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Limit theory of restricted range approximations of complex-valued continuous functions

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Abstract This paper is concerned with the problem of the best restricted range approximations of complex-valued continuous functions. Several properties for the approximating set $\mathcal{P}_\Omega$ such that the classical characterization results and/or the uniqueness results of the best approximations hold are introduced. Under the very mild conditions, we prove that these properties are equivalent that $\mathcal{P}$ is a Haar subspce.

Keywords: restricted range approximation, Haar subspace, characterization, uniqueness

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1 Introduction

Recent attention is focused on the problem of the best restricted range approximations in the space of complex-valued continuous functions. The setting is as follows. Let Q be a compact metric space and let $C(Q)$ denote the Banach space of all complex-valued continuous functions on Q endowed with the uniform norm:

$$\|f\| = \max_{t\in Q} |f(t)|, \quad \forall f \in C(Q).$$

Let $\{\Omega_t : t \in Q\}$ be a family of nonempty closed convex sets in the complex plane $\mathbb{C}$ and let $\mathcal{P}$ be a finite dimensional subspace of $C(Q)$. Set $\mathcal{P}_\Omega = \{p \in \mathcal{P} : p(t) \in \Omega_t, \ \forall\, t \in Q\}$. Then, for a given $f \in C(Q)$, an element p^* of $\mathcal{P}_\Omega$ is called a best (restricted range) approximation to f from $\mathcal{P}_\Omega$ if and only if $\|f - p^*\| = d(f, \mathcal{P}_\Omega)$, where $d(f, \mathcal{P}_\Omega)$ is defined by $d(f, \mathcal{P}_\Omega) = \inf_{p\in\mathcal{P}_\Omega} \|f - p\|$. This problem was first introduced and formulated by Smirnov and Smirnov in [1, 2], where the characterization theorems and the uniqueness theorems of the best restricted range approximation were obtained under the special case when $\{\Omega_t : t \in Q\}$ is a system of closed disks with centers and radii continuously depending on t, and $\mathcal{P}$ is a Haar subspace. These results were extended to some more general cases in [3–5]. In particular, this problem was considered in [6] for a very general case when the strong interior-point condition and the lower semicontinuity of the set-valued function $t \mapsto \Omega_t$ on Q are assumed.

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The interest of the present paper is in the converse problem, that is, characterizing the approximating set $\mathcal{P}_\Omega$ for which the corresponding characterization and/or the uniqueness theorems hold. A similar problem in the simple case for uniqueness was also considered in [4]. In the present paper, making use of the introduced new notion of an extremal bi-support, we define several properties such as $\mathcal{C}$, $\underline{\mathcal{C}}$, $\mathcal{C}_i$, $\underline{\mathcal{C}}_i$, $\mathcal{C}^*$ and $\underline{\mathcal{C}}^*$ as well as $\mathcal{U}$ and $\mathcal{U}_i$. The characterization results for the approximating set $\mathcal{P}_\Omega$ to have these properties are provided, respectively.

The paper is motivated by the ideas in [7, 8] where similar properties were introduced and characterized respectively for the non-restricted and restricted range approximations of real-valued continuous functions by means of the notion of an alternation. However, this problem for the case of complex-valued continuous function approximations is more difficult than that for the case of real-valued continuous function approximations because the characterization of the alternation type, which is a powerful tool and plays a key role in the case of real-valued continuous function approximations, is invalid in the case of complex-valued continuous function approximations. In fact, the technique used in the present paper is completely different from that in [7, 8].

2 Notions and preliminary results

In this paper, we use $\mathbb{C}$ to denote the complex plane and $\mathbf{B}(z_0, \delta)$ the open ball of $\mathbb{C}$ with a center z_0 and a radius δ. Let Z be a closed subset of $\mathbb{C}$. The interior (resp. boundary, convex hull) of Z is denoted by $\mathrm{int}Z$ (resp. $\mathrm{bd}Z$, $\mathrm{co}Z$). The normal cone of Z at z_0 is denoted by $N_Z(z_0)$ and defined by $N_Z(z_0) = \{\tau \in \mathbb{C} : \ \mathrm{Re}\overline{\tau}(z - z_0) \leqslant 0, \ \forall z \in Z\}$. The distance from z_0 to Z is denoted by $d(z_0, Z)$ and defined by $d(z_0, Z) = \inf\{|z_0 - z| : \ z \in Z\}$.

One basic assumption in the study of the restricted range approximation problem of complex-valued continuous functions is that $\mathcal{P}_\Omega$ has an interior point or a strong interior point which are defined as follows, see [1–6] for details.

Definition 2.1. *A point $\overline{p} \in \mathcal{P}_\Omega$ is called*

(i) *An interior point of $\mathcal{P}_\Omega$ if*

$$\overline{p}(t) \in \mathrm{int}\Omega_t, \quad \forall\ t \in Q; \tag{2.1}$$

(ii) *A strong interior point of $\mathcal{P}_\Omega$ if there exists a positive number δ such that*

$$\mathbf{B}(\overline{p}(t), \delta) \subseteq \Omega_t, \quad \forall\ t \in Q. \tag{2.2}$$

Moreover, $\mathcal{P}_\Omega$ is said to satisfy the interior-point (*resp. strong interior-point*) *condition if there exists $\overline{p} \in \mathcal{P}_\Omega$ such that* (2.1) (*resp.* (2.2)) *holds.*

Clearly, a strong interior point is an interior point. Proposition 2.1 below shows that the converse is true provided that the set-valued function $\Omega : t \to \Omega_t$ is lower semicontinuous on Q, which is defined as follows (cf. [9]).

Definition 2.2. *Let $\Omega : t \mapsto \Omega_t$ be a set-valued function defined on Q. Then Ω is said to be*

(i) *Lower semicontinuous at $t_0 \in Q$ if, for any $z_0 \in \Omega_{t_0}$ and any $\epsilon > 0$, there exists an open neighborhood $U(t_0)$ of t_0 such that $\Omega_t \cap B(z_0, \epsilon) \neq \emptyset$ for all $t \in U(t_0)$;*

(ii) *Lower semicontinuous on Q if it is lower semicontinuous at each point $t_0 \in Q$.*

Proposition 2.1. *Suppose that the set-valued function $\Omega : t \mapsto \Omega_t$ is lower semicontinuous on Q. Let $\overline{p} \in \mathcal{P}_\Omega$. Then $\overline{p}$ is an interior point of $\mathcal{P}_\Omega$ if and only if it is a strong interior point of $\mathcal{P}_\Omega$.*

Proof. We only need to prove the necessity part. Suppose that $\overline{p}$ is an interior point of $\mathcal{P}_\Omega$ but not a strong interior point of $\mathcal{P}_\Omega$. Then, for every positive integer n, there exists $t_n \in Q$ such that $\mathbf{B}(\overline{p}(t_n), \frac{1}{n}) \nsubseteq \Omega_{t_n}$. Pick $z_n \in \mathbf{B}(\overline{p}(t_n), \frac{1}{n})\backslash\Omega_{t_n}$. Then $|z_n - \overline{p}(t_n)| < \frac{1}{n}$. Without loss of generality, we may assume $t_n \to t_0$. Then $\overline{p}(t_n) \to \overline{p}(t_0)$ and $z_n \to \overline{p}(t_0) \in \Omega_{t_0}$. By the assumption, $\mathbf{B}(\overline{p}(t_0), \delta) \subseteq \Omega_{t_0}$ for some $\delta > 0$. Let $0 < \epsilon < \frac{1}{4}\delta$. Then there exists a natural number N_1 such that

$$|\overline{p}(t_n) - \overline{p}(t_0)| < \frac{1}{4}\delta \quad \text{and} \quad |z_n - \overline{p}(t_0)| < \epsilon, \quad \forall\, n > N_1. \tag{2.3}$$

For each $n > N_1$, define

$$x_n = \overline{p}(t_n) + \lambda_n(z_n - \overline{p}(t_n)) \tag{2.4}$$

and

$$y_n = \overline{p}(t_n) + \mu_n(z_n - \overline{p}(t_n))\mathbf{i}, \tag{2.5}$$

where $\mathbf{i} = \sqrt{-1}$. Choose $\lambda_n > 0$ and $\mu_n > 0$ such that

$$|x_n - \overline{p}(t_0)| = |y_n - \overline{p}(t_0)| = \frac{3}{4}\delta. \tag{2.6}$$

Then, it follows from (2.3) and (2.6) that $\lambda_n > 1$ and $\mu_n > 1$. Without loss of generality, we may assume that $x_n \to w_1$ and $y_n \to w_2$. Then by (2.6) one has that

$$|w_1 - \overline{p}(t_0)| = |w_2 - \overline{p}(t_0)| = \frac{3}{4}\delta. \tag{2.7}$$

Since

$$y_n - \overline{p}(t_n) = \frac{\mu_n}{\lambda_n}(x_n - \overline{p}(t_n))\mathbf{i}, \tag{2.8}$$

it follows from (2.8) that $\frac{\mu_n}{\lambda_n} \to 1$ and

$$w_2 - \overline{p}(t_0) = (w_1 - \overline{p}(t_0))\mathbf{i}. \tag{2.9}$$

Set $u_n = 2\overline{p}(t_n) - x_n, v_n = 2\overline{p}(t_n) - y_n$. Then $u_n \to w_3 := 2\overline{p}(t_0) - w_1$ and $v_n \to w_4 := 2\overline{p}(t_0) - w_2$. Clearly,

$$w_3 - \overline{p}(t_0) = -(w_1 - \overline{p}(t_0)), \quad w_4 - \overline{p}(t_0) = -(w_1 - \overline{p}(t_0))\mathbf{i}, \tag{2.10}$$

where the second equality is by (2.9). Furthermore, by (2.7) and (2.10),

$$|w_3 - \overline{p}(t_0)| = |w_4 - \overline{p}(t_0)| = \frac{3}{4}\delta. \tag{2.11}$$

Note that, by (2.7), (2.9), (2.10) and (2.11), $\text{co}\{w_k\}_{k=1}^4$ is a square with the center $\overline{p}(t_0)$ and the length of each side $\frac{3}{4}\sqrt{2}\delta$. Hence, there exists $\epsilon > 0$ such that, for any set $\{w'_k\}_{k=1}^4$ with $w'_k \in \mathbf{B}(w_k, \epsilon)$ for $k = 1, \ldots, 4$,

$$\mathbf{B}(\overline{p}(t_0), \epsilon) \subseteq \text{co}(\{w'_k\}_{k=1}^4). \tag{2.12}$$

Since $\mathbf{B}(\overline{p}(t_0),\delta)\subseteq\Omega_{t_0}$, it follows from (2.7) and (2.11) that

$$w_k\in\mathbf{B}\Big(w_k,\frac{1}{4}\delta\Big)\subseteq\Omega_{t_0}\quad\forall\ k=1,\dots,4.$$

As the set-valued function $t\to\Omega_t$ is lower semicontinuous at t_0, there exists natural a number $N_2>N_1$ such that for any $n>N_2$, $\mathbf{B}(w_k,\epsilon)\cap\Omega_{t_n}\neq\emptyset$ for each $k=1,\dots,4$. Take $w_k'\in\mathbf{B}(w_k,\epsilon)\cap\Omega_{t_n}$ ($\forall\ k=1,\dots,4$). Then, (2.12) and the fact that Ω_{t_n} is convex,

$$\mathbf{B}(\overline{p}(t_0),\epsilon)\subseteq\mathrm{co}\{w_k'\}_{k=1}^4\subseteq\Omega_{t_n}.\tag{2.13}$$

Thanks to Combining (2.13) and (2.3) implies that $z_n\in\Omega_{t_n}$, which contradicts the choice of z_n. Hence $\overline{p}$ is a strong interior point of $\mathcal{P}_\Omega$ and the proof is complete.

Throughout the whole paper, we always assume that $\{\Omega_t\}$ satisfies the following Hypotheses.

Hypothesis 1. *The set-valued function $t\mapsto\Omega_t$ is lower semicontinuous on Q.*

Hypothesis 2. *$\mathcal{P}_\Omega$ has an interior point $\overline{p}$.*

Let $f\in C(Q)$ and $p^*\in\mathcal{P}_\Omega$. Following [3, 6], set

$$M(f-p^*)=\{t\in Q:\ |f(t)-p(t)|=\|f-p^*\|\},\quad B(p^*)=\{t\in Q:\ p^*(t)\in\mathrm{bd}\Omega_t\}.$$

Moreover, define

$$\sigma(p^*,\ t)=\mathrm{sign}\,(f(t)-p^*(t)),\quad\forall\ t\in Q\tag{2.14}$$

and

$$\tau(p^*,t)=\{\overline{\tau}:\ \tau\in-N_{\Omega_t}(p^*(t))\setminus 0\},\quad\forall\ t\in Q,\tag{2.15}$$

where $\mathrm{sign}z=\overline{z}/|z|$ if $z\neq 0$ and 0 if $z=0$. Note that $\tau(p^*,t)\neq\emptyset$ since $N_{\Omega_t}(p^*(t))\neq\{0\}$ if $t\in B(p^*)$, and that, for each $t\in B(p^*)$, $\tau\in\tau(p^*,\ t)$ if and only if $\overline{\tau}\in-N_{\Omega_t}(p^*(t))\setminus\{0\}$.

The following two classes of admissible functions were introduced by Smirnov and Smirnov in [2, 4] respectively for the study of the uniqueness problem of the best restricted range approximation of complex-valued continuous functions.

Definition 2.4. *A function $f\in C(Q)$ is called*

(i) *Admissible of type* I *if*

$$f(t)\in\Omega_t,\quad\forall\ t\in Q\tag{2.16}$$

or there exists a best approximation p^* to f from $\mathcal{P}_\Omega$ such that

$$M(f-p^*)\cap B(p^*)=\emptyset;\tag{2.17}$$

(ii) *Admissible of type* II *if $f\in\mathcal{P}_\Omega$ or*

$$d(f,\mathcal{P}_\Omega)>\sup_{t\in Q}d(f(t),\Omega_t).\tag{2.18}$$

The set of all admissible functions of type I *(resp.* II*) is denoted by $C_a^1(Q)$ (resp. $C_a^2(Q)$).*

Below we will prove that $C_a^1(Q)$ is contained in $C_a^2(Q)$. For this end, we first give two lemmas.

Lemma 2.1. *Let F be the function on Q defined by $F(t)=d(f(t),\Omega_t)$ for each $t\in Q$. Then F is upper semicontinuous on Q.*

Proof. Let $t_0 \in Q$. Then there exists $z_0 \in \Omega_{t_0}$ such that $F(t_0) = |f(t_0) - z_0|$. Let $\epsilon > 0$. Since f is continuous at t_0, it follows from Hypothesis 1 that there exists an open neighborhood $U(t_0)$ of t_0 such that $\Omega_t \cap \mathbf{B}(z_0, \frac{\epsilon}{2}) \neq \emptyset$ and $|f(t) - f(t_0)| < \frac{\epsilon}{2}$ for all $t \in U(t_0)$. Thus, for each $t \in U(t_0)$ and $z_t \in \Omega_t \cap \mathbf{B}(z_0, \frac{\epsilon}{2})$, we have that

$$F(t) \leqslant |f(t) - z_t| \leqslant |f(t) - f(t_0)| + |f(t_0) - z_0| + |z_0 - z_t| < F(t_0) + \epsilon.$$

This shows that $F(t)$ is upper semicontinuous at t_0 since $\epsilon > 0$ is arbitrary.

Lemma 2.2. *$f \in C_a^2(Q) \setminus \mathcal{P}_\Omega$ if and only if there exists a best approximation p^* to f from $\mathcal{P}_\Omega$ such that*

$$\|f - p^*\| > d(f(t), \Omega_t), \quad \forall\, t \in M(f - p^*). \tag{2.19}$$

Proof. It suffices to prove the "if" part since the proof of the "only if" part is trivial. Suppose that there exists a best approximation p^* to f from $\mathcal{P}_\Omega$ such that (2.19) holds. Noting that the function $t \mapsto d(f(t), \Omega_t)$ is upper semicontinuous on Q by Lemma 2.1, one has that

$$d(f, \mathcal{P}_\Omega) = \|f - p^*\| > \max\{d(f(t), \Omega_t) :\ t \in M(f - p^*)\} \tag{2.20}$$

since $M(f-p^*)$ is compact. Consequently, there exists an open set $Q_1 \subseteq Q$ with $Q_1 \supseteq M(f-p^*)$ such that

$$d(f, \mathcal{P}_\Omega) > \sup_{t \in Q_1} d(f(t), \Omega_t). \tag{2.21}$$

Let $Q_2 = Q \setminus Q_1$. Then, for each $t \in Q_2$, one has $t \notin M(f - p^*)$. It follows that

$$\|f - p^*\| > |f(t) - p^*(t)| \geqslant d(f(t), \Omega_t), \quad \forall\, t \in Q_2.$$

Since Q_2 is compact and the function $t \mapsto d(f(t), \Omega_t)$ is upper semicontinuous by Lemma 2.1, we have that $d(f, \mathcal{P}_\Omega) > \max_{t \in Q_2} d(f(t), \Omega_t)$. This together with (2.21) gives (2.18) and so $f \in C_a^2(Q) \setminus \mathcal{P}_\Omega$.

Proposition 2.2. *$C_a^1(Q) \subseteq C_a^2(Q)$.*

Proof. Let $f \in C_a^1(Q)$. Then (2.16) or (2.17) holds. Without loss of generality, we may assume that $f \notin \mathcal{P}_\Omega$. Hence $d(f, \mathcal{P}_\Omega) > 0$. Thus, to complete the proof, it suffices to show (2.19). Since (2.19) is clear in the case when (2.16) holds, it remains to show (2.19) in the case when (2.17) holds. Let $t \in M(f - p^*)$. Then $t \notin B(p^*)$ by (2.17); hence $p^*(t) \in \mathrm{int}\Omega_t$, which implies that $|f(t)-p^*(t)| > d(f(t), \Omega_t)$. Consequently, $\|f-p^*\| = |f(t)-p^*(t)| > d(f(t), \Omega_t), \forall\, t \in M(f-p^*)$, that is, (2.19) holds.

Remark 2.1. In general, $C_a^1(Q) \neq C_a^2(Q)$ as shown by the following

Example 2.1. Let $Q = \{-1, 0, 1\}$, $\Omega_{-1} = \Omega_1 = \{z : \mathrm{Re} z \geqslant 1\}$ and $\Omega_0 = \mathbb{C}$. Let $\mathcal{P} = \mathrm{span}\{1, t\}$. Then $\mathcal{P}_\Omega = \{p : p(t) = a + bt,\ \mathrm{Re}(a+b) \geqslant 1,\ \mathrm{Re}(a-b) \geqslant 1\}$. Now, define $f \in C(Q)$ by $f(-1) = 1/2,\ f(0) = 0$ and $f(1) = 2$. Then $d(f, \mathcal{P}_\Omega) = 1$. In fact, for each $p(t) = a+bt \in \mathcal{P}_\Omega$, we have

$$\mathrm{Re}(a+b) \geqslant 1, \quad \mathrm{Re}(a-b) \geqslant 1; \tag{2.22}$$

hence $\mathrm{Re}\, a \geqslant 1$ and

$$\|f - p\| \geqslant |f(0) - p(0)| = |a| \geqslant \mathrm{Re}\, a \geqslant 1. \tag{2.23}$$

Set $p^*=1$. Then $p^*\in\mathcal{P}_\Omega$ and $\|f-p^*\|=1$. This together with (2.23) implies that $d(f,\mathcal{P}_\Omega)=1$ and p^* is a best approximation to f from $\mathcal{P}_\Omega$. Moreover, we have that $f\in C_a^2(Q)$ because

$$d(f,\mathcal{P}_\Omega)=1>\frac{1}{2}=\sup_{t\in Q}d(f(t),\Omega_t).$$

However $f\notin C_a^1(Q)$. In fact, note that (2.16) does not hold and $M(f-p^*)\cap B(p^*)=\{1\}$. It suffices to prove that p^* is a unique best approximation to f from $\mathcal{P}_\Omega$. For this purpose, let $p\in\mathcal{P}_\Omega$ satisfy $\|f-p\|=1$. Assume that $p(t)=a+bt$. Then $a=1$ by (2.23) and so Re$b=0$ by (2.22). Thus

$$1=\|f-p\|\geqslant|f(1)-p(1)|=|b-1|=\sqrt{1+(\mathrm{Im}\ b)^2}\geqslant 1.$$

This yields Im $b=0$, and hence $p=p^*=1$.

Let

$$\tilde{M}(p^*)=\{t\in M(f-p^*)\cap B(p^*):\ \overline{\sigma(p^*,t)}\in N_{\Omega_t}(p^*(t))\}.$$

The following proposition shows that the elements of $C_a^2(Q)$ can be characterized by the emptyness property of the set $\tilde{M}(p^*)$.

Proposition 2.3. *Let $f\in C(Q)\setminus\mathcal{P}_\Omega$. Then the following statements are equivalent*:

(i) $f\in C_a^2(Q)$;

(ii) $\tilde{M}(p^*)=\emptyset$ *for any best approximation p^* to f from $\mathcal{P}_\Omega$*;

(iii) $\tilde{M}(p^*)=\emptyset$ *for some best approximation p^* to f from $\mathcal{P}_\Omega$.*

Proof. (i) $\Longrightarrow$ (ii). Let $f\in C_a^2(Q)\setminus\mathcal{P}_\Omega$. Then (2.18) holds. Suppose on the contrary that $\tilde{M}(p^*)\neq\emptyset$ for some best approximation p^* to f from $\mathcal{P}_\Omega$. Let $t\in\tilde{M}(p^*)$. Then $\|f-p^*\|=|f(t)-p^*(t)|$ and $\overline{\sigma(p^*,t)}\in N_{\Omega_t}(p^*(t))$. Thus, for any $z\in\Omega_t$, we have that

$$\begin{aligned}d(f,\mathcal{P}_\Omega)=\|f-p^*\|&=\mathrm{Re}\sigma(p^*,t)(z-p^*(t))+\mathrm{Re}\sigma(p^*,t)(f(t)-z)\\&\leqslant\mathrm{Re}\sigma(p^*,t)(f(t)-z)\leqslant|f(t)-z|.\end{aligned}$$

This implies that $d(f,\mathcal{P}_\Omega)\leqslant d(f(t),\Omega_t)\leqslant\sup_{t\in Q}d(f(t),\Omega_t)$, which contradicts (2.18). Thus the implication (i)$\Longrightarrow$(ii) is proved.

(ii)$\Longrightarrow$(iii). It is trivial.

(iii)$\Longrightarrow$(i). Suppose that (iii) holds. By Lemma 2.2, we only need to verify that $d(f,\mathcal{P}_\Omega)>d(f(t),\Omega_t)$ for each $t\in M(f-p^*)$. To do this, suppose on the contrary that there exists some $t_0\in M(f-p^*)$ such that $d(f,\mathcal{P}_\Omega)\leqslant d(f(t_0),\Omega_{t_0})$. Then $|f(t_0)-p^*(t_0)|=\|f-p^*\|\leqslant d(f(t_0),\Omega_{t_0})$ so that $p^*(t_0)$ is a best approximation to $f(t_0)$ from Ω_{t_0} since $p^*(t_0)\in\Omega_{t_0}$. Thus $f(t_0)-p^*(t_0)\in N_{\Omega_{t_0}}(p^*(t_0))$ by [10, Theorem 2.1, p. 6]. In particular, $p^*(t_0)\in\mathrm{bd}\Omega_{t_0}$. Furthermore, $\overline{\sigma(p^*,t_0)}\in N_{\Omega_{t_0}}(p^*(t_0))$. Hence $t_0\in\tilde{M}(p^*)$. The proof is complete.

Recall that the notion of the Chebeshev alternation plays an important role in characterizing the best approximations of real-valued continuous functions on a bounded interval $[a,b]$, see for example [7, 8]. Unfortunately, this notion does not make sense for the approximations of complex-valued continuous functions. Motivated by the equivalent relationship between the Chebeshev alternation and the extremal signature in a real case, we extend the notion of the extremal support to study the problem of the best approximations of complex-valued continuous functions. We first recall the notion of the extremal signature, see for exampl [11, 12].

Let $A \subseteq Q$ be a finite subset and σ a function defined on Q. σ is said to have the finite support A if $\sigma(t) \neq 0$ for each $t \in A$ and 0 for each $t \in Q \setminus A$. Furthermore, σ is said to be a signature with the support A if $|\sigma(t)| = 1$ for each $t \in A$. Then, σ is said to be an extremal signature, if there exists a function μ with the support A such that $\operatorname{sign} \mu(t) = \sigma(t)$ for each $t \in A$ and $\sum_{t \in A} p(t)\overline{\mu(t)} = 0$ for each $p \in \mathcal{P}$. The following definition is an extension of this notion.

Definition 2.5. *Let (A, B) be a pair of finite subsets of Q. Let σ and τ be the functions defined on Q. Then*

(i) *(σ, τ) is called a bi-signature with the support (A, B) if σ and τ are the signatures with the supports A and B, respectively*;

(ii) *A bi-signature (σ, τ) with the support (A, B) is called an extremal bi-signature if there exist two functions μ and ν on Q such that*

$$(\operatorname{sign} \mu(t_1), \operatorname{sign} \nu(t_2)) = (\sigma(t_1), \tau(t_2)), \quad \forall\ (t_1, t_2) \in A \times B \tag{2.24}$$

and

$$\sum_{t \in A} p(t)\overline{\mu(t)} + \sum_{t \in B} p(t)\overline{\nu(t)} = 0, \quad \forall\ p \in \mathcal{P}. \tag{2.25}$$

We still require the notion of the extremal bi-support with respect to (f, p^*).

Definition 2.6. *Let A and B be the finite subsets of Q and $A \cup B \neq \emptyset$. (A, B) is said to be an extremal bi-support with respect to (f, p^*) if $A \subseteq M(f - p^*)$, $B \subseteq B(p^*)$ and there exists an extremal bi-signature (σ, τ) with the support (A, B) such that*

$$(\sigma(t_1), \tau(t_2)) \in (\sigma(p^*, t_1), \tau(p^*, t_2)), \quad \forall\ (t_1, t_2) \in A \times B. \tag{2.26}$$

It is clear that if (A, B) is an extremal bi-support with respect to (f, p^*), then $A \neq \emptyset$ by the interior-point condition. Thus, using Proposition 2.1 and [6, Theorem 5.1] (noting that the strong interior-point condition mentioned there is satisfied, thanks to Hypothesis 2), we obtain the following characterization theorem of the best restricted range approximation in view of an extremal bi-signature.

Theorem 2.1. *Let $f \in C(Q) \setminus \mathcal{P}_\Omega$ and $p^* \in \mathcal{P}_\Omega$. Then p^* is a best restricted range approximation to f from $\mathcal{P}_\Omega$ if and only if there exists an extremal bi-support with respect to (f, p^*).*

In particular, when $\mathcal{P}$ is a Haar subspace, Theorem 2.1 can be improved to Theorem 2.2 below. Recall that an n-dimensional subspace $\mathcal{P} \subseteq C(Q)$ is called a Haar subspace if every element $p \in \mathcal{P} \setminus \{0\}$ has at most $n - 1$ zeros in Q.

Theorem 2.2. *Let $f \in C(Q) \setminus \mathcal{P}_\Omega$ and $p^* \in \mathcal{P}_\Omega$. If $\mathcal{P}$ is a Haar subspace, then the following statements are equivalent*:

(i) *p^* is a best restricted range approximation to f from $\mathcal{P}_\Omega$*;

(ii) *Either $\tilde{M}(p^*) \neq \emptyset$ or any extremal bi-support (A, B) with respect to (f, p^*) satisfies $|A \cup B| \geqslant n + 1$*;

(iii) *Either $\tilde{M}(p^*) \neq \emptyset$ or there exists an extremal bi-support (A, B) with respect to (f, p^*) satisfying $|A \cup B| \geqslant n + 1$.*

Proof. (i)$\Longrightarrow$(ii) Suppose that p^* is a best approximation to f from $\mathcal{P}_\Omega$ and $\tilde{M}(p^*) = \emptyset$. Suppose on the contrary that there exists an extremal bi-support (A, B) with respect to (f, p^*) such that $|A \cup B| \leqslant n$. In view of Definition 2.6, we have that $A \subseteq M(f - p^*)$, $B \subseteq B(p^*)$ and there exists an extremal bi-signature (σ, τ) with the support (A, B) such that (2.26) holds. Thus, by Definition 2.5, there are two functions μ and ν on Q such that (2.24) and (2.25) hold. It follows from (2.24) and (2.26) that

$$(\operatorname{sign}\mu(t_1),\ \operatorname{sign}\nu(t_2)) \in (\sigma(p^*, t_1),\ \tau(p^*, t_2)), \quad \forall\ (t_1, t_2) \in (A, B). \tag{2.27}$$

Also from (2.25), we have that

$$\sum_{t\in A\cap B} p(t)\overline{\mu(t)+\nu(t)} + \sum_{t\in A\setminus B} p(t)\overline{\mu(t)} + \sum_{t\in B\setminus A} p(t)\overline{\nu(t)} = 0, \quad \forall\, p \in \mathcal{P}. \tag{2.28}$$

Noting that $\mathcal{P}$ is a Haar subspace and that $|A \cap B| + |A \setminus B| + |B \setminus A| \leqslant n$, we may pick $p_1 \in \mathcal{P}$ such that

$$p_1(t) = \mu(t) + \nu(t), \quad \forall\ t \in A \cap B, \tag{2.29}$$

$$p_1(t) = \mu(t), \quad \forall\ t \in A \setminus B, \tag{2.30}$$

$$p_1(t) = \nu(t), \quad \forall\ t \in B \setminus A. \tag{2.31}$$

Since $\tilde{M}(p^*) = \emptyset$, it follows that $p_1(t) \neq 0$ for each $t \in A \cap B$. In fact, suppose on the contrary that $p_1(t_1) = 0$ for some $t_1 \in A \cap B$. Then $\mu(t_1) + \nu(t_1) = 0$. Note that $\operatorname{sign}\mu(t_1) = \sigma(p^*, t_1)$ and $\overline{\operatorname{sign}\nu(t_1)} \in -N_{\Omega_{t_1}}(p^*(t_1))$ by (2.27). We have that

$$\overline{\sigma(p^*, t_1)} = \overline{\operatorname{sign}\mu(t_1)} = -\overline{\operatorname{sign}\nu(t_1)} \in N_{\Omega_{t_1}}(p^*(t_1)),$$

so that $t_1 \in \tilde{M}(p^*)$, which is a contradiction. Thus, (2.28) is not true for $p = p_1$. The implication (i)$\Longrightarrow$(ii) is proved.

(ii)$\Longrightarrow$(iii) It is trivial.

(iii)$\Longrightarrow$(i) Suppose that (iii) holds. Then (i) holds by Theorem 2.1 in the case when $\tilde{M}(p^*) = \emptyset$. It remains to prove (i) in the case when $\tilde{M}(p^*) \neq \emptyset$. Take $t_2 \in \tilde{M}(p^*)$. Then $t_2 \in M(f - p^*) \cap B(p^*)$ and $\overline{\sigma(p^*, t_2)} \in N_{\Omega_{t_2}}(p^*(t_2))$. Let $A = B = \{t_2\}$. Define $\mu(t_2) = \overline{\sigma(p^*, t_2)}$ and $\nu(t_2) = -\overline{\sigma(p^*, t_2)}$. Then $\operatorname{sign}\mu(t_2) = \sigma(p^*, t_2), \operatorname{sign}\nu(t_2) = -\sigma(p^*, t_2) \in \tau(p^*, t_2)$ and $p(t_2)\overline{\mu(t_2)} + p(t_2)\overline{\nu(t_2)} = 0, \forall\ p \in \mathcal{P}$. This means that (A, B) is an extremal bi-support with respect to (f, p^*), and hence p^* is a best approximation to f from $\mathcal{P}_\Omega$ by Theorem 2.1. The proof is complete.

3 The limit theory for characterizations

We begin with some additional notions which will be used in this section.

Definition 3.1. *$\mathcal{P}_\Omega$ is said to have Property $\mathcal{C}$ (resp. $\underline{\mathcal{C}}$) if, for any $f \in C(Q) \setminus \mathcal{P}_\Omega$, p^* is a best approximation to f from $\mathcal{P}_\Omega$ if and only if any (resp. at least one) extremal bi-support (A, B) with respect to (f, p^*) satisfies $|A \cup B| \geqslant n + 1$.*

Remark 3.1. Clearly Property $\mathcal{C}$ implies Property $\underline{\mathcal{C}}$, but the converse is not true, in general, as shown by the following

Example 3.1. Let $Q=\{-1,1\}$, $\Omega_{-1}=\{z\in\mathbb{C}:|z+1|\leqslant 1\}$ and $\Omega_1=\{z\in\mathbb{C}:|z-1|\leqslant 1\}$. Let $\bar{p}\in C(Q)$ be defined by $\bar{p}(t)=t$ for each $t\in Q$ and let $\mathcal{P}=\{p:p=a\bar{p},\ a\in\mathbb{C}\}$. Then $\mathcal{P}$ is a 1-dimensional Haar subspace, and $\mathcal{P}_\Omega=\{p:p(t)=at,\ |a-1|\leqslant 1\}$. Hence Hypotheses 1 and 2 in the previous section are satisfied (noting that $\bar{p}$ is an interior point of $\mathcal{P}_\Omega$). Let $f\in C(Q)\backslash\mathcal{P}_\Omega$, and let p^* be a best approximation to f from $\mathcal{P}_\Omega$. We claim that there exists an extremal bi-support (A,B) with respect to (f,p^*) such that $|A\cup B|\geqslant n+1$. Note that the claim holds by Theorem 2.2 in the case when $\tilde{M}(p^*)=\emptyset$. It remains to consider the case when $\tilde{M}(p^*)\neq\emptyset$. Without loss of generality, we may assume that $1\in\tilde{M}(p^*)$. Then $1\in M(f-p^*)\cap B(p^*)$ and $\overline{\sigma(p^*,1)}\in N_{\Omega_1}(p^*(1))$. Therefore, $-1\in B(p^*)$. Let $A=\{1\}$, $B=\{1,-1\}$. Then $A\subseteq M(f-p^*)$ and $B\subseteq B(p^*)$. Thus to complete the proof of the claim, it suffices to verify that (A,B) is an extremal bi-support with respect to (f,p^*). To do this, define

$$\mu_A(1)=2\overline{\sigma(p^*,1)},\quad \nu_B(1)=-\overline{\sigma(p^*,1)},\quad \nu_B(-1)=\overline{\sigma(p^*,1)}.$$

Then

$$p(1)\overline{\mu_A(1)}+p(1)\overline{\nu_B(1)}+p(-1)\overline{\nu_B(-1)}=0,\quad \forall\ p\in\mathcal{P}. \tag{3.1}$$

Define

$$\sigma_A(t)=\begin{cases}\sigma(p^*,1), & t=1,\\ 0, & t\in Q\setminus\{1\}\end{cases}$$

and $\tau_B(t)=-t\sigma(p^*,1), t\in Q=\{-1,1\}$. Then

$$(\operatorname{sign}\mu_A(t_1),\operatorname{sign}\nu_B(t_2))=(\sigma_A(t_1),\tau_B(t_2)),\quad \forall\ (t_1,t_2)\in A\times B.$$

This and (3.1) imply that (σ_A,τ_B) is an extremal bi-signature with the support (A,B). Moreover, it is easy to see that

$$N_{\Omega_1}(p^*(1))=\{\lambda\overline{\sigma(p^*,1)}:\lambda\geqslant 0\},\quad N_{\Omega_{-1}}(p^*(-1))=\{-\lambda\overline{\sigma(p^*,1)}:\lambda\geqslant 0\}.$$

Hence

$$\tau(p^*,1)=\{-\lambda\sigma(p^*,1):\lambda>0\},\quad \tau(p^*,-1)=\{\lambda\sigma(p^*,1):\lambda>0\},$$

which implies that

$$(\sigma_A(t_1),\tau_B(t_2))\in(\sigma(p^*,t_1),\tau(p^*,t_2)),\forall\ (t_1,t_2)\in A\times B. \tag{3.2}$$

Thus the claim stands. Consequently, $\mathcal{P}_\Omega$ has Property $\underline{\mathcal{C}}$.

Below we will show that $\mathcal{P}_\Omega$ does not have Property $\mathcal{C}$. Define f by $f(t)=-t$ for each $t\in Q$. Then $f\in C(Q)\setminus\mathcal{P}_\Omega$. Furthermore, for any $p=a\bar{p}\in\mathcal{P}_\Omega$, one gets that $\mathrm{Re}a\geqslant 0$ because $1\geqslant|a-1|\geqslant 1-\mathrm{Re}a$. Hence $\|f-p\|=|1+a|\geqslant\mathrm{Re}(1+a)\geqslant 1=\|f-q^*\|$ and $q^*=0$ is a best approximation to f from $\mathcal{P}_\Omega$. Let $A_1=B_1=\{1\}$. Then $A_1\subseteq M(f-q^*)$ and $B_1\subseteq B(q^*)$. Define $\mu_{A_1}(1)=\overline{\sigma(q^*,1)}$ and $\nu_{B_1}(1)=-\overline{\sigma(q^*,1)}$. Then $p(1)\overline{\mu_{A_1}(1)}+p(1)\overline{\nu_{B_1}(1)}=0,\forall\ p\in\mathcal{P}$. This shows that (A_1,B_1) is an extremal bi-support with respect to (f,q^*) since $\operatorname{sign}\mu_{A_1}(1)=\sigma(q^*,1)$ and $\operatorname{sign}\nu_{B_1}(1)\in\tau(q^*,1)$. However, $|A\cup B|=1<n+1$. In view of Definition 3.1, $\mathcal{P}_\Omega$ does not have Property $\mathcal{C}$.

Definition 3.2. *$\mathcal{P}_\Omega$ is said to have Property $\mathcal{C}_i$ (resp. $\underline{\mathcal{C}}_i$) if, for each $f \in C_a^i(Q) \setminus \mathcal{P}_\Omega$, p^* is a best approximation to f from $\mathcal{P}_\Omega$ if and only if any (resp. at least one) extremal bi-support (A, B) with respect to (f, p^*) satisfies $|A \cup B| \geqslant n+1$ for $i = 1, 2$.*

Definition 3.3. *$\mathcal{P}_\Omega$ is said to have Property $\mathcal{C}^*$ (resp. $\underline{\mathcal{C}}^*$) if, for each $f \in C(Q) \setminus \mathcal{P}_\Omega$, p^* is a best approximation to f from $\mathcal{P}_\Omega$ if and only if either $\tilde{M}(p^*) \neq \emptyset$ or any (resp. at least one) extremal bi-support (A, B) with respect to (f, p^*) satisfies $|A \cup B| \geqslant n+1$.*

Remark 3.2. (i) $\mathcal{P}_\Omega$ has Property $\mathcal{C}$ (resp. $\underline{\mathcal{C}}$) $\Longrightarrow$ $\mathcal{P}_\Omega$ has Property $\mathcal{C}_2$ (resp. $\underline{\mathcal{C}}_2$) $\Longrightarrow$ $\mathcal{P}_\Omega$ has Property $\mathcal{C}_1$ (resp. $\underline{\mathcal{C}}_1$);

(ii) $\mathcal{P}_\Omega$ has Property $\mathcal{C}^*$(resp. $\underline{\mathcal{C}}^*$) $\Longrightarrow$ $\mathcal{P}_\Omega$ has Property $\mathcal{C}_2$ (resp. $\underline{\mathcal{C}}_2$) $\Longrightarrow$ $\mathcal{P}_\Omega$ has Property $\mathcal{C}_1$ (resp. $\underline{\mathcal{C}}_1$).

To establish the characterizations for $\mathcal{P}_\Omega$ to have Property $\mathcal{C}$ (resp. $\mathcal{C}_i$, $\underline{\mathcal{C}}_i$, $\mathcal{C}^*$ and $\underline{\mathcal{C}}^*$), we need to verify a lemma, which is also used in Sec. 4.

Lemma 3.1. *Suppose that $\mathcal{P}$ is not a Haar subspace. Then there exists $f \in C_a^1(Q) \setminus \mathcal{P}_\Omega$ which has the following properties*:

(i) *There are at least two best approximations to f from $\mathcal{P}_\Omega$;*

(ii) *There exists a best approximation p^* to f from $\mathcal{P}_\Omega$ such that $|M(f - p^*) \cup B(p^*)| \leqslant n$.*

Proof. By the assumption, there exists $p_1 \in \mathcal{P} \setminus \{0\}$ such that $p_1(t)$ has n distinct zeros $t_1, \ldots, t_n$ in Q. Let $\{\phi_1, \ldots, \phi_n\}$ be a basis of $\mathcal{P}$. Consider the following system of the equations with the unknown complex variable $(c_1, \ldots, c_n)$

$$\sum_{k=1}^{n} c_k \phi_i(t_k) = 0, \quad i = 1, \ldots, n. \tag{3.3}$$

Then (3.3) has a nonzero solution $(c_1, \ldots, c_n)$ because $\det(\phi_i(t_k))_{i,k=1}^n = 0$. Let $N = \{k : \ c_k \neq 0\}$. Then $N \neq \emptyset$. Since Q is a compact metric space, by the Tietze Extension Theorem, there exists $f_0 \in C(Q)$ such that

$$f_0(t_k) = \frac{\overline{c_k}}{|c_k|}, \quad \forall\, k \in N \tag{3.4}$$

and

$$|f_0(t)| < 1, \quad \forall\, t \in Q \setminus \{t_k : \ k \in N\}. \tag{3.5}$$

Recall that $\overline{p}$ is an interior point of $\mathcal{P}_\Omega$. Thus, by Proposition 2.1, there is a positive number δ such that

$$\mathbf{B}(\overline{p}(t), \delta) \subseteq \Omega_t, \quad \forall\, t \in Q. \tag{3.6}$$

Set $M = \max_{t \in Q} |p_1(t)|$. Define the function f_1 on Q by

$$f_1(t) = \Big(1 - \frac{1}{M}|p_1(t)|\Big) f_0(t), \quad \forall\, t \in Q$$

and set $f = \delta f_1 + \overline{p}$. Note that $|f(t) - \overline{p}(t)| \leqslant \delta$ for all $t \in Q$ by (3.4) and (3.5). Hence $f(t) \in \Omega_t$ for each $t \in Q$ by the closeness of Ω_t. Furthermore, we claim that $d(f, \mathcal{P}_\Omega) = \delta$ and $\overline{p}$ is a best approximation to f from $\mathcal{P}_\Omega$. In fact, for each $p \in \mathcal{P}_\Omega$, write $p - \overline{p} = \sum_{i=1}^n b_i \phi_i$. Then by (3.3) we have that

$$\mathrm{Re} \sum_{k=1}^{n} c_k [p(t_k) - \overline{p}(t_k)] = \mathrm{Re} \sum_{k=1}^{n} c_k \sum_{i=1}^{n} b_i \phi_i(t_k) = \mathrm{Re} \sum_{i=1}^{n} b_i \sum_{k=1}^{n} c_k \phi_i(t_k) = 0.$$

Thus there exists $k_0 \in N$ such that $\mathrm{Re}c_{k_0}[p(t_{k_0}) - \overline{p}(t_{k_0})] \leqslant 0$. It follows that $\mathrm{Re}\overline{f_1(t_{k_0})}[\overline{p}(t_{k_0}) - p(t_{k_0})] \geqslant 0$ due to (3.4). This implies that

$$\begin{aligned}\|f-p\|^2 &\geqslant |f(t_{k_0}) - p(t_{k_0})|^2 = |\delta f_1(t_{k_0}) + \overline{p}(t_{k_0}) - p(t_{k_0})|^2 \\ &= \delta^2 |f_1(t_{k_0})|^2 + 2\delta \mathrm{Re}\overline{f_1(t_{k_0})}[\overline{p}(t_{k_0}) - p(t_{k_0})] + |\overline{p}(t_{k_0}) - p(t_{k_0})|^2 \\ &\geqslant \delta^2.\end{aligned}$$

Hence $d(f, \mathcal{P}_\Omega) \geqslant \delta$ and the claim stands because $\|f - \overline{p}\| \leqslant \delta \leqslant d(f, \mathcal{P}_\Omega)$. Consequently, $f \in C_a^1(Q) \setminus \mathcal{P}_\Omega$ since $B(\overline{p}) = \emptyset$ by (3.6). Below we will show that f has at least two best approximations from $\mathcal{P}_\Omega$. To do this, define $p^\lambda = \frac{\lambda}{M} p_1 + \overline{p}, \forall\ \lambda \in [0, \delta)$. Then $|p^\lambda(t) - \overline{p}(t)| < \delta$ for each $t \in Q$ and so $p^\lambda \in \mathcal{P}_\Omega$ by (3.6). Moreover,

$$|f(t) - p^\lambda(t)| \leqslant \delta \left(1 - \frac{1}{M}|p_1(t)|\right) + \frac{\lambda}{M}|p_1(t)| = \delta - \frac{\delta - \lambda}{M}|p_1(t)| \leqslant \delta.$$

Hence $\|f - p^\lambda\| \leqslant \delta = d(f, \mathcal{P}_\Omega)$ and p^λ is a best approximation to f from $\mathcal{P}_\Omega$ for each $\lambda \in [0, \delta)$. Thus (i) holds. It remains to show that f satisfies (ii). To this end, let $p^* = \overline{p}$. Then (3.4), (3.5) and (3.6) imply that $M(f - p^*) = \{t_k : k \in N\}$ and $B(p^*) = \emptyset$. Therefore,

$$|M(f - p^*) \cup B(p^\lambda)| = |M(f - p^*)| = |N| \leqslant n,$$

and (ii) is proved. The proof is complete.

The first result of this section is as follows.

Theorem 3.1. *The following statements are equivalent*:

(i) $\mathcal{P}$ *is a Haar subspace*;

(ii) $\mathcal{P}$ *has Property* $\mathcal{C}^*$;

(iii) $\mathcal{P}$ *has Property* $\underline{\mathcal{C}}^*$;

(iv) $\mathcal{P}_\Omega$ *has Property* $\mathcal{C}_2$;

(v) $\mathcal{P}_\Omega$ *has Property* $\mathcal{C}_1$;

(vi) $\mathcal{P}_\Omega$ *has Property* $\underline{\mathcal{C}}_2$;

(vii) $\mathcal{P}_\Omega$ *has Property* $\underline{\mathcal{C}}_1$.

Proof. By Theorem 2.2, Proposition 2.3 and Remark 3.2, the following implications hold:

$$\begin{array}{ccccccc} \text{(i)} & \Longrightarrow & \text{(ii)} & \Longrightarrow & \text{(iv)} & \Longrightarrow & \text{(vi)} \\ & & \Downarrow & & \Downarrow & & \Downarrow \\ \text{(vii)} & \Longleftarrow & \text{(iii)} & & \text{(v)} & \Longrightarrow & \text{(vii)}. \end{array}$$

Thus, to complete the proof of Theorem 3.1, it suffices to prove the implication (vii)$\Longrightarrow$(i).

Suppose on the contrary that (i) is not true. Then, by Lemma 3.1, there exist $f \in \mathcal{C}_a^1(Q) \setminus \mathcal{P}_\Omega$ and a best approximation p^* to f from $\mathcal{P}_\Omega$ such that $|M(f - p^*) \cup B(p^*)| \leqslant n$. Thus if (A, B) is an extremal bi-support with respect to (f, p^*) then $|A \cup B| \leqslant n$. Hence $\mathcal{P}_\Omega$ does not have Property $\underline{\mathcal{C}}_1$. The proof is complete.

Remark 3.3. By Remark 3.2 (i) and Theorem 3.1, $\mathcal{P}_\Omega$ has Property $\underline{\mathcal{C}}$ such that $\mathcal{P}$ is a Haar subspace. However, the converse is not true, in general, as illustrated in the following.

Example 3.2. Let $Q=\{-1,0,1\}$ and let $\Omega_{-1}=\Omega_0=\Omega_1=\{z:\ |z|\leqslant 1\}$. Let $\mathcal{P}=\{p: p(t)=a+bt,\ a,b\in\mathbb{C}\}$. Then $\mathcal{P}$ is a Haar subspace and

$$\mathcal{P}_\Omega=\{p:p(t)=a+bt,\ |a|\leqslant 1,\ |a-b|\leqslant 1,\ |a+b|\leqslant 1\}.$$

Clearly Hypotheses 1 and 2 in sec. 2 are satisfied. To verify that $\mathcal{P}_\Omega$ does not have Property $\underline{\mathcal{C}}$, define f by $f(0)=1,\ f(-1)=0$ and $f(1)=2+2\mathbf{i}$. Then, for each $p\in\mathcal{P}_\Omega$, assuming $p(t)=a+bt$, one has that

$$\begin{aligned}\|f-p\|&=\max\{|1-a|,|a-b|,|2+2\mathbf{i}-(a+b)|\}\\&\geqslant|2+2\mathbf{i}|-|a+b|\geqslant 2\sqrt{2}-1.\end{aligned}$$

Set $p^*(t)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathbf{i}t,\ t\in Q$. Then $p^*\in\mathcal{P}_\Omega$ and $\|f-p^*\|=2\sqrt{2}-1$. Hence p^* is a best approximation to f from $\mathcal{P}_\Omega$. Furthermore, $M(f-p^*)=\{1\}$ and $B(p^*)=\{-1,1\}$. Hence $|M(f-p^*)\cup B(p^*)|=2$, which implies that $\mathcal{P}_\Omega$ does not have Property $\underline{\mathcal{C}}$.

For the characterization of Property $\mathcal{C}$, we need to establish the following.

Proposition 3.1. *Suppose that $\mathcal{P}$ is a Haar subspace. Then the following statements are equivalent*:

(i) $\mathcal{P}_\Omega\neq\mathcal{P}$;

(ii) *There exist $p^*\in\mathcal{P}_\Omega$ and $t_0\in Q$ such that $p^*(t_0)\in\mathrm{bd}\Omega_{t_0}$*;

(iii) *There exists $t_1\in Q$ such that $\Omega_{t_1}\neq\mathbb{C}$.*

Proof. (i)$\Longrightarrow$(ii) Suppose that (i) holds and set $\mathcal{P}_{\mathrm{int}\Omega}=\{p\in\mathcal{P}:p(t)\in\mathrm{int}\Omega_t,\forall\ t\in Q\}$. Then $\mathcal{P}_{\mathrm{int}\Omega}$ is a nonempty subset of $\mathcal{P}_\Omega$ since $\overline{p}\in\mathcal{P}_{\mathrm{int}\Omega}$. We claim that $\mathcal{P}_{\mathrm{int}\Omega}$ is an open subset of $\mathcal{P}$ in the uniform norm. Indeed, let $p_0\in\mathcal{P}_{\mathrm{int}\Omega}$. Then p_0 is an interior point of $\mathcal{P}_\Omega$ in the sense of Definition 2.1; hence p_0 is a strong interior point of $\mathcal{P}_\Omega$ by Proposition 2.1. This means that there is $\delta>0$ such that $\mathbf{B}(p_0(t),\delta)\subseteq\Omega_t$ for each $t\in Q$. Let $p\in\mathcal{P}$ with $\|p-p_0\|<\frac{\delta}{2}$. Then $|p(t)-p_0(t)|<\frac{\delta}{2}$; hence $\mathbf{B}\big(p(t),\frac{\delta}{2}\big)\subseteq\mathbf{B}(p_0(t),\delta)\subseteq\Omega_t,\forall\ t\in Q$.

It follows that $p(t)\in\mathrm{int}\Omega_t$ for each $t\in Q$ and so $p\in\mathcal{P}_{\mathrm{int}\Omega}$. Consequently, $\mathbf{B}(p_0,\frac{\delta}{2})\subseteq\mathcal{P}_{\mathrm{int}\Omega}$ and the claim stands. Since $\mathcal{P}_\Omega$ is a proper closed subset of $\mathcal{P}$, we get that $\mathcal{P}_{\mathrm{int}\Omega}\neq\mathcal{P}_\Omega$. Pick $p^*\in\mathcal{P}_\Omega\setminus\mathcal{P}_{\mathrm{int}\Omega}$. Then there exists $t_0\in Q$ such that $p^*(t_0)\in\mathrm{bd}\Omega_{t_0}$ by the definition of $\mathcal{P}_{\mathrm{int}\Omega}$. Therefore, (ii) holds.

(ii)$\Longrightarrow$(iii). It is trivial.

(iii)$\Longrightarrow$(i). By the assumption, we may take $z_1\in\mathbb{C}\setminus\Omega_{t_1}$. Since $\mathcal{P}$ is a Haar subspace, there exists $p_1\in\mathcal{P}$ such that $p_1(t_1)=z_1$. Hence $p_1\notin\mathcal{P}_\Omega$ as $p_1(t_1)\notin\Omega_{t_1}$, that is, $\mathcal{P}_\Omega\neq\mathcal{P}$. The proof is complete.

The second result of this section is stated as follows:

Theorem 3.2. *$\mathcal{P}_\Omega$ has Property $\mathcal{C}$ if and only if $\mathcal{P}$ is a Haar subspace and $\mathcal{P}_\Omega=\mathcal{P}$.*

Proof. Suppose that $\mathcal{P}$ is a Haar subspace and $\mathcal{P}_\Omega=\mathcal{P}$. Then by Proposition 3.1, $\Omega_t=\mathbb{C}$ for each $t\in Q$; hence $B(p^*)=\emptyset$ for each $p^*\in\mathcal{P}$. Therefore, $\mathcal{P}_\Omega$ has Property $\mathcal{C}$ by Theorem 2.2.

Conversely, suppose that $\mathcal{P}_\Omega$ has Property $\mathcal{C}$. Then $\mathcal{P}$ is a Haar subspace by Remark 3.2 (i) and Theorem 3.1. Thus, it remains to prove that $\mathcal{P}_\Omega=\mathcal{P}$. To do this, suppose on the contrary that $\mathcal{P}_\Omega\neq\mathcal{P}$. Then, by Proposition 3.1, there exist $p^*\in\mathcal{P}_\Omega$ and $t_0\in Q$ such that $p^*(t_0)\in\mathrm{bd}\Omega_{t_0}$, which implies that $N_{\Omega_{t_0}}(p^*(t_0))\neq\{0\}$. Let $\tau\in N_{\Omega_{t_0}}(p^*(t_0))\setminus\{0\}$. By the

Tietze Extension Theorem there exists a function $g \in C(Q)$ such that $g(t_0) = \tau$ and $|g(t)| \leqslant |\tau|$ for each $t \in Q$. Define $f \in C(Q)$ by $f = p^* + g$. Then $t_0 \in M(f - p^*)$. Noting that $p^*(t_0) \in \mathrm{bd}\Omega_{t_0}$ and $\overline{\sigma(p^*, t_0)} = \frac{\tau}{|\tau|} \in N_{\Omega_{t_0}}(p^*(t_0))$, we have that $t_0 \in \tilde{M}(p^*)$. Hence p^* is a best approximation to f from $\mathcal{P}_\Omega$ by Theorem 2.2. Furthermore, let $A = B = \{t_0\}$. It is easy to see that (A, B) is an extremal bi-support with respect to (f, p^*). This implies that $\mathcal{P}_\Omega$ does not have Property $\mathcal{C}$ since $|A \cup B| = 1$, which is a contradiction. Thus $\mathcal{P}_\Omega = \mathcal{P}$.

4 The limit theory for uniqueness

We begin with the following definitions.

Definition 4.1. *$\mathcal{P}_\Omega$ is said to have Property $\mathcal{U}$ (resp. $\mathcal{U}_1$, $\mathcal{U}_2$) if, for each $f \in C(Q)$ (resp. $C_a^1(Q)$, $C_a^2(Q)$), f has a unique best approximation from $\mathcal{P}_\Omega$.*

Definition 4.2. *$\mathcal{P}_\Omega$ is said to have Property K with respect to Q if, for any $p_1, p_2 \in \mathcal{P}_\Omega$ and $t_0 \in Q$, the condition $p_1(t_0) = p_2(t_0) \in \mathrm{bd}\Omega_{t_0}$ implies that $p_1 = p_2$.*

Remark 4.1. Property K was first introduced by Shi[9] in the case of real-valued continuous function approximations and by Smirnov and Smirnov[4,5] in the case of complex-valued continuous function approximations.

The first theorem addresses Property $\mathcal{U}$, the proof of which is almost the same as that of Theorem 4.2 in [4] and so omitted here.

Theorem 4.1. *$\mathcal{P}_\Omega$ has Property $\mathcal{U}$ if and only if the following conditions hold*:

(i) *$\mathcal{P}$ is a Haar subspace*;

(ii) *$\mathcal{P}_\Omega$ has Property K with respect to Q.*

The second theorem addresses Properties $\mathcal{U}_1$ and $\mathcal{U}_2$. Let G be a subset of $\mathbb{C}$. Recall that G is said to be strictly convex if, for any distinct elements $g_1, g_2 \in G$, $(g_1 + g_2)/2 \in \mathrm{int}G$. Note that the notion of the strict convexity plays a basic role in the study of the uniqueness of approximations of complex-valued continuous functions, see, for example [3–5].

Hypothesis 3. Ω_t is strictly convex for each $t \in Q$.

Theorem 4.2. *Consider the following statements*:

(i) *$\mathcal{P}$ is a Haar subspace*;

(ii) *$\mathcal{P}_\Omega$ has Property $\mathcal{U}_1$*;

(iii) *$\mathcal{P}_\Omega$ has Property $\mathcal{U}_2$.*

Then (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i). *If, in addition, Hypothesis* 3 *is satisfied, then* (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii).

Proof. (iii)$\Longrightarrow$(ii) This results from Proposition 2.2.

(ii)$\Longrightarrow$(i). Suppose that (ii) holds but (i) does not. Then $\mathcal{P}$ is not a Haar subspace. By Lemma 3.1, there exists $f \in C_a^1(Q) \setminus \mathcal{P}_\Omega$ such that f has at least two best approximations from $\mathcal{P}_\Omega$, which contradicts (ii). Hence (ii)$\Longrightarrow$(i) holds.

Finally, suppose that, in addition, Hypothesis 3 is satisfied. To complete the proof of the theorem, it suffices to prove (i)$\Longrightarrow$(iii). To this end, suppose that (i) holds. Let $f \in C_a^2(Q) \setminus \mathcal{P}_\Omega$ and let p^* and q^* be the best approximations to f from $\mathcal{P}_\Omega$. Set $r^* = \frac{1}{2}(p^* + q^*)$. Then r^* is a best approximation to f from $\mathcal{P}_\Omega$. Using the standard technique, we get the following inclusions:

$$M(f - r^*) \subseteq M(f - p^*) \cap M(f - q^*) \subseteq Z(p^* - q^*), \tag{4.1}$$

$$B(r^*) \subseteq B(p^*) \cap B(q^*), \tag{4.2}$$

where $Z(p)$ stands for the set of all zeros of p in Q. By Proposition 2.3 we have that $\tilde{M}(r^*) = \emptyset$. Thus, Theorem 2.2 implies that

$$|M(f - r^*) \cup B(r^*)| \geqslant n + 1. \tag{4.3}$$

Since each Ω_t is strictly convex, (4.2) yields that $B(r^*) \subseteq Z(p^* - q^*)$. Combining this and (4.1) gives that $M(f - r^*) \cup B(r^*) \subseteq Z(p^* - q^*)$. Hence, $|Z(p^* - q^*)| \geqslant n + 1$ due to (4.3) and so $p^* = q^*$ because $\mathcal{P}$ is a Haar subspace. Thus $\mathcal{P}_\Omega$ has Property $\mathcal{U}_2$ and the proof of Theorem 4.2 is complete.

The following corollary, which was obtained in [2–5] respectively under some stronger conditions, is now an immediate consequence of Theorem 4.1.

Corollary 4.1. *Suppose that $\mathcal{P}$ is a Haar subspace and that Hypothesis* 3 *is satisfied. Let $f \in C_a^2(Q)$ and let p^* be a best approximation to f from $\mathcal{P}_\Omega$. Then p^* is the unique best approximation to f from $\mathcal{P}_\Omega$ provided that $\tilde{M}(p^*) = \emptyset$.*

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