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RESTRICTED p-CENTERS FOR SETS IN REAL LOCALLY CONVEX SPACES

Xian-Fa Luo □ *School of Mathematics and Computer Science, Jishou University, Jishou, People's Republic of China*

Chong Li □ *Department of Mathematics, Zhejiang University, Hangzhou, People's Republic of China*

Jin-Su He □ *Department of Mathematics, Zhejiang Normal University, Jinhua, People's Republic of China*

□ *Let p be a continuous seminorm on a real locally convex space X. The current paper is concerned with the problem of restricted p-centers in X. Characterizations of restricted p-centers and strongly unique restricted p-centers with respect to a "sun" are provided, and then some recent results due to Laurent and Pai [9] are extended and improved.*

Keywords Locally convex spaces; Restricted p-centers; RS-sets; Suns.

AMS Subject Classification 41A65; 41A28.

1. INTRODUCTION

Let X, Z be a pair of real linear spaces put in duality by a separating bilinear form $\langle , \rangle$, and endowed with compatible locally convex topologies respectively. A subset F of X is said to be p-bounded if $\sup\{p(x) : x \in F\} < \infty$. We denote the collection of all nonempty p-bounded subsets of X by $\mathscr{B}_p(X)$. Given $x \in X,\ F \in \mathscr{B}_p(X)$, and $V \subseteq X$, write

$$r_p(F; x) = \sup\{p(y - x) : y \in F\},$$
$$\mathrm{rad}_p(F; V) = \inf\{r_p(F; v) : v \in V\},$$

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Address correspondence to Xian-Fa Luo, School of Mathematics and Computer Science, Jishou University, Jishou 416000, People's Republic of China. E-mail: luoxianfaseu@163.com

and

$$\mathrm{Cent}_p(F;V) = \{v_0 \in V : r_p(F;v_0) = \mathrm{rad}_p(F;V)\}.$$

Then according to [14], $\mathrm{rad}_p(F;V)$ is called the restricted p-radius of F with respect to V, and any element of the (possible void) set $\mathrm{Cent}_p(F;V)$ is called a restricted p-center of F with respect to V.

In the special case when p is a norm on X, the notion of a restricted p-center clearly coincides with that of the well-known restricted Chebyshev center (i.e., best simultaneous approximation). The study of the restricted Chebyshev center problem in a Banach space X has a long history, see, for example, [1, 3–7, 10, 11, 13, 15]. In particular, in the case when V is a nonempty convex subset of X, the Kolmogorov type's characterization of restricted Chebyshev centers with respect to V is obtained in [3] by Freilich and Mclaughlin. This result is further extended in [15] to the case when V is a sun. Some results on the uniqueness and strong uniqueness of the restricted Chebyshev centers with respect to an interpolating subspace are given in [10]. For the general setting of restricted p-centers in real locally convex topological spaces, the existence theorem for the set V with some suitable compactness and the characterization theorem as well as the uniqueness theorem for the set V with some suitable Haar-like property are given in [14], while the existence theorem and characterization theorem with respect to a p-interpolating subspace with some compactness is given in [9].

In the current paper, we will continue to study the restricted p-center problem in a real locally convex space but with a completely different technique. More precisely, we first convert the restricted p-center problem into a restricted Chebyshev center problem in the corresponding quotient space $X/\ker(p)$ with respect to the kernel $\ker(p)$, which is done in Section 2. In Section 3, we recall some known and establish some new results on restricted Chebyshev centers in normed linear spaces. Finally, the main results on restricted p-centers with respect to nonlinear sets in real locally convex spaces, including characterization theorems and strong uniqueness theorems, are provided in Section 4. We remark that most part of the results for nonlinear cases obtained in this paper extend and improve those in [9], and some of them are new even for the linear cases, see, for example, Corollaries 4.4 and 4.7.

2. PRELIMINARIES AND DEFINITIONS

We begin with the following notions, one of which is a generalization of the corresponding one in [14]. In what follows, we always assume that F and V are nonempty subsets of X.

Definition 2.1. Let $F \in \mathscr{B}_p(X)$ and $V \subseteq X$. F is called

(i) *wp*-sup-compact with respect to V if, for each net $\{y_\alpha\} \subseteq F$ and each $v_0 \in V$, $\lim_\alpha p(y_\alpha - v_0) = r_p(F; v_0)$ implies that there exist a subnet $\{y_\lambda\}$ of $\{y_\alpha\}$ and an element $y \in F$ such that $p(y_\lambda - y) \to 0$;
(ii) *p*-sup-compact with respect to V if, for each net $\{y_\alpha\} \subseteq F$ and each $v_0 \in V$, $\lim_\alpha p(y_\alpha - v_0) = r_p(F; v_0)$ implies that there exists a subnet $\{y_\lambda\}$ of $\{y_\alpha\}$ that converges to some element in F;
(iii) *wp*-compact if for each net $\{y_\alpha\} \subseteq F$, there exist a subnet $\{y_\lambda\}$ of $\{y_\alpha\}$ and an element $y \in F$ such that $p(y_\lambda - y) \to 0$;
(iv) *wp*-sup-compact (*resp.* *p*-sup-compact) if it is *wp*-sup-compact (*resp.* *p*-sup-compact) with respect to $V = X$.

Remark 2.2. With respect to V, the following assertions hold.

(a) F is *p*-sup-compact implies that it is *wp*-sup-compact, but the converse is not true. For example, let $\mathbb{R}^2$ denote the two-dimensional Euclidean space. For every $x = (t_1, t_2) \in \mathbb{R}^2$, let $p(x) = |t_1|$ and $F = \{(t_1, t_2) : t_1 \in [0, 1]\}$. Then F is *wp*-sup-compact but not *p*-sup-compact.
(b) F is *wp*-sup-compact if and only if it is *p*-sup-compact when X is a normed linear space with the norm p and, in this case, F is also called M-compact with respect to V, which was introduced by Panda and Kapoor [12] in the special case when $V = X$.

Let *wp*-sup-$K(X, V)$ [*resp.* *p*-sup-$K(X, V)$, *wp*-$K(X)$] denote the collection of all subsets $F \in \mathscr{B}_p(X)$, which is *wp*-sup-compact with respect to V (*resp.* *p*-sup-compact with respect to V, *wp*-compact). Then, by Remark 2.2,

$$p - \sup -K(X, V) \subseteq wp - \sup -K(X, V)$$

and

$$wp - K(X) \subseteq wp - \sup -K(X, V).$$

Let $\ker(p)$ stand for the kernel of p, defined by $\ker(p) = \{x \in X : p(x) = 0\}$, and write $N = \ker(p)$. Then N is a $\sigma(X, Z)$-closed subspace of X. Recall that the quotient space X/N of X with respect to N is a normed linear space with the norm defined by

$$\|[x]\|_p = \inf\{p(y) : y \in [x]\}, \quad [x] = x + N \in X/N. \tag{2.1}$$

Let $Q_N : X \to X/N$ denote the quotient mapping (i.e., $Q_N(x) = [x]$) while its dual mapping is denoted by $Q_N^* : (X/N)^* \to Z$. Denote the compatible locally convex topology on X by Γ and the quotient topology on X/N

by Γ_N. Then the following result is known and easy to prove (cf. [8, §10.8.5, p. 94]).

Lemma 2.3. *Let*

$$N^{\perp} = \{z \in Z : \langle x, z\rangle = 0,\ \forall x \in N\}.$$

Let $(X/N, \Gamma_N)^*$ and $N^{\perp}$ be respectively endowed with the weak topology and the restricted $\sigma(Z, X)$-topology. Then Q_N^* is a topological isomorphism from $(X/N, \Gamma_N)^*$ onto $N^{\perp}$.*

For convenience, we write $J = (Q_N^*)^{-1}$. Then, for each $z \in N^{\perp}$,

$$\langle [x], J(z)\rangle = \langle Q_N(y), J(z)\rangle = \langle y, Q_N^* \circ J(z)\rangle = \langle y, z\rangle, \quad \forall y \in [x]. \tag{2.2}$$

In addition, for a set A in a locally convex topological space, let $[A]$ denote the image of A under the mapping Q_N (i.e., $[A] = Q_N(A) = \{[a] : a \in A\}$) and let $\operatorname{ext} A$ stand for the set of all extreme points from A. Then we have the following two propositions; the first one establishes the connection between $\operatorname{ext}\partial p(\theta)$ and $\operatorname{ext}\mathbf{B}[(X/N)^*]$ while the second between the restricted p-centers in X and the restricted Chebyshev centers in X/N. Let θ denote the zero in X and let f be a convex real-valued function on X. Recall that the subdifferential of f at x is denoted by $\partial f(x)$ and defined by

$$\partial f(x) = \{z \in Z : \langle y - x, z\rangle \le f(y) - f(x), \forall y \in X\}.$$

Note that $\partial f(x)$ is bounded if f is continuous at x. Then $\partial p(\theta)$ is bounded and compact with the restricted $\sigma(Z, X)$-topology.

Proposition 2.4. *Let $\mathbf{B}[(X/N)^*]$ denote the closed unit ball of $(X/N, \|\cdot\|_p)^*$. Then the following assertions hold.*

(i) $\partial p(\theta) \subseteq N^{\perp}$.
(ii) $\mathbf{B}[(X/N)^*] = J[\partial p(\theta)]$.
(iii) $\operatorname{ext}\mathbf{B}[(X/N)^*] = J[\operatorname{ext}\partial p(\theta)]$; $\overline{\operatorname{ext}\mathbf{B}[(X/N)^*]}^{*} = J[\overline{\operatorname{ext}\partial p(\theta)}^{\sigma(Z,X)}]$.

Proposition 2.5. *Let $F \in \mathscr{B}_p(X)$, $V \subseteq X$ and $v \in V$. Then*

$$r_p(F; v) = r_{\|.\|_p}([F]; [v]); \quad \operatorname{rad}_p(F; V) = \operatorname{rad}_{\|.\|_p}([F]; [V]).$$

The proofs of the above two propositions are direct and so we omit them. The following corollary is a direct consequence of Proposition 2.5.

Corollary 2.6. *Let $F \in \mathscr{B}_p(X)$, $V \subseteq X$, and $v_0 \in V$. Then $v_0 \in \operatorname{Cent}_p(F; V)$ if and only if $[v_0] \in \operatorname{Cent}_{\|.\|_p}([F]; [V])$.*

The following two propositions describe the relationships between the wp-sup-compactness, the p-sup-compactness in X and the M-compactness in the corresponding quotient space X/N.

Proposition 2.7. *Let N have a topological complement in X and $F \in \mathcal{B}_p(X)$. Then $[F]$ is* M*-compact with respect to $[V]$ in X/N implies that there exists $F_1 \subseteq X$ such that $[F_1] = [F]$ and F_1 is p-*sup*-compact with respect to V in X.*

Proof. Suppose that $F \in \mathcal{B}_p(X)$ and $X = N \oplus N_1$, where N_1 is a topological complement of N in X. Let F_1 be the projection of F in N_1. Then $[F_1] = [F]$ and F_1 is p-sup-compact with respect to V in X. In fact, the first assertion is trivial, and hence we only need to show the second one. For this purpose, let $\{x_\alpha\} \subseteq F_1$ and $v_0 \in V$ satisfy $p(x_\alpha - v_0) \to r_p(F_1; v_0)$. Then $\|[x_\alpha] - [v_0]\|_p \to r_{\|.\|_p}([F_1]; [v_0]) = r_{\|.\|_p}([F]; [v_0])$. Because $[F]$ is M-compact with respect to $[V]$ in X/N, there is a subnet $\{[x_\lambda]\}$ of $\{[x_\alpha]\}$ such that $[x_\lambda] \to [x_0] \in [F]$. By $[F_1] = [F]$, without loss of generality, we may assume that $x_0 \in F_1$. Note that each x in X can be uniquely expressed as $x = x_N \oplus x_{N_1}$, where $x_N \in N$ and $x_{N_1} \in N_1$. Define the mapping $T : X/N \to N_1$ by

$$T([x]) = x_{N_1}, \quad \forall [x] \in X/N.$$

Then T is well-defined because $x_{N_1} = y_{N_1}$ if $[x] = [y]$. Noting that the mapping $T \circ Q_N$ is the projection of X onto N_1 and that $X = N \oplus N_1$ is a topological direct sum, $T \circ Q_N$ is continuous on X. This implies that T is continuous by [8, §10.7.(5), p. 92]. Therefore $x_\lambda = T([x_\lambda]) \to T([x_0]) = x_0 \in F_1$, and $F_1 \in p\text{-sup-}K(X, V)$. This completes the proof. □

Proposition 2.8. *Let $F \in \mathcal{B}_p(X)$. Then F is wp-*sup*-compact with respect to V in X if and only if $[F]$ is* M*-compact with respect to $[V]$ in X/N.*

Proof. Suppose that F is wp-sup-compact with respect to V in X. Let a net $\{[x_\alpha]\} \subseteq [F]$ and an element $[v_0] \in [V]$ be such that $\lim_\alpha \|[x_\alpha] - [v_0]\|_p = r_{\|.\|_p}([F]; [v_0])$. Without loss of generality, we may assume that $\{x_\alpha\} \subseteq F$ and $v_0 \in V$. Then $\lim_\alpha p(x_\alpha - v_0) = r_p(F; v_0)$. By the assumptions, there exist a subnet $\{x_\lambda\}$ of $\{x_\alpha\}$ and $x \in F$ such that $p(x_\lambda - x) \to 0$. Hence $\|[x_\lambda] - [x]\|_p = p(x_\lambda - x) \to 0$. Therefore $[F]$ is M-compact with respect to $[V]$ in X/N.

Conversely, let $F \in \mathcal{B}_p(X)$ be such that $[F]$ is M-compact with respect to $[V]$ in X/N. Let $\{x_\alpha\}$ be a net in F and v_0 in V. Suppose that $\lim_\alpha p(x_\alpha - v_0) = r_p(F; v_0)$. Then $\lim_\alpha \|[x_\alpha] - [v_0]\|_p = r_{\|.\|_p}([F]; [v_0])$; hence, by the M-compactness of $[F]$, there exist a subnet $\{[x_\lambda]\}$ of $\{[x_\alpha]\}$ and $[x] \in [F]$ (we may assume that $x \in F$) such that $\|[x_\lambda] - [x]\|_p \to 0$. By (2.1), we can take $\tilde{x}_\lambda \in N$ such that $p(x_\lambda - x + \tilde{x}_\lambda) \to 0$. Hence $p(x_\lambda - x) \to 0$ and F is wp-sup-compact with respect to V in X. The proof is complete. □

We end this section with some other notions that will be used in the remainder of this paper.

Definition 2.9. A subset V of X is said to be a p-S-sun (*resp.* wp-CS-sun, wp-MS-sun) if, for each $v_0 \in V$ and each $F \in \mathscr{B}_p(X)$ [*resp.* $F \in wp\text{-K(X)}$, $F \in wp\text{-sup-}K(X, V)$], $v_0 \in \mathrm{Cent}_p(F; V)$ implies $v_0 \in \mathrm{Cent}_p(F_\alpha; V)$, where $F_\alpha = v_0 + \alpha(F - v_0)$, $\forall \alpha > 0$.

Clearly, V is a p-S-sun $\Rightarrow$ V is a wp-MS-sun $\Rightarrow$ V is a wp-CS-sun. In the case when p is a norm on X, the notions of a p-S-sun and a wp-CS-sun were introduced and studied by Xu and Li [15], where they were called the simultaneous sun and the compact simultaneous sun, respectively. For simplicity, we call a p-S-sun (*resp.* wp-CS-sun, wp-MS-sun) a S-sun (*resp.* CS-sun, MS-sun) in the case when p is a norm.

Definition 2.10. Let $v_0 \in V$ and $F \in \mathscr{B}_p(X)$. v_0 is called a strongly unique restricted p-center of F with respect to V if $N \cap (V - v_0) = \{\theta\}$ and there exists $\gamma > 0$ such that

$$r_p(F; v) \geq r_p(F; v_0) + \gamma p(v - v_0), \quad \forall v \in V.$$

Note that a strongly unique restricted p-center of F with respect to V is unique. Let $\gamma_V(F, v_0)$ denote the largest constant γ for which the above inequality holds. Then v_0 is a strongly unique restricted p-center of F with respect to V if and only if $N \cap (V - v_0) = \{\theta\}$ and $\gamma_V(F, v_0) > 0$. We call $\gamma_V(F, v_0)$ the strong uniqueness constant of F with respect to V.

Definition 2.11. Let $\{F\}$ be a subset of $\mathscr{B}_p(X)$ and $v_0 \in V$. v_0 is called a uniformly strongly unique restricted p-center of $\{F\}$ with respect to V if $N \cap (V - v_0) = \{\theta\}$ and $\inf_{F \in \{F\}} \gamma_V(F, v_0) > 0$.

Definition 2.12. A subset V of X is said to be a strong p-S-sun (*resp.* strong wp-CS-sun, strong wp-MS-sun) if, for every $v_0 \in V$ and every $F \in \mathscr{B}_p(X)$ [*resp.* $F \in wp\text{-K(X)}, F \in wp\text{-sup-}K(X, V)$], the fact that v_0 is a strongly unique restricted p-center of F with respect to V implies that v_0 is a uniformly strongly unique restricted p-center of $\{F_\alpha\}_{\alpha > 0}$ with respect to V.

Similarly, it is easy to see that V is a strong p-S-sun $\Rightarrow$ V is a strong wp-MS-sun $\Rightarrow$ V is a strong wp-CS-sun. In the case when p is a norm on X, a strong p-S-sun (*resp.* strong wp-CS-sun, strong wp-MS-sun) is also simply called a strong S-sun (*resp.* strong CS-sun, strong MS-sun).

The relationships between p-S-suns in a locally convex space and S-suns in the corresponding quotient space are described in the following propositions. The proofs are just direct consequences of Propositions 2.5 and 2.8.

Proposition 2.13. *Let $V \subseteq X$. Then V is a p-*S-sun *(resp. wp-*CS-sun*, wp-*MS-sun*) of X if and only if $[V]$ is an* S-sun *(resp.* CS-sun, MS-sun*) of X/N.*

Proposition 2.14. *Let $V \subseteq X$. Suppose that $N \cap (V - v) = \{\theta\}$ for each $v \in V$. Then V is a strong p-*S-sun *(resp. strong wp-*CS-sun*, strong wp-*MS-sun*) of X if and only if $[V]$ is a strong* S-sun *(resp. strong* CS-sun*, strong* MS-sun*) of X/N.*

Finally, we introduce the notions of p-interpolating spaces and p-RS-sets, which are respectively generalizations of the corresponding notions in a normed linear space, see, for example, [1, 2]. Let W_1 and W_2 denote the subspaces spanned by $\operatorname{ext} \partial p(\theta)$ and $\overline{\operatorname{ext} \partial p(\theta)}^{\sigma(Z;X)}$, respectively. We throughout the paper assume that

$$\dim W_i \geq n + 1, \quad i = 1, 2.$$

Definition 2.15. Let V be an n-dimensional subspace of X spanned by $\operatorname{span}\{v_1, \ldots, v_n\}$. V is called an n-dimensional p-interpolating subspace (*resp.* strictly p-interpolating subspace) if

$$\det(\langle v_i, z_j \rangle)_{i,j=1}^{n} \neq 0$$

holds for any n linearly independent elements $z_1, \ldots, z_n$ in $\operatorname{ext} \partial p(\theta)$ [*resp.* $\overline{\operatorname{ext} \partial p(\theta)}^{\sigma(Z;X)}$].

Definition 2.16. Let $v_1, \ldots, v_n$ be n linearly independent elements of X and let

$$V = \left\{ v = \sum_{i=1}^{n} c_i v_i : c_i \in J_i \right\}. \tag{2.3}$$

Then V is called a p-RS-set (*resp.* strict p-RS-set) of X if each J_i is a subset of $\mathbb{R}$ of one of the following types:

(I) the whole of $\mathbb{R}$,
(II) a nontrivial proper closed (bounded or unbounded) interval of $\mathbb{R}$,
(III) a singleton;

and in addition every subset of $v_1, \ldots, v_n$ consisting of all v_i with J_i of type (I) and some v_i with J_i of type (II) spans a p-interpolating (*resp.* strictly p-interpolating) subspace of X.

3. RESTRICTED CHEBYSHEV CENTERS IN NORMED SPACES

Throughout this section, we always assume that Y is a real normed linear space with the norm $\|\cdot\|$, and its dual is denoted by Y^*. The closed unit ball of Y^* is denoted by $\mathbf{B}(Y^*)$ and endowed with the restricted weak* topology. Let V be a nonempty subset of Y. We use $\mathscr{B}(Y)$ [*resp.* $\mathscr{K}(Y)$, $\mathscr{M}(Y,V)$] to denote the set of all nonempty bounded subsets (*resp.* compact subsets, M-compact subsets with respect to V) of Y. Letting $F \in \mathscr{B}(Y)$, we define the function U_F on $\mathbf{B}(Y^*)$ by

$$U_F(x^*) = \sup\{\langle y, x^*\rangle : y \in F\}, \quad x^* \in \mathbf{B}(Y^*) \tag{3.1}$$

and U_F^+ by

$$U_F^+(x^*) = \inf_{O\in N_{x^*}} \sup_{\omega\in O} U_F(\omega), \quad x^* \in \mathbf{B}(Y^*),$$

where N_{x^*} is the family of all open neighborhoods of x^* in $\mathbf{B}(Y^*)$. Then, by [3], U_F^+ is upper semicontinuous on $\mathbf{B}(Y^*)$. Moreover, if F is compact, then $U_F^+ = U_F$ and so U_F is continuous on $\mathbf{B}(Y^*)$. It also follows from [3] that, for each $v \in Y$,

$$[U_F(y^*) - \langle v, y^*\rangle]^+|_{y^*=x^*} = U_F^+(x^*) - \langle v, x^*\rangle$$

and

$$\sup_{x^*\in\mathbf{B}(Y^*)} (U_F^+(x^*) - \langle v, x^*\rangle) = r_{\|.\|}(F; v). \tag{3.2}$$

Set

$$M_{F-v} = \{x^* \in \overline{\operatorname{ext}\mathbf{B}(Y^*)}^* : U_F^+(x^*) - \langle v, x^*\rangle = r_{\|.\|}(F, v)\};$$

$$E_{F-v} = \{x^* \in \operatorname{ext}\mathbf{B}(Y^*) : U_F(x^*) - \langle v, x^*\rangle = r_{\|.\|}(F, v)\}.$$

It is easy to see that M_{F-v} is nonempty. Furthermore, if F is compact, E_{F-v} is nonempty too.

The following two theorems are known, see, for example, [15].

Theorem 3.1. *Let $V \subseteq Y$. Then the following statements are equivalent.*

(i) *V is an* S-sun.
(ii) *For each $v_0 \in V$ and each $F \in \mathscr{B}(Y)$,*

$$v_0 \in \operatorname{Cent}_{\|.\|}(F; V) \iff \max\{\langle v_0 - v, x^*\rangle : x^* \in M_{F-v_0}\} \geq 0, \quad \forall v \in V.$$

Theorem 3.2. *Let $V \subseteq Y$. Then the following statements are equivalent.*

(i) *V is a* CS-sun.
(ii) *For each $v_0 \in V$ and each $F \in \mathcal{K}(Y)$,*

$$v_0 \in \mathrm{Cent}_{\|.\|}(F; V) \Longleftrightarrow \max\{\langle v_0 - v, x^* \rangle : x^* \in E_{F-v_0}\} \geq 0, \quad \forall v \in V.$$

Theorems 3.1 and 3.2 characterize respectively the restricted Chebyshev centers of bounded subsets and compact subsets F with respect to a "sun" V of Y. Theorem 3.4 below extends Theorem 3.2 from the class of compact subsets to the class of M-compact subsets. For the proof we need to establish the following lemma.

Lemma 3.3. *Let $v_0 \in V$ and let $F \in \mathcal{M}(Y, V)$. Set*

$$\widehat{M}^+_{F-v_0} = \{x^* \in \mathbf{B}(Y^*) : U^+_F(x^*) - \langle v_0, x^* \rangle = r_{\|.\|}(F; v_0)\}$$

and

$$\widehat{M}_{F-v_0} = \{x^* \in \mathbf{B}(Y^*) : U_F(x^*) - \langle v_0, x^* \rangle = r_{\|.\|}(F; v_0)\}.$$

Then

$$\widehat{M}^+_{F-v_0} = \widehat{M}_{F-v_0}. \tag{3.3}$$

Moreover, the function U_F on $\mathbf{B}(Y^*)$ is upper semicontinuous at each point of $\widehat{M}_{F-v_0}$.

Proof. Because

$$U_F(x^*) - \langle v_0, x^* \rangle \leq U^+_F(x^*) - \langle v_0, x^* \rangle \leq r_{\|.\|}(F; v_0),$$

by (3.2), it is trivial that $\widehat{M}^+_{F-v_0} \supseteq \widehat{M}_{F-v_0}$. In order to show $\widehat{M}^+_{F-v_0} \subseteq \widehat{M}_{F-v_0}$, let $x^* \in \widehat{M}^+_{F-v_0}$. It suffices to show that

$$U_F(x^*) - \langle v_0, x^* \rangle = r_{\|.\|}(F; v_0). \tag{3.4}$$

Let N_{x^*} denote the collection of all open neighborhoods of x^*. Then, for each $O \in N_{x^*}$,

$$\sup_{y^* \in O}[U_F(y^*) - \langle v_0, y^* \rangle] \geq U^+_F(x^*) - \langle v_0, x^* \rangle.$$

Hence, for any $\epsilon > 0$, there exists $y^*_{O,\epsilon} \in O$ such that

$$U_F(y^*_{O,\epsilon}) - \langle v_0, y^*_{O,\epsilon} \rangle > U^+_F(x^*) - \langle v_0, x^* \rangle - \epsilon = r_{\|.\|}(F; v_0) - \epsilon. \tag{3.5}$$

Define a partial order "$\geq$" on N_{x^*} as follows. Let $O_1, O_2 \in N_{x^*}$. Then $O_1 \geq O_2$ if and only if $O_1 \subseteq O_2$. Thus, for each $\epsilon > 0$, $\{y^*_{O,\epsilon}\}_{\geq}$ is a net on $\mathbf{B}(Y^*)$ and it is easy to see that $y^*_{O,\epsilon} \to x^*$ weakly* when ϵ fixed. By (3.1) and (3.5), we may select $x_{O,\epsilon} \in F$ such that

$$\langle x_{O,\epsilon} - v_0, y^*_{O,\epsilon}\rangle > r_{\|.\|}(F; v_0) - \epsilon. \tag{3.6}$$

Hence,

$$r_{\|.\|}(F; v_0) \geq \lim_{O,\epsilon\to 0} \|x_{O,\epsilon} - v_0\| \geq \lim_{O,\epsilon\to 0} \langle x_{O,\epsilon} - v_0, y^*_{O,\epsilon}\rangle \geq r_{\|.\|}(F; v_0).$$

Because F is M-compact with respect to V, there exists a subnet of $\{x_{O,\epsilon}\}$, denoted by itself, and $x_0 \in F$ such that $\lim_{O,\epsilon\to 0} \|x_{O,\epsilon} - x_0\| = 0$. Noting that

$$|\langle x_{O,\epsilon} - v_0, y^*_{O,\epsilon}\rangle - \langle x_0 - v_0, y^*_{O,\epsilon}\rangle| \leq \|x_{O,\epsilon} - x_0\| \to 0,$$

we obtain, by (3.6),

$$r_{\|.\|}(F; v_0) \leq \lim_{O,\epsilon\to 0} \langle x_0 - v_0, y^*_{O,\epsilon}\rangle = \langle x_0 - v_0, x^*\rangle \leq U_F(x^*) - \langle v_0, x^*\rangle.$$

Thus (3.4) holds. Let $x^* \in \widehat{M}_{F-v_0}$. To prove the continuity of U_F at x^*, let $\epsilon > 0$. Because U_F^+ is upper semicontinuous at x^*, there exists $O \in N_{x^*}$ such that $U_F^+(y^*) < U_F^+(x^*) + \epsilon$ for each $y^* \in O$. It follows that

$$U_F(y^*) \leq U_F^+(y^*) < U_F(x^*) + \epsilon, \quad \forall y^* \in O.$$

Hence U_F is upper semicontinuous at x^*. The proof is complete. □

Theorem 3.4. *Let $V \subseteq Y$. Then the following statements are equivalent.*

(i) *V is a* MS-sun.
(ii) *For each $v_0 \in V$ and each $F \in \mathcal{M}(Y, V)$,*

$$v_0 \in \mathrm{Cent}_{\|.\|}(F; V) \Longleftrightarrow \max\{\langle v_0 - v, x^*\rangle : x^* \in E_{F-v_0}\} \geq 0, \quad \forall v \in V. \tag{3.7}$$

Proof. By Theorem 3.1 and its proof given in [15], we have that (i) holds if and only if the following assertion holds:

(ii′) for each $v_0 \in V$ and each $F \in \mathcal{M}(Y, V)$,

$$v_0 \in \mathrm{Cent}_{\|.\|}(F; V) \Longleftrightarrow \max\{\langle v_0 - v, x^*\rangle : x^* \in M_{F-v_0}\} \geq 0, \quad \forall v \in V. \tag{3.8}$$

By Lemma 3.3, $\widehat{M}_{F-v_0}$ is weakly * compact. Noting that the function h on $\mathbf{B}(Y^*)$ defined by $h(x^*) = U_F(x^*) - \langle v_0, x^*\rangle$ is convex, we see that $\widehat{M}_{F-v_0}$ is an extremal subset of $\mathbf{B}(Y^*)$. Hence

$$\begin{aligned} M_{F-v_0} &\supseteq E_{F-v_0} = \widehat{M}_{F-v_0} \cap \operatorname{ext}\mathbf{B}(Y^*) = \operatorname{ext}\widehat{M}_{F-v_0} = \operatorname{ext}(\overline{\widehat{M}_{F-v_0}})^* \\ &\supseteq \operatorname{ext}(\overline{\operatorname{co}\widehat{M}_{F-v_0}})^* \neq \phi, \end{aligned}$$

where $\operatorname{co}\widehat{M}_{F-v_0}$ stands for the convex hull of the set $\widehat{M}_{F-v_0}$. Then by the Krein-Milman theorem, (3.7) holds if and only if (3.8) holds. The proof is complete. □

Now let us consider the strong uniqueness of the Chebyshev center. We require the following lemma, which was proved in [10].

Lemma 3.5. *Let $F \in \mathscr{B}(Y)$ and let $x \in Y$. Set*

$$\tau(F, x) = \lim_{t\to 0+} \frac{\sup_{y\in F}\|y + tx\| - \sup_{y\in F}\|y\|}{t}.$$

Then

$$\tau(F, x) = \max\{\langle x, x^*\rangle : x^* \in M_F\}.$$

Theorem 3.6. *Let $V \subseteq Y$. Then the following statements are equivalent.*

(i) *V is a strong* S-sun.
(ii) *For each $v_0 \in V$ and each $F \in \mathscr{B}(Y)$, v_0 is a strongly unique restricted Chebyshev center of F with respect to V if and only if there exists $\gamma > 0$ such that*

$$\max\{\langle v_0 - v, x^*\rangle : x^* \in M_{F-v_0}\} \geq \gamma\|v - v_0\|, \quad \forall v \in V. \tag{3.9}$$

Proof. (i) ⇒ (ii) Let $v_0 \in V$ and $F \in \mathscr{B}(Y)$. Suppose that (i) holds and that v_0 is a strongly unique restricted Chebyshev center of F with respect to V. Then $\gamma = \inf_{\alpha>0}\gamma_V(F_\alpha, v_0) > 0$. Consequently, for each $\alpha > 0$ and each $v \in V$, we have

$$r_{\|.\|}(F_\alpha; v) \geq r_{\|.\|}(F_\alpha; v_0) + \gamma_V(F_\alpha, v_0)\|v - v_0\| \geq r_{\|.\|}(F_\alpha; v_0) + \gamma\|v - v_0\|.$$

That is,

$$\sup_{x\in F}\|v_0 + \alpha(x - v_0) - v\| \geq \sup_{x\in F}\|v_0 + \alpha(x - v_0) - v_0\| + \gamma\|v - v_0\|.$$

Letting $\alpha = 1/t$, we obtain that

$$\frac{\sup_{x\in F} \|x - v_0 + t(v_0 - v)\| - \sup_{x\in F} \|x - v_0\|}{t} \geq \gamma \|v - v_0\|.$$

Letting $t \to 0+$, we get (3.9) by Lemma 3.5, and the necessity part of (ii) is proved.

Conversely, suppose that (3.9) holds. Then, for each $v \in V\backslash\{v_0\}$, we may take $x_v^* \in M_{F-v_0}$ such that $\langle v_0 - v, x_v^*\rangle \geq \gamma\|v - v_0\|$. Hence,

$$\begin{aligned} r_{\|.\|}(F; v) &\geq U_F^+(x_v^*) - \langle v, x_v^*\rangle = U_F^+(x_v^*) - \langle v_0, x_v^*\rangle + \langle v_0 - v, x_v^*\rangle \\ &\geq r_{\|.\|}(F; v_0) + \gamma\|v - v_0\|. \end{aligned} \tag{3.10}$$

This shows that v_0 is a strongly unique restricted Chebyshev center of F with respect to V, and the sufficiency part of (ii) is proved. Hence (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i) Let $F \in \mathscr{B}(Y)$. Suppose that (ii) holds and that $v_0 \in V$ is a strongly unique restricted Chebyshev center F with respect to V. Then (3.9) holds by (ii). Note that, for each $\alpha > 0$, $F_\alpha \in \mathscr{B}(Y)$ and $M_{F_\alpha - v_0} = M_{F-v_0}$. It follows from (3.9) that

$$\max_{x^*\in M_{F_\alpha - v_0}} \langle v_0 - v, x^*\rangle = \max_{x^*\in M_{F-v_0}} \langle v_0 - v, x^*\rangle \geq \gamma\|v - v_0\|, \quad \forall v \in V.$$

Again by (ii) and (3.10) , we get $\gamma_V(F_\alpha, v_0) \geq \gamma$. Therefore $\inf_{\alpha>0} \gamma_V(F_\alpha, v_0) \geq \gamma > 0$, that is, v_0 is a uniformly strongly unique restricted Chebyshev center of $\{F_\alpha\}_{\alpha>0}$ with respect to V. Hence (i) holds and the proof of Theorem 3.6 is complete. □

Similarly, we have the following result.

Theorem 3.7. *Let $V \subseteq Y$. Then the following statements are equivalent.*

(i) *V is a strong* CS-sun (*resp. strong* MS-sun).
(ii) *For each $v_0 \in V$ and each $F \in \mathscr{K}(Y)$ [resp. $F \in \mathscr{M}(Y, V)$], v_0 is a strongly unique restricted Chebyshev center of F with respect to V if and only if there exists $\gamma > 0$ such that*

$$\max\{\langle v_0 - v, x^*\rangle : x^* \in E_{F-v_0}\} \geq \gamma\|v - v_0\|, \quad \forall v \in V.$$

The following theorem is proved in [11, Theorem 3.2] (the proof given there is valid for M-compact subsets thanks to Lemma 3.3 although it was stated for totally bounded subsets).

Theorem 3.8. *Let V be an* RS-*set* (*resp. a strict* RS-*set*) *and let* $F \in \mathscr{M}(Y, V)$ [*resp.* $F \in \mathscr{B}(Y)$]. *Then F has a strongly unique restricted* Chebyshev *center with respect to V provided that* $\mathrm{rad}_{\|.\|}(F; V) > \mathrm{rad}_{\|.\|}(F; Y)$.

4. RESTRICTED p-CENTERS IN REAL LOCALLY CONVEX SPACES

In this section, we assume that X is a real locally convex topology linear space with its dual Z. Let p be a continuous seminorm defined on X. Let $F \in \mathscr{B}_p(X)$. According to [9], define

$$s_F(z) = \sup\{\langle y, z\rangle : y \in F\}, \quad z \in \partial p(\theta)$$

and

$$s_F^+(z) = \inf_{O\in N_z} \sup_{\omega\in O} s_F(\omega), \quad z \in \partial p(\theta),$$

where N_z is the family of all the open neighborhoods of z in $\partial p(\theta)$. Let $V \subseteq X$ and let $v \in V$. Set

$$\Lambda_{F,v} = \{z \in \overline{\mathrm{ext}\,\partial p(\theta)}^{\sigma(Z;X)} : s_F^+(z) - \langle v, z\rangle = r_p(F; v)\}$$

and

$$\Delta_{F,v} = \{z \in \mathrm{ext}\,\partial p(\theta) : s_F(z) - \langle v, z\rangle = r_p(F; v)\}.$$

Recalling that $N = \ker(p)$, let $Y = X/N$ be the quotient space of X with respect to N. Also recall that $J = (Q_N^*)^{-1}$ as in Section 2. Then, by Proposition 2.4,

$$s_F(z) = U_{[F]}[J(z)], \quad s_F^+(z) = U_{[F]}^+[J(z)], \ \ z \in \partial p(\theta). \tag{4.1}$$

Hence the following assertion is immediate.

Lemma 4.1. *Let* $F \in \mathscr{B}_p(X)$, $V \subseteq X$ *and* $v \in V$. *Then*

$$M_{[F]-[v]} = J(\Lambda_{F,v}); \qquad E_{[F]-[v]} = J(\Delta_{F,v}).$$

Thus the first result of this section, which characterizes a p-center of $F \in \mathscr{B}_p(X)$ with respect to a p-S-sun of X, can be stated as follows.

Theorem 4.2. *Let $V \subseteq X$. Then the following statements are equivalent.*

(i) *V is a p*-S-sun *of X.*
(ii) *For each $v_0 \in V$ and each $F \in \mathscr{B}_p(X)$,*

$$v_0 \in \mathrm{Cent}_p(F; V) \Longleftrightarrow \max\{\langle v_0 - v, z\rangle : z \in \Lambda_{F,v_0}\} \geq 0, \quad \forall v \in V. \tag{4.2}$$

Proof. By Proposition 2.13, (i) is equivalent to the following statement.

(i′) $[V]$ is an S-sun of X/N.
By Theorem 3.1, (i′) holds if and only if, for each $[v_0] \in [V]$ (we may assume that $v_0 \in V$) and each $F \in \mathscr{B}(X)$,

$$[v_0] \in \mathrm{Cent}_p([F];[V]) \Longleftrightarrow \max\{\langle [v_0]-[v], f\rangle : f \in M_{[F]-[v_0]}\} \geq 0, \quad \forall [v] \in [V]. \tag{4.3}$$

Applying Lemma 4.1 and Proposition 2.5 to (4.3), we get that (4.3) is equivalent to (4.2). Hence, (i) and (ii) are equivalent and the proof is complete. □

Similarly, we have the following results for *wp*-CS-sun and *wp*-MS-sun of X.

Theorem 4.3. *Let $V \subseteq X$. Then the following statements are equivalent.*

(i) *V is a wp-*CS-sun (*resp. wp*-MS-sun) *of X.*
(ii) *For each $v_0 \in V$ and each $F \in wp$-$K(X)$ [resp. $F \in wp$-*sup-*$K(X, V)$],*

$$v_0 \in \mathrm{Cent}_p(F; V) \Longleftrightarrow \max\{\langle v_0 - v, z\rangle : z \in \Delta_{F,v_0}\} \geq 0. \tag{4.4}$$

Note that a linear subspace of X is a p-S-sun (hence a *wp*-MS-sun). Thus from Theorem 4.3 we immediately have the following corollary, which was proved by Laurent and Pai in the case when F is p-sup-compact (*cf*. [9, Theorem 2.4]).

Corollary 4.4. *Let V be a subspace of X and $F \in \mathscr{B}_p(X)$ [resp. $F \in wp$-*sup-*$K(X, V)$]. Then* (4.2) *[resp.* (4.4)*] holds.*

By Theorems 3.6, 3.7 and Lemma 4.1, Theorems 4.5 and 4.6 below can be verified by a similar manner.

Theorem 4.5. *Let $V \subseteq X$ be such that $N \cap (V - v) = \{\theta\}$ for each $v \in V$. Then the following statements are equivalent.*

(i) *V is a strong p-*S-sun *of X.*
(ii) *For each $v_0 \in V$ and each $F \in \mathscr{B}_p(X)$, v_0 is a strongly unique restricted p-center of F with respect to V if and only if there exists $\gamma > 0$ such that*

$$\max\{\langle v_0 - v, z\rangle : z \in \Lambda_{F,v_0}\} \geq \gamma p(v - v_0), \quad \forall v \in V.$$

Theorem 4.6. *Let $V \subset X$ be such that $N \cap (V - v) = \{\theta\}$ for each $v \in V$. Then the following statements are equivalent.*

(i) *V is a strong wp-*CS-sun (*resp. strong wp-*MS-sun) *of X.*
(ii) *For each $v_0 \in V$ and each $F \in wp\text{-}K(X)$ [resp. $F \in wp\text{-}\mathrm{sup}\text{-}K(X, V)$], v_0 is a strongly unique restricted p-center of F with respect to V if and only if there exists $\gamma > 0$ such that*

$$\max\{\langle v_0 - v, z\rangle : z \in \Delta_{F,v_0}\} \geq \gamma p(v - v_0), \quad \forall v \in V.$$

Let $v, v_0 \in X$, and $F \in \mathscr{B}_p(X)$. Let $r_p'(F; v_0, v)$ denote the directional derivative of the convex function $r_p(F; \cdot)$ at v_0 in the direction v defined by

$$r_p'(F; v_0, v) = \lim_{t \to 0+} \frac{r_p(F; v_0 + tv) - r_p(F; v_0)}{t}.$$

Then, by Proposition 2.5, we have

$$r_p'(F; v_0, v) = \tau([F] - [v_0], -[v]). \tag{4.5}$$

Consequently, by Lemma 3.5 and Lemma 4.1,

$$r_p'(F; v_0, v) = \max\{\langle -v, z\rangle : z \in \Lambda_{F,v_0}\}. \tag{4.6}$$

Because a linear subspace V of X satisfying $N \cap V = \{\theta\}$ is a strong p-S-sun, by Theorem 4.5 and (4.6), we have the following corollary, which was proved in [9, Theorem 4.4].

Corollary 4.7. *Let V be a subspace of X such that $V \cap N = \{\theta\}$. Let $v_0 \in V$ and $F \in \mathscr{B}_p(X)$. Then the following statements are equivalent.*

(i) *v_0 is a strongly unique restricted p-center of F with respect to V.*
(ii) *There exists $\gamma > 0$ such that*

$$r_p'(F; v_0, v) \geq \gamma p(v), \quad \forall v \in V.$$

(iii) *There exists $\gamma > 0$ such that*

$$\max\{\langle v, z\rangle : z \in \Lambda_{F,v_0}\} \geq \gamma p(v), \quad \forall v \in V. \tag{4.7}$$

(iv) *There exists $\gamma > 0$ such that*

$$\max\{\langle v, z\rangle : z \in \Sigma_{F,v_0}\} \geq \gamma p(v), \quad \forall v \in V,$$

where

$$\Sigma_{F,v_0} = \{z \in \partial p(\theta) : s_F^+(z) - \langle v_0, z\rangle = r_p(F; v_0)\}.$$

Furthermore, in the case when $F \in wp\text{-sup-}K(X, V)$, *the set* Λ_{F,v_0} *in* (4.7) *can be replaced by* Δ_{F,v_0}.

In order to establish the result on the strong uniqueness of a restricted p-center with respect to a p-RS-set, we need the following lemmas.

Lemma 4.8. *Let* $z_1, \dots, z_n \in N^{\perp}$. *Then* $\{z_i\}_{i=1}^n$ *is linearly independent in* Z *if and only if* $\{J(z_i)\}_{i=1}^n$ *is linearly independent in* $(X/N)^*$.

Proof. It follows from the fact that J is a topological isomorphism from $N^{\perp}$ onto $(X/N)^*$. □

Lemma 4.9. (i) *Let* V *be an* n*-dimensional subspace of* X. *Then* V *is an* n*-dimensional* p*-interpolating* (*resp. strictly* p*-interpolating*) *subspace of* X *if and only if* $[V]$ *is an* n*-dimensional interpolating* (*resp. strictly interpolating*) *subspace of* X/N. *Moreover, in this case,* $N \cap V = \{\theta\}$.

(ii) *Let* V *be defined by* (2.3). *Then* V *is a* p-RS*-set* (*resp. strict* p-RS*-set*) *of* X *implies that* $[V]$ *is an* RS*-set* (*resp. a strict* RS*-set*) *of* X/N.

Proof. (i) Suppose that $v_1, \dots, v_n \in X$ are n linearly independent elements and that V is spanned by $\{v_1, \dots, v_n\}$. Suppose that V is an n-dimensional p-interpolating (*resp*. strictly p-interpolating) space of X. We claim that $N \cap V = \{\theta\}$. In fact, let $v = \sum_{i=1}^n a_i v_i \in N \cap V$. Choose n-linearly independent members

$$z_i \in \operatorname{ext} \partial p(\theta) \left(\textit{resp. } z_i \in \overline{\operatorname{ext} \partial p(\theta)}^{\sigma(Z;X)}\right).$$

Then $\langle v, z_j\rangle = 0$ for each $j = 1, \dots, n$ by Proposition 2.4(i), which means that $\sum_{i=1}^n \langle v_i, z_j\rangle a_i = 0$ for each $j = 1, \dots, n$. Because V is an n-dimensional p-interpolating (*resp*. strictly p-interpolating) subspace of X, we have that $a_i = 0$ for each $i = 1, 2, \dots, n$. Therefore, the claim is proved. In particular, $[v_1], \dots, [v_n]$ are linearly independent in X/N. By Definition 2.15, Proposition 2.4 and Lemma 4.8, it is easy to verify that $[V] = \operatorname{span}\{[v_1], \dots, [v_n]\}$ is an n-dimensional interpolating (*resp*. strictly interpolating) subspace of X/N. Hence the necessity part is proved. The sufficiency part can be proved similarly.

(ii) Let

$$V = \left\{v = \sum_{i=1}^n c_i v_i : c_i \in J_i\right\},$$

where $v_1, v_2, \ldots, v_n$ are linearly independent and each J_i is a subset of $\mathbb{R}$ of one of types (I), (II), and (III) as given in Definition 2.16. Suppose that V is a p-RS-set (*resp.* strict p-RS-set) of X. Without loss of generality, we assume that $J_1, \ldots, J_k$ are of type (I) or type (II) and $J_{k+1}, \ldots, J_n$ type (III), where k with $0 \le k \le n$ is a fixed integer. Let $\{[v_1], \ldots, [v_m]\}$ be a maximum linearly independent subsystem of $\{[v_1], \ldots, [v_n]\}$. Because V is a p-RS-set (*resp.* strict p-RS-set) of X, the subspace spanned by $\{v_1, \ldots, v_k\}$ is a k-dimensional p-interpolating (*resp.* strictly p-interpolating) subspace of X. This together with (i) implies that the subspace spanned by $[v_1], \ldots, [v_k]$ is a k-dimensional interpolating (*resp.* strict interpolating) subspace of X/N. In particular, $[v_1], \ldots, [v_k]$ are linearly independent and $k \le m$. Now assume that

$$[v_i] = \sum_{j=1}^{m} \alpha_{ij}[v_j], \quad \forall i = m+1, \ldots, n.$$

Let $[v] \in [V]$ and assume that $[v] = \sum_{i=1}^{n} c_i[v_i]$. Because for each $k < i \le n$, J_i is a singleton, we may assume that $J_i = \{\overline{c}_i\}$. Then, $[v]$ can be rewritten as

$$\begin{aligned} [v] &= \sum_{i=1}^{m} \left(c_i + \sum_{j=m+1}^{n} c_j \alpha_{ji} \right) [v_i] \\ &= \sum_{i=1}^{k} \left(c_i + \sum_{j=m+1}^{n} \overline{c}_j \alpha_{ji} \right) [v_i] + \sum_{i=k+1}^{m} \left(\overline{c}_i + \sum_{j=m+1}^{n} \overline{c}_j \alpha_{ji} \right) [v_i]. \end{aligned}$$

Let

$$J_i' = \begin{cases} \sum_{j=m+1}^{n} \overline{c}_j \alpha_{ji} + J_i & \text{if } 1 \le i \le k, \\ \{\overline{c}_i + \sum_{j=m+1}^{n} \overline{c}_j \alpha_{ji}\} & \text{if } k < i \le n. \end{cases}$$

Then

$$[V] = \left\{ [v] = \sum_{i=1}^{m} c_i'[v_i] : c_i' \in J_i' \right\}.$$

Moreover, by (i), it is easy to show that every subset of $\{[v_1], \ldots, [v_k]\}$ consisting of all $[v_i]$ with J_i' of type (I) and some $[v_i]$ with J_i' of type (II) spans an interpolating (*resp.* a strictly interpolating) subspace of X/N. Therefore, $[V]$ is an RS-set (*resp.* a strict RS-set) of X/N. The proof is complete. □

Remark 4.10. In general, the converse of Lemma 4.9(ii) is not true. For example, let $\mathbb{R}^3$ be the three-dimensional Euclidean space. Define

$$p(x) = |t_1| + |t_2|, \quad \forall x = (t_1, t_2, t_3) \in \mathbb{R}^3.$$

Then p is a continuous seminorm on X,

$$N = \ker(p) = \{(0,0,t) : t \in \mathbb{R}\} \quad \text{and} \quad \partial p(\theta) = \{(t_1, t_2, 0) : |t_1| \leq 1, |t_2| \leq 1\}.$$

It is easy to see that

$$\text{ext}\partial p(\theta) = \{(1,1,0), (1,-1,0), (-1,1,0), (-1,-1,0)\}$$

and the dimension of the subspace spanned by $\text{ext}\partial p(\theta)$ is 2. Set

$$V = \left\{\left(t, 0, \frac{1}{1+t}\right) : t \geq 0\right\}.$$

Then $[V] = \{c[(1,0,0)] : c \geq 0\}$ is a strict RS-set, but V is not. However, the following result holds: If $[V]$ is an RS-set (*resp.* a strict RS-set) of X/N, then there exists a subset V_1 of X such that $[V_1] = [V]$ and V_1 is an RS-set (*resp.* a strict RS-set) of X. In fact, suppose that

$$[V] = \left\{\sum_{i=1}^{n} c_i[v_i] : c_i \in J_i\right\}$$

is an RS-set (*resp.* a strict RS-set) of X/N, where $[v_1], \ldots, [v_n]$ are linearly independent in X/N, and $J_1, \ldots, J_n$ a subset of $\mathbb{R}$ of one of types (I), (II), and (III) as given in Definition 2.16. Then $v_1, \ldots, v_n$ are linearly independent in X because $[v_1], \ldots, [v_n]$ are in X/N. Let

$$V_1 = \left\{\sum_{i=1}^{n} c_i v_i : c_i \in J_i\right\}.$$

It is easy to verify that V_1 is an RS-set (*resp.* a strict RS-set) of X and $[V_1]=[V]$.

Theorem 4.11. *Let V be a p-RS-set (resp. strict p-RS-set) of X. Let $F \in wp$-sup-$K(X, V)$ [resp. $F \in \mathcal{B}_p(X)$] be such that $\text{rad}_p(F; V) > \text{rad}_p(F; X)$. Then F has a strongly unique restricted p-center with respect to V.*

Proof. Suppose that V is a p-RS-set (*resp.* strict p-RS-set) of X. Then $[V]$ is an RS-set (*resp.* a strict RS-set) of X/N by Lemma 4.9(ii). Let $F \in wp$-sup-$K(X, V)$ [*resp.* $F \in \mathcal{B}_p(X)$] be such that $\text{rad}_p(F; V) > \text{rad}_p(F; X)$. Then, we have that $[F]$ is an M-compact subset with respect to $[V]$ (*resp.* a bounded subset) of X/N by Proposition 2.8 (*resp.* 2.1), and that $\text{rad}_{\|.\|_p}([F]; [V]) > \text{rad}_{\|.\|_p}([F]; [X])$ by Proposition 2.5. By Theorem 3.8, $[F]$ has a strongly unique restricted Chebyshev center with respect to $[V]$, which is denoted by $[v_0]$. Consequently, there exists $\gamma > 0$ such that

$$r_{\|\cdot\|_p}([F];[v]) \geq r_{\|\cdot\|_p}([F];[v_0]) + \gamma\|[v]-[v_0]\|_p, \quad \forall [v] \in [V]. \tag{4.8}$$

Without loss of generality, we may assume that $v_0 \in V$. Then (4.8) is equivalent that

$$r_p(F;v) \geq r_p(F;v_0) + \gamma p(v-v_0), \quad \forall v \in V.$$

To complete the proof, it suffices to show that $N \cap (V - v_0) = \{\theta\}$. For this purpose, assuming that V is defined by (2.3), let V_0 be the subspace of X spanned by the subset of $\{v_1, \ldots, v_n\}$ consisting of all v_i with J_i of type (I) or (II). Then V_0 is an interpolating (*resp.* a strict interpolating) subspace of X and hence, by Lemma 4.9(i), $N \cap V_0 = \{\theta\}$. Noting that, if $v \in V$, then $v - v_0 \in V_0$ and so $N \cap (V - v_0) = \{\theta\}$. This completes the proof. □

In particular, for a p-interpolating (*resp.* strictly p-interpolating) subspace V of X, we have the following corollary from Theorem 4.11.

Corollary 4.12. *Let V be a p-interpolating (resp. strictly p-interpolating) subspace of X. Let $F \in wp\text{-}\mathrm{sup}\text{-}K(X,V)$ [resp. $F \in \mathcal{B}_p(X)$] be such that* $\mathrm{rad}_p(F;V) > \mathrm{rad}_p(F;X)$. *Then F has a strongly unique restricted p-center with respect to V.*

Remark 4.13. In the case when p is a norm and F is a totally bounded subset of X, Corollary 4.12 was first obtained by Li in [10]. For a general seminorm p, Corollary 4.12 was established by Laurent and Pai in [9] under the additional assumptions that V is p-inf-bounded (i.e., for each $\lambda > 0$, $\{x \in V : p(x) \leq \lambda\}$ is bounded in X) and that $\ker(p) \cap V = \{\theta\}$. It should be noted that the proof used in [9] would be invalid if the above assumptions are dropped.

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