0} - \mathbf{I}_{T_{p_0}M}\| \leqslant \sum_{k=1}^{\infty} \frac{\|\mathrm{D}\xi(p_0)^{-1}\mathcal{D}^{k+1}\xi(p_0)\|}{k!}\|v\|^k.$$

It follows from (6.1), (6.15) and (6.20) that

$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathrm{D}\xi(p)P_{c,p,p_0} - \mathbf{I}_{T_{p_0}M}\| \leqslant \sum_{k=1}^{\infty}(k+1)\gamma^k\|v\|^k = \frac{1}{(1-u)^2} - 1 < 1.$$

Thus, $\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathrm{D}\xi(p)P_{c,p,p_0}$ is invertible by the Banach lemma. Consequently, $\mathrm{D}\xi(p)^{-1}$ exists and the first inequality of (6.18) holds because P_{c,p,p_0} is an isometry. To establish the second inequality of (6.18), let $k = 2, 3, \ldots$ and observe that

$$\left\|\mathrm{D}\xi(p)^{-1}\frac{\mathcal{D}^k\xi(p)}{k!}\right\| \leqslant \|\mathrm{D}\xi(p)^{-1}\mathcal{P}_{c,p,p_0}\mathrm{D}\xi(p_0)\| \left\|\frac{\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathcal{D}^k\xi(p)}{k!}\right\|. \tag{6.21}$$

In view of (6.19) and (6.20), one can apply Proposition 6.1 (with $j = k$) to conclude that

$$\begin{aligned}\left\|\frac{\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathcal{D}^k\xi(p)}{k!}\right\| &= \left\|\frac{1}{k!}\sum_{j=0}^{\infty}\frac{1}{j!}\mathrm{D}\xi(p_0)^{-1}\mathcal{D}^{k+j}\xi(p_0)v^j P^k_{c,p_0,p}\right\| \\ &\leqslant \sum_{j=0}^{\infty}\frac{(k+j)!}{k!j!}\left\|\frac{\mathrm{D}\xi(p_0)^{-1}\mathcal{D}^{k+j}\xi(p_0)}{(k+j)!}\right\|\|v\|^j \\ &\leqslant \sum_{j=0}^{\infty}\frac{(k+j)!}{k!j!}\gamma^{k+j-1}\|v\|^j.\end{aligned} \tag{6.22}$$

This, together with (6.15), implies that

$$\frac{\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathcal{D}^k\xi(p)\|}{k!}\leqslant\frac{\gamma^{k-1}}{(1-\gamma\|v\|)^{k+1}}. \tag{6.23}$$

Combining this with the first inequality of (6.18) and (6.21) gives that

$$\left\|\mathrm{D}\xi(p)^{-1}\frac{\mathcal{D}^k\xi(p)}{k!}\right\|\leqslant\frac{1}{\psi(u)}\left(\frac{\gamma}{1-u}\right)^{k-1} \tag{6.24}$$

thanks to (6.20). Consequently,

$$\gamma(\xi,p)=\sup_{k\geqslant 2}\left\|\mathrm{D}\xi(p)^{-1}\frac{\mathcal{D}^k\xi(p)}{k!}\right\|_p^{\frac{1}{k-1}}\leqslant\frac{\gamma(\xi,p_0)}{1-u}\sup_{k\geqslant 2}\frac{1}{\psi(u)^{\frac{1}{k-1}}}=\frac{\gamma(\xi,p_0)}{(1-u)\psi(u)},$$

where the last equality holds because the supremum attains at $k=2$ as $0<\psi(u)\leqslant 1$ by (6.17). □

Proposition 6.2. *Let* $0<r\leqslant\frac{2-\sqrt{2}}{2\gamma}$. *Then* ξ *satisfies the* 2*-piece* γ*-condition at* p_0 *in* $\mathbf{B}(p_0,r)$.

Proof. Let $p,q\in\mathbf{B}(p_0,r)$. Let c_1 be a geodesic connecting p_0, p and c_2 a geodesic connecting p,q such that, for some $v_1\in T_{p_0}M$ and $v_2\in T_pM$, $p=\exp_{p_0}(v_1)$, $q=\exp_p(v_2)$, $l(c_1)=\|v_1\|, l(c_2)=\|v_2\|$ with $l(c_1)+l(c_2)<r$. Since $u:=\gamma l(c_1)\leqslant\gamma r<\frac{2-\sqrt{2}}{2}$, Lemma 6.1 is applicable. It follows that

$$\gamma(\xi,p)\leqslant\frac{\gamma}{(1-u)\psi(u)}. \tag{6.25}$$

Since

$$\|v_1\|+\|v_2\|<r\leqslant\frac{1-\frac{\sqrt{2}}{2}}{\gamma},$$

it follows (6.25) that

$$\|v_2\|<\frac{1-\frac{\sqrt{2}}{2}}{\gamma}-\|v_1\|=\frac{1-\frac{\sqrt{2}}{2}-u}{\gamma}\leqslant\frac{(1-u)\psi(u)}{\gamma}\leqslant\frac{1}{\gamma(\xi,p)},$$

where the second inequality holds because, for each $u\in[0,1]$, $(1-u)\psi(u)\geqslant 1-\frac{\sqrt{2}}{2}-u$ by a simple calculation. Hence, $q\in\mathbf{B}(p,\frac{1}{\gamma(\xi,p)})$. Thus, Proposition 6.1 is applicable to concluding that

$$\begin{aligned}\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}\circ\mathcal{P}_{c_2,p,q}\mathcal{D}^2\xi(q)&=\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}\sum_{i=0}^{\infty}\frac{1}{i!}\mathcal{D}^{i+2}\xi(p)v_2^iP_{c_2,p,q}^2\\&=\mathrm{D}\xi(p_0)^{-1}\sum_{i=0}^{\infty}\frac{1}{i!}\sum_{j=0}^{\infty}\frac{1}{j!}\mathcal{D}^{j+i+2}\xi(p_0)\\&\quad\times v_1^jP_{c_1,p_0,p}^{i+2}v_2^iP_{c_2,p,q}^2.\end{aligned} \tag{6.26}$$

Since

$$\left\| \frac{\mathrm{D}\xi(p_0)^{-1}\mathcal{D}^{j+i+2}\xi(q_0)}{(j+i+2)!} \right\| \leqslant \gamma^{j+i+1},$$

one has from (6.26) that

$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}\circ\mathcal{P}_{c_2,p,q}\mathcal{D}^2\xi(q)\| \leqslant \sum_{i=0}^{\infty}\frac{1}{i!}\sum_{j=0}^{\infty}\frac{(j+i+2)!}{j!}\gamma^{j+i+1}\|v_1\|^j\|v_2\|^i. \tag{6.27}$$

Using (6.15) to calculate the quantity on the right-hand side of the inequality (6.27), we get that

$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}\circ\mathcal{P}_{c_2,p,q}\mathcal{D}^2\xi(q)\| \leqslant \frac{2\gamma}{(1-\gamma(\|v_1\|+\|v_2\|))^3} = \frac{2\gamma}{(1-\gamma(l(c_1)+l(c_2)))^3}$$

and the proof is complete. □

Recall that $\beta = \| \mathrm{D}\xi(p_0)^{-1}\xi(p_0) \|$ and $\gamma = \gamma(\xi, p_0)$. Then, by Proposition 6.2, Corollaries 5.3–5.5 are applied to deduce Corollaries 6.1 and 6.2.

Corollary 6.1. *Suppose that* $\alpha = \beta\gamma \leqslant 3 - 2\sqrt{2}$. *Then the following assertions hold.*

(i) *Newton's method* (2.19) *with initial point* p_0 *is well-defined and the generated sequence* $\{p_n\}$ *converges to a singular point* p^* *of* ξ *in* $\overline{\mathbf{B}(p_0, r_1)}$.

(ii) *If* $\alpha = \beta\gamma < 3 - 2\sqrt{2}$, *then*

$$d(p_{n+1}, p_n) \leqslant \mu^{2^n-1}\beta \quad \textit{for each } n = 0, 1, 2, \ldots . \tag{6.28}$$

(iii) *There exists a unique singular point* $p^* \in \overline{\mathbf{B}(p_0, r_1)}$ *of* ξ *in* $\mathbf{B}(p_0, \frac{2-\sqrt{2}}{2\gamma})$.

Corollary 6.2. *Suppose that* $\alpha = \beta\gamma \leqslant \frac{13-3\sqrt{17}}{4} \approx 0.157671$. *Then Newton's method* (2.19) *with initial point* p_0 *is well-defined and the generated sequence* $\{p_n\}$ *converges to a singular point* p^* *of* ξ *in* $\overline{\mathbf{B}(p_0, r_1)}$. *Moreover,*

$$d(p_{n+1}, p_n) \leqslant \left(\frac{1}{2}\right)^{2^n-1}\beta \quad \textit{for each } n = 0, 1, 2, \ldots . \tag{6.29}$$

Remark 6.1. Recall that $\mathbf{r}_{p_0}$ denotes the radius of injectivity of the exponential map at p_0. In the case when ξ is a vector field or a mapping from M to $\mathbb{R}^m$ and D is the Levi–Civita connection, Dedieu et al. proved in [6] that if

$$\beta \leqslant s_0\mathbf{r}_{p_0} \quad \text{and} \quad \alpha = \beta\gamma(\xi, p_0) \leqslant \alpha_0, \tag{6.30}$$

then the conclusion of Corollary 6.2 holds, where $\alpha_0 = 0.130716944\cdots$ is the unique root of the equation $2u = (1 - 4u + 2u^2)^2$ in $(0, 1 - \frac{\sqrt{2}}{2})$; while

$$s_0 = \frac{1}{\sigma + \frac{(1-\sigma\alpha_0)^2}{\psi(\sigma\alpha_0)}(1 + \frac{\sigma}{1-\sigma\alpha_0})} = 0.103621842\cdots \quad \text{and}$$

$$\sigma = \sum_{k\geqslant 0}\left(\frac{1}{2}\right)^{2^k-1} = 1.632843018\cdots.$$

This result was improved in a recent paper [19] in such way that the criterion (6.30) replaced by the weaker one:

$$\beta \leqslant (2-\sqrt{2})\mathbf{r}_{p_0} \quad \text{and} \quad \alpha = \beta\gamma(\xi, p_0) \leqslant \frac{13-3\sqrt{17}}{4} \approx 0.157671. \tag{6.31}$$

As mentioned in Remark 5.1, their proofs are based on the key Lemma 4.4 in [6], which is not correct. Corollaries 6.1 and 6.2 improve excellently the corresponding results in [6,19]. Note that the convergence criterions in Corollaries 6.1 and 6.2 are independent of the parameter $\mathbf{r}_{p_0}$.

The remainder of this section is devoted to the determination of an approximate singular point of an analytic section. For the purpose, we first recall the notion of the approximate zero of an analytic mapping f from the domain U in a Banach space to another. The following unified definition is taken from [28]. Consider Newton's iteration with initial point x_0:

$$x_{n+1} = x_n - f'(x_n)^{-1} f(x_n) \quad \text{for each } n = 0, 1, 2, \ldots . \tag{6.32}$$

Definition 6.1. Suppose $x_0 \in U$ is such that Newton's iteration (6.32) is well-defined for f and satisfies

$$e(x_n) \leqslant \left(\frac{1}{2}\right)^{2^{n-1}} e(x_{n-1}) \quad \text{for all } n = 1, 2, \ldots,$$

where $e(x_n)$ denotes some measurement of the approximation degree between x_n and the zero point x^*. Then x_0 is said to be an approximate zero of f in the sense of $e(x_n)$.

The notion of the approximate zero in the sense of $\|x_{n+1} - x_n\|$ was introduced in [21]; while the second kind of approximate zero is defined in the sense of $\|x_n - x^*\|$ in [21] and a more reasonable definition for the second kind was given in [3,22], which is also presented and studied by Wang in [31]. The notion of the approximate zero in the sense of $\|f'(x_0)^{-1} f(x_n)\|$ was defined in [4] and, as shown in [28], it is equivalent to that in the sense of $\|x_{n+1} - x_n\|$, or equivalently, in the sense of $\|f'(x_n)^{-1} f(x_n)\|$. Definition 6.2 in the following extends the notion of the approximate zero to the case of sections in Remannian manifolds.

Definition 6.2. Suppose $p_0 \in M$ is such that Newton's method (2.19) is well-defined for ξ and satisfies

$$\Theta(p_n) \leqslant \left(\frac{1}{2}\right)^{2^{n-1}} \Theta(p_{n-1}) \quad \text{for all } n = 1, 2, \ldots,$$

where $\Theta(p_n)$ denotes some measurement of the approximation degree between p_n and the singular point p^*. Then p_0 is said to be an approximate singular point of ξ in the sense of $\Theta(p_n)$.

Note that, in general, the approximate zero in the sense of $\|\mathrm{D}\xi(p_n)^{-1}\xi(p_n)\|$ is not equivalent to that in the sense of $d(p_{n+1}, p_n)$ on Riemannian manifolds. In view of Proposition 6.2, Corollary 5.4 implies the following corollary.

Corollary 6.3. *Suppose that* $\alpha = \beta\gamma \leqslant \frac{13-3\sqrt{17}}{4} \approx 0.157671$. *For each* $n = 1, 2, \ldots$, *let* c_n *be the geodesic in M defined by* (4.6). *Then* p_0 *is an approximate singular point of* ξ *in the senses of* $\|\mathrm{D}\xi(p_n)^{-1}\xi(p_n)\|$ *and* $\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,p_{n-1}}\mathcal{P}_{c_n,p_{n-1},p_n}\xi(p_n)\|$.

7. Concluding remark

Recently, in the case when $\xi = X$ is a vector field and D is the Levi–Civita connection, Alvarez et al. introduced in [2] a Lipschitz-type continuity of $\mathrm{D}X$ which is a Riemannian analogue of the property used in [33] by Zabrejko and Nguen:

$$\| \mathrm{D}X(p_0)^{-1}[P_{c,c(0),c(b)}\mathrm{D}X(c(b)) - P_{c,c(0),c(a)}\mathrm{D}X(c(a))] \| \leqslant L(u)l(c|_{[a,b]}) \tag{7.1}$$

a holds for each $u \in [0, R]$, $0 \leqslant a \leqslant b$, and $c \in C_2(p_0, r)$, where R is a positive real number and $C_2(p_0, r)$ is the set of all the piecewise geodesics $c: [0, T] \to M$ such that $c(0) = p_0$ and $l(c) < r$, and there exists $\tau \in (0, T]$ satisfying $c|_{[0,\tau]}$ is a minimizing geodesic and $c|_{[\tau,T]}$ is a geodesic. The same convergence criterion was established for Newton's method of vector fields on Riemannian manifolds in [2]. As applications, Corollary 6.1 was also obtained for analytic vector fields and analytic mappings on Riemannian manifolds in [2], respectively. Taylor formula for any geodesic plays a crucial role, which, however, was claimed in [2] without the proof and without the reference.

The approach used in the present paper is clearly different from that in [2], and it is not difficult to see that the Lipschitz-type continuity (7.1) implies the 2-piece L-average Lipschitz condition. The results in the present paper are deeper and more general. In particular, the convergence criterion of Newton's method for sections was established for any affine connections (not necessary the Levi–Civita connection), which has not been explored before. Furthermore, even in the case when the section is an analytic vector field and the affine connection is the Levi–Civita connection, the criterion of the present paper to judge a point to be an approximate singular point of an analytic section is new to the best of our knowledge.

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