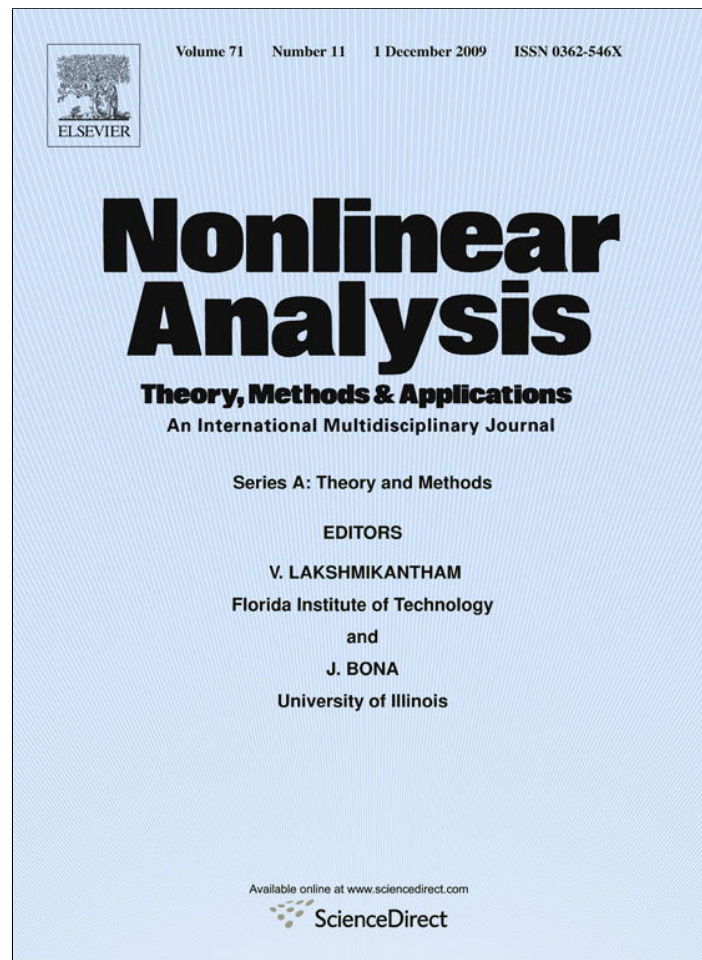


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Nonlinear Analysis

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ARTICLE INFO

Article history:

Received 1 October 2008

Accepted 29 April 2009

MSC:

primary 34A55

secondary 34B24

Keywords:

Variational inequalities

Riemannian manifold

Monotone vector fields

ABSTRACT

We establish the existence and uniqueness results for variational inequality problems on Riemannian manifolds and solve completely the open problem proposed in [S.Z. Németh, Variational inequalities on Hadamard manifolds, *Nonlinear Anal.* 52 (2003) 1491–1498]. Also the relationships between the constrained optimization problem and the variational inequality problems as well as the projections on Riemannian manifolds are studied.

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1. Introduction

Variational inequality problems on finite-dimensional Banach spaces are powerful tools for studying constrained optimization problems and equilibrium problems as well as complementary problems, and have been studied extensively, see for example the survey [1,2]. One basic problem for variational inequality problems is the existence issue of solutions, which has a lot of applications in these fields mentioned above (cf. [2,1]).

Applied problems posed on manifolds arise in many natural contexts. Classical examples are given by some numerical problems such as eigenvalue problems and invariant subspace computations, constrained minimization problems and boundary value problems on manifolds, etc, see for example [3–9,31,34]. Recent interests are focused on extending some classical and important results for solving these problems on Banach spaces to the setting of manifolds. For example, the well-known Kantorovich theorem and the famous Smale point estimate theory for Newton's method (for solving equations) on Banach spaces were established respectively in [10,11] and [10,12,13] for vector fields on Riemannian manifolds; while the convergence results of the proximal point methods for optimization problems on Hilbert spaces were extended to the setting of Hadamard manifolds in [14,15]. In particular, Németh introduced the variational inequalities on Hadamard manifolds and obtained some basic existence theorems and uniqueness theorems in [6], where an open problem on how to extend the existence and uniqueness results on Hadamard manifolds to Riemannian manifolds is also proposed.

In the present paper, we study the existence and uniqueness problems of the solutions for variational inequality problems on Riemannian manifolds. The main purpose is to solve completely the open problem proposed in [6]. Some equivalences

[☆] The second author is partially supported by DGES, Grant SAB 2006-0195, Spain; the National Natural Science Foundation of China (Grant No.10671175; 10731060). The third author was partially supported by a grant from NSC and the fourth author was supported by the grant NSC 97-2115-M-110-001.

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between the variational inequality problem and the constrained minimization problems as well as the projections on Riemannian manifolds are established in Section 3, which will play a key role in the study of the existence problem. We introduce new notion of a weak pole for a subset of the underlying manifold and then establish the existence and uniqueness results for the variational inequality problem on locally convex subsets with weak pole interior points. Also we provide a simple counterexample to illustrate that, even on a compact totally convex subset, the solution of the variational inequality problem with a continuous vector field may not exist.

2. Preliminaries

We begin with some standard notions about Riemannian manifolds. The readers are referred to some text books for details, for example, [16–19].

Let (M, g) be a complete finite-dimensional Riemannian manifold with the Levi-Civita connection ∇ on M . Let $x \in M$ and let $T_x M$ denote the tangent space at x to M . We denote by $\langle \cdot, \cdot \rangle_x$ the scalar product on $T_x M$ with the associated norm $\| \cdot \|_x$, where the subscript x is sometimes omitted. For $x, y \in M$, let $c : [0, 1] \rightarrow M$ be a piecewise smooth curve joining x to y . Then the arc-length of c is defined by $l(c) := \int_0^1 \|\dot{c}(t)\| dt$, while the Riemannian distance from x to y is defined by $d(x, y) := \inf_c l(c)$, where the infimum is taken over all piecewise smooth curves $c : [0, 1] \rightarrow M$ joining x to y . Recall that a curve $c : [0, 1] \rightarrow M$ joining x to y is a geodesic if

$$c(0) = x, \quad c(1) = y \quad \text{and} \quad \nabla_{\dot{c}} \dot{c} = 0 \quad \text{on} \quad [0, 1], \tag{2.1}$$

and a geodesic $c : [0, 1] \rightarrow M$ joining x to y is minimal if its arc-length equals its Riemannian distance between x and y . By the Hopf–Rinow Theorem (cf. [17]), if M is additionally connected, then (M, d) is a complete metric space, and there is at least one minimal geodesic joining x to y . Moreover, the exponential map at x $\exp_x : T_x M \rightarrow M$ is well-defined on $T_x M$. Clearly, a curve $c : [0, 1] \rightarrow M$ is a minimal geodesic joining x to y if and only if there exists a vector $v \in T_x M$ such that $\|v\| = d(x, y)$ and $c(t) = \exp_x(tv)$ for each $t \in [0, 1]$.

For a Banach space or a Riemannian manifold Z , we use $\mathbf{B}_Z(p, r)$ and $\overline{\mathbf{B}}_Z(p, r)$ to denote, respectively, the open metric ball and the closed metric ball at p with radius r , that is,

$$\mathbf{B}_Z(p, r) = \{q \in Z : d(p, q) < r\} \quad \text{and} \quad \overline{\mathbf{B}}_Z(p, r) = \{q \in Z : d(p, q) \leq r\}.$$

We often omit the subscript Z if no confusion arises.

We denote by $\Gamma_{x,y}$ the set of all geodesics $c := \gamma_{xy} : [0, 1] \rightarrow M$ satisfying (2.1). Note that each $c \in \Gamma_{x,y}$ can be extended naturally to a geodesic defined on \mathbb{R} . Definition 2.1 presents the notions of the different kinds of convexity, where items (a) and (b) are known in [20] while items (c) and (d) are, respectively, known in [9,21,32].

Definition 2.1. Let A be a nonempty subset of M . A is called

- (a) a *weakly convex* subset if, for any $p, q \in A$, there is a minimal geodesic of M joining p to q lying in A ;
- (b) a *strongly convex* subset if, for any $p, q \in A$, there is just one minimal geodesic of M joining p to q and it is in A ;
- (c) a *locally convex* subset if, for any $p \in A$, there is a positive $\varepsilon > 0$ such that $A \cap \mathbf{B}(p, \varepsilon)$ is strongly convex;
- (d) a *totally convex* subset if, for any $p, q \in A$, every geodesic of M joining p to q is in A .

Clearly, the following implications hold for a nonempty set A in M :

$$\text{the strong convexity} \implies \text{the total convexity} \implies \text{the weak convexity} \implies \text{the local convexity.} \tag{2.2}$$

Remark 2.1. Recall (cf. [19]) that M is a *Hadamard manifold* if it is a simple connected and complete Riemannian manifold with nonpositive sectional curvature. In a Hadamard manifold, the geodesic between any two points is unique and the exponential map at each point of M is a global diffeomorphism. Therefore, all convexities in a Hadamard manifold coincide.

In the following definition, we introduce the notions of convex functions. The notion of the convex function is taken from [9], where it was defined in a totally convex subset; while the one of the weakly convex function is new. Let $A \subseteq M$ be a nonempty set and let $\Gamma_{x,y}^A$ denote the set of all $\gamma_{xy} \in \Gamma_{x,y}$ such that $\gamma_{xy} \subseteq A$.

Definition 2.2. Let $A \subset M$ be a weakly convex set and $f : A \rightarrow \mathbb{R}$ be a real-valued function. f is said to be

- (a) *weakly convex* on A if, for any $x, y \in A$, there is a geodesic $\gamma_{xy} \in \Gamma_{x,y}^A$ such that the composition $f \circ \gamma_{xy}$ is convex on $[0, 1]$;
- (b) *convex* on A if, for any $x, y \in A$ and any geodesic $\gamma_{xy} \in \Gamma_{x,y}^A$, the composition $f \circ \gamma_{xy}$ is convex on $[0, 1]$.

3. Variational inequality problems

Let $A \subseteq M$ be a nonempty set and let V be a vector field on A . The variational inequality problem considered here on Riemannian manifold M is of finding $\bar{x} \in A$ such that

$$\langle V(\bar{x}), \dot{\gamma}_{\bar{x}y}(0) \rangle \geq 0, \quad \text{for each } y \in A \text{ and each } \gamma_{\bar{x}y} \in \Gamma_{\bar{x},y}^A. \tag{3.1}$$

Any point $\bar{x} \in A$ satisfying (3.1) is called a solution of the variational inequality problem (3.1). Clearly, the following equivalence holds for each $r > 0$:

$$\bar{x} \text{ is a solution of (3.1)} \iff [\langle rV(\bar{x}), \dot{\gamma}_{\bar{x}y}(0) \rangle \geq 0 \ \forall y \in A \text{ and } \forall \gamma_{\bar{x}y} \in \Gamma_{\bar{x},y}^A]. \quad (3.2)$$

The variational inequality problem is closely connected with the optimization problem on Riemannian manifolds. Let f be a real-valued function on A . Consider the optimization problem

$$\min_{x \in A} f(x). \quad (3.3)$$

Then we have the following proposition, which describes the equivalence between the optimization problem (3.3) and the variational inequality problem (3.1).

Proposition 3.1. *Let $A \subset M$ be a closed weakly convex set and $\bar{x} \in A$. Suppose that $f : M \rightarrow \mathbb{R}$ is weakly convex and differentiable. Then \bar{x} is a solution of the optimization problem (3.3) if and only if \bar{x} is a solution of the variational inequality problem (3.1) with $V = \text{grad} f$.*

Proof. Suppose that \bar{x} is a solution of the optimization problem (3.3). Then, for any $y \in A$ and any $\gamma_{\bar{x}y} \in \Gamma_{\bar{x},y}^A$, one has that

$$(f \circ \gamma_{\bar{x}y})(0) = f(\bar{x}) = \min_{t \in [0,1]} (f \circ \gamma_{\bar{x}y})(t)$$

because $\gamma_{\bar{x}y}(t) \in A$ for each $t \in [0, 1]$. Consequently, $t = 0$ is a minimizer of the function $f \circ \gamma_{\bar{x}y}$ on $[0, 1]$ and so $(f \circ \gamma_{\bar{x}y})'(0) \geq 0$. Note that

$$(f \circ \gamma_{\bar{x}y})'(0) = \langle \text{grad} f(\bar{x}), \dot{\gamma}_{\bar{x}y}(0) \rangle. \quad (3.4)$$

It follows that \bar{x} is a solution of the variational inequality problem (3.1) with $V = \text{grad} f$.

Conversely, let $V = \text{grad} f$ and suppose that \bar{x} is such that (3.1) holds. Let $y \in A$. Since f is weakly convex, there exists $\gamma_{\bar{x}y} \in \Gamma_{\bar{x},y}^A$ such that $f \circ \gamma_{\bar{x}y}$ is convex on $[0, 1]$. Hence

$$f(y) = (f \circ \gamma_{\bar{x}y})(1) \geq (f \circ \gamma_{\bar{x}y})(0) + (f \circ \gamma_{\bar{x}y})'(0) = f(\bar{x}) + (f \circ \gamma_{\bar{x}y})'(0).$$

By (3.1) and (3.4),

$$(f \circ \gamma_{\bar{x}y})'(0) = \langle \text{grad} f(\bar{x}), \dot{\gamma}_{\bar{x}y}(0) \rangle = \langle V(\bar{x}), \dot{\gamma}_{\bar{x}y}(0) \rangle \geq 0.$$

It follows that $f(y) \geq f(\bar{x})$ and so \bar{x} is a solution of the problem (3.3). The proof is complete. \square

Let P_A denote the projection on A , that is,

$$P_A x = \{\bar{x} \in A : d(x, \bar{x}) = \inf_{z \in A} d(x, z)\} \text{ for each } x \in M. \quad (3.5)$$

Then $P_A x \neq \emptyset$ for each $x \in M$ if A is closed (because A is locally compact). In general, P_A is a set-valued map. The following proposition provides the case when $P_A x$ is a singleton, which was proved in [20].

Proposition 3.2. *Let A be a locally convex closed subset of M . Then there exists an open subset U of M with $A \subseteq U$ such that P_A is single-valued and Lipschitz continuous on U .*

Recall that, for a point $x \in M$, the convexity radius at x is defined by

$$r(x) := \sup \left\{ r > 0 : \begin{array}{l} \text{each ball in } \mathbf{B}(x, r) \text{ is strongly convex} \\ \text{and each geodesic in } \mathbf{B}(x, r) \text{ is minimal} \end{array} \right\}.$$

Clearly, if M is a Hadamard manifold, then $r(x) = +\infty$ for each $x \in M$. The following lemma lists some useful properties for our study, where item (i) is a direct consequence of [19, Theorem 5.3, p.169]; item (ii) is given in [22]; while item (iii) is referred to [21,20].

Lemma 3.1. *Let $D \subseteq M$ be compact and set*

$$r(D) := \inf\{r(x) : x \in D\}. \quad (3.6)$$

Then the following assertions hold.

- (i) The function $x \mapsto r(x) \in [0, +\infty]$ is continuous on M and $r(D) > 0$.
- (ii) There exists a real number $\theta \in [r(D), +\infty]$ (the largest one is denoted by $\theta(D)$) such that for each $0 < r \leq \theta$ and $x \in D$, the exponential map $\exp_x : \mathbf{B}(0, r) \rightarrow \mathbf{B}(x, r)$ is diffeomorphic.
- (iii) There exists a real number $\rho \in (0, r(D)]$ (the largest one is denoted by $\rho(D)$) such that for each $0 < r \leq \rho$, $x \in D$, each nonconstant geodesic $c : [0, 1] \rightarrow \mathbf{B}(x, r)$ and each minimal geodesic $c_0 : [0, 1] \rightarrow \mathbf{B}(x, \rho(D))$ joining x to $c(0)$, if $\langle c'(0), c'_0(1) \rangle \geq 0$, then $s \mapsto d(c(s), x)$ is strictly increasing on $[0, 1]$.

Theorems 3.1 and **3.2**, which provide the sufficient and necessary conditions for a point \bar{x} to be in P_{AX} , will play crucial roles in our study of the existence of solutions of the variational inequality problems.

Theorem 3.1. *Let $A \subset M$ be a nonempty closed set and let $\bar{x} \in A$. Let $x \in M$ and $\gamma_{x\bar{x}} \in \Gamma_{x,\bar{x}}$ be a minimal geodesic. If $\bar{x} \in P_{AX}$, then*

$$\langle \dot{\gamma}_{x\bar{x}}(1), \dot{\gamma}_{\bar{x}z}(0) \rangle \geq 0 \quad \text{for each } z \in A \text{ and each } \gamma_{\bar{x}z} \in \Gamma_{\bar{x},z}^A. \tag{3.7}$$

Proof. Without loss of generality, we assume that $x \notin A$. Write $x_t = \gamma_{x\bar{x}}(t)$ for each $t \in [0, 1]$. Since $\bar{x} \in P_{AX}$, it follows that $\bar{x} \in P_{Ax_t}$ for each $t \in (0, 1)$. In fact, otherwise, there exists $y \in A$ such that $d(y, x_t) < d(\bar{x}, x_t)$. Then,

$$d(x, y) \leq d(x, x_t) + d(x_t, y) < d(x, \gamma_{x\bar{x}}(t)) + d(\gamma_{x\bar{x}}(t), \bar{x}) = d(x, \bar{x}),$$

which contradicts that $\bar{x} \in P_{Ax}$. Fix $r_0 > 0$ and apply **Lemma 3.1** to a compact subset $D := \overline{\mathbf{B}(\bar{x}, r_0)}$ to get $r := \min\{\theta(D), r_0\} > 0$. Let $t_0 \in (0, 1)$ be such that $\bar{x} \in \mathbf{B}(p, \frac{r}{2})$, where $p := x_{t_0} = \gamma_{x\bar{x}}(t_0)$. Then $\exp_p : \mathbf{B}(0, r) \rightarrow \mathbf{B}(p, r)$ is diffeomorphic. Now suppose that (3.7) does not hold. Then there exist $z \in A$ and $\gamma_{\bar{x}z} \in \Gamma_{\bar{x},z}^A$ such that

$$\langle \dot{\gamma}_{x\bar{x}}(1), \dot{\gamma}_{\bar{x}z}(0) \rangle < 0. \tag{3.8}$$

Let $s_0 \in (0, 1)$ be such that $q := \gamma_{\bar{x}z}(s_0) \in \mathbf{B}(\bar{x}, \frac{r}{2})$. Then $q \in \mathbf{B}(p, r)$. Define the geodesics $\gamma_{p\bar{x}}$ and $\gamma_{\bar{x}q}$ by

$$\gamma_{p\bar{x}}(t) = \gamma_{x\bar{x}}(t_0 + (1 - t_0)t) \quad \text{for each } t \in [0, 1]$$

and

$$\gamma_{\bar{x}q}(t) = \gamma_{\bar{x}z}(s_0 t) \quad \text{for each } t \in [0, 1],$$

respectively. Then, $\gamma_{\bar{x}q} \in \Gamma_{\bar{x},q}^A$. Moreover, $\dot{\gamma}_{p\bar{x}}(1) = (1 - t_0)\dot{\gamma}_{x\bar{x}}(1)$ and $\dot{\gamma}_{\bar{x}q}(0) = s_0\dot{\gamma}_{\bar{x}z}(0)$. It follows from (3.8) that

$$\langle \dot{\gamma}_{p\bar{x}}(1), \dot{\gamma}_{\bar{x}q}(0) \rangle < 0. \tag{3.9}$$

Since M is complete, $\gamma_{\bar{x}q}(t)$ is defined for all $t \in (-\infty, +\infty)$. Noting $\bar{x} \in \mathbf{B}(p, \frac{r}{2})$, there exists $0 < \varepsilon < 1$ such that $\gamma_{\bar{x}q}(-\varepsilon, \varepsilon) \subset \mathbf{B}(p, r)$. Let $\gamma : (-\varepsilon, \varepsilon) \times [0, 1] \rightarrow M$ be defined by

$$\gamma(s, t) = \exp_p t(\exp_p^{-1} \gamma_{\bar{x}q}(s)) \quad \text{for each } (s, t) \in (-\varepsilon, \varepsilon) \times [0, 1].$$

Thus, for each $s \in (-\varepsilon, \varepsilon)$, $\gamma(s, \cdot)$ is the geodesic joining p to $\gamma_{\bar{x}q}(s)$; hence $\|\frac{\partial \gamma}{\partial t}(s, \cdot)\|$ is a constant. Note that $\gamma(0, \cdot) = \gamma_{p\bar{x}}(\cdot)$ and that \exp_p is diffeomorphic by the choice of p . Hence γ is a variation of $\gamma_{p\bar{x}}$ and $V(t) = \frac{\partial \gamma}{\partial s}(0, t)$ is the variational field of γ . In particular,

$$V(0) = \frac{\partial \gamma}{\partial s}(0, 0) = 0 \quad \text{and} \quad V(1) = \frac{\partial \gamma}{\partial s}(0, 1) = \dot{\gamma}_{\bar{x}q}(0). \tag{3.10}$$

Let $L : (-\varepsilon, \varepsilon) \rightarrow [0, +\infty)$ be defined by

$$L(s) = \int_0^1 \left\| \frac{\partial \gamma}{\partial t}(s, t) \right\| dt \quad \text{for each } s \in (-\varepsilon, \varepsilon)$$

that is, $L(s)$ is the arc-length of the geodesic $\gamma(s, \cdot)$. Since $\|\frac{\partial \gamma}{\partial t}(s, \cdot)\|$ is a constant for each fixed $s \in (-\varepsilon, \varepsilon)$, it follows that

$$L^2(s) = \int_0^1 \left\| \frac{\partial \gamma}{\partial s}(s, t) \right\|^2 dt \quad \text{for each } s \in (-\varepsilon, \varepsilon).$$

Applying the first variation formula (cf. [17, Proposition 2.4, p.195]), we get that

$$\begin{aligned} \frac{1}{2} \frac{dL^2}{ds}(0) &= - \int_0^1 \langle V(t), \nabla_{\dot{\gamma}_{p\bar{x}}(t)} \dot{\gamma}_{p\bar{x}}(t) \rangle dt - \langle V(0), \dot{\gamma}_{p\bar{x}}(0) \rangle + \langle V(1), \dot{\gamma}_{p\bar{x}}(1) \rangle \\ &= \langle \dot{\gamma}_{\bar{x}q}(0), \dot{\gamma}_{p\bar{x}}(1) \rangle \end{aligned}$$

thanks to (3.10). This together with (3.9) yields that $\frac{dL^2}{ds}(0) < 0$; hence, the function $L(\cdot)$ is strictly decreasing in a neighborhood of 0. Consequently, there exists $\bar{s} \in (0, \varepsilon)$ such that $L(\bar{s}) < L(0)$ and so

$$d(p, \gamma_{\bar{x}q}(\bar{s})) \leq L(\bar{s}) < L(0) = d(p, \bar{x}).$$

This contradicts that $\bar{x} \in P_A(p)$ as $\gamma_{\bar{x}q}(\bar{s}) \in A$ and completes the proof. \square

The converse of **Theorem 3.1** is as follows.

Theorem 3.2. Let $A \subset M$ be a nonempty closed, weakly convex set and let $\bar{x} \in A$. Then there exists $\hat{r}_{\bar{x}} > 0$ such that for each $x \in \mathbf{B}(\bar{x}, \hat{r}_{\bar{x}})$ and each minimal geodesic $\gamma_{x\bar{x}} \in \Gamma_{x,\bar{x}}$,

$$\bar{x} \in P_A x \iff (3.7) \text{ holds.} \tag{3.11}$$

Proof. Fix $r_0 > 0$. Then $D := \overline{\mathbf{B}(\bar{x}, r_0)}$ is compact. Set

$$\hat{r}_{\bar{x}} := \min\{r_0, \rho(\overline{\mathbf{B}(\bar{x}, r_0)})\}.$$

Then $\hat{r}_{\bar{x}} > 0$ by Lemma 3.1(iii). Below we show that $\hat{r}_{\bar{x}}$ is as desired.

Let $x \in \mathbf{B}(\bar{x}, \hat{r}_{\bar{x}}) \subseteq \mathbf{B}(\bar{x}, r_0)$ and $\gamma_{x\bar{x}} \in \Gamma_{x,\bar{x}}$ be a minimal geodesic. By Theorem 3.1, it suffices to verify the implication that

$$(3.7) \text{ holds} \implies \bar{x} \in P_A x. \tag{3.12}$$

To do this, suppose that (3.7) holds but there is $\bar{y} \in A$ such that

$$d(x, \bar{y}) < d(x, \bar{x}) < \hat{r}_{\bar{x}}. \tag{3.13}$$

As $\bar{x}, \bar{y} \in A$ and A is weakly convex, there exists a minimal geodesic $\gamma_{\bar{x}\bar{y}} \in \Gamma_{\bar{x},\bar{y}}^A$. It follows from (3.7) that

$$\langle \gamma'_{x\bar{x}}(1), \gamma'_{\bar{x}\bar{y}}(0) \rangle \geq 0. \tag{3.14}$$

Since $x \in \overline{\mathbf{B}(\bar{x}, r_0)}$, by the definition of $\hat{r}_{\bar{x}}$, $\mathbf{B}(x, \hat{r}_{\bar{x}})$ is strongly convex. It follows that $\gamma_{x\bar{x}}, \gamma_{\bar{x}\bar{y}} \subset \mathbf{B}(x, \hat{r}_{\bar{x}})$ because $\bar{x}, \bar{y} \in \mathbf{B}(x, \hat{r}_{\bar{x}})$ by (3.13). By (3.14), Lemma 3.1(iii) is applicable to $\gamma_{x\bar{x}}, \gamma_{\bar{x}\bar{y}}$ in place of c, c_0 to conclude that the function $s \mapsto d(\gamma_{\bar{x}\bar{y}}(s), x)$ is strictly increasing on $[0, 1]$. This implies that $d(\bar{x}, x) < d(\bar{y}, x)$, which contradicts (3.13). Hence (3.12) is proved. \square

Corollary 3.1. Let $A \subset M$ be a nonempty compact, weakly convex set. Then there exists $\hat{r}_A > 0$ such that (3.11) holds for any $\bar{x} \in A, x \in M$ with $d(x, A) \leq \hat{r}_A$ and each minimal geodesic $\gamma_{x\bar{x}} \in \Gamma_{x,\bar{x}}$.

Proof. By Theorem 3.2, for each $\bar{x} \in A$, there exists $\hat{r}_{\bar{x}}$ such that (3.11) holds for each $x \in \mathbf{B}(\bar{x}, \hat{r}_{\bar{x}})$ and each minimal geodesic $\gamma_{x\bar{x}} \in \Gamma_{x,\bar{x}}$. Since A is compact, there exist finite points, say $\{\bar{x}_1, \dots, \bar{x}_n\} \subseteq A$, together with the corresponding positive numbers $\hat{r}_{\bar{x}_1}, \dots, \hat{r}_{\bar{x}_n}$, such that $A \subseteq \cup_{i=1}^n \mathbf{B}(\bar{x}_i, \frac{1}{2}\hat{r}_{\bar{x}_i})$. Define $\hat{r}_A = \frac{1}{2} \min_{1 \leq i \leq n} \hat{r}_{\bar{x}_i}$. Then it is easy to see that \hat{r}_A is as desired and the proof is complete. \square

Proposition 3.3. Let $A \subset M$ be a nonempty closed set and V be a vector field on A . Let $\bar{x} \in A$ and $r \in (0, \frac{r(\bar{x})}{\|V(\bar{x})\|})$. Then

$$\bar{x} \in P_A(\exp_{\bar{x}}(-rV(\bar{x}))) \implies \bar{x} \text{ is a solution of (3.1)}. \tag{3.15}$$

Proof. Set $x = \exp_{\bar{x}}(-rV(\bar{x}))$ and suppose that $\bar{x} \in P_A x$. Since $d(x, \bar{x}) \leq \|rV(\bar{x})\| < r(\bar{x})$, it follows that the minimal geodesic $\gamma_{x\bar{x}} \in \Gamma_{x,\bar{x}}$ is unique and satisfies that $\dot{\gamma}_{x\bar{x}}(1) = rV(\bar{x})$. By Theorem 3.1, (3.7) holds; hence

$$\langle rV(\bar{x}), \dot{\gamma}_{\bar{x}z}(0) \rangle \geq 0 \quad \text{for each } z \in A \text{ and } \gamma_{\bar{x}z} \in \Gamma_{\bar{x},z}^A.$$

By (3.2), \bar{x} is a solution of the problem (3.1) and the proof is complete. \square

Similar to the proof of Corollary 3.1, one has the following corollary.

Corollary 3.2. Let $A \subset M$ be a nonempty compact set. Let V be a bounded vector field on A . Then there exists $\hat{r}_A > 0$ such that (3.15) holds for each $r \in (0, \hat{r}_A]$ and each $\bar{x} \in A$.

Theorem 3.3. Let $A \subset M$ be a nonempty weakly convex set and let $\bar{x} \in A$. Let V be a vector field on A . Then there exists $\bar{r}_{\bar{x}} > 0$ such that for each $r \in (0, \bar{r}_{\bar{x}})$,

$$\bar{x} \in P_A(\exp_{\bar{x}}(-rV(\bar{x}))) \iff \bar{x} \text{ is a solution of (3.1)}. \tag{3.16}$$

Proof. Let $\hat{r}_{\bar{x}}$ be given in Theorem 3.2 and set

$$\bar{r}_{\bar{x}} = \frac{\min\{r(\bar{x}), \hat{r}_{\bar{x}}\}}{\|V(\bar{x})\|}.$$

Let $r \in (0, \bar{r}_{\bar{x}})$. Then

$$\|rV(\bar{x})\| < \min\{r(\bar{x}), \hat{r}_{\bar{x}}\}.$$

Write $x = \exp_{\bar{x}}(-rV(\bar{x}))$. Then $d(x, \bar{x}) < \min\{r(\bar{x}), \hat{r}_{\bar{x}}\}$. Thus, the unique minimal geodesic $\gamma_{x\bar{x}} \in \Gamma_{x,\bar{x}}$ satisfies that $\dot{\gamma}_{x\bar{x}}(1) = rV(\bar{x})$ and Theorem 3.2 is applicable. Consequently,

$$\bar{x} \in P_A(\exp_{\bar{x}}(-rV(\bar{x}))) \iff \langle rV(\bar{x}), \dot{\gamma}_{\bar{x}z}(0) \rangle \geq 0 \quad \text{for each } z \in A \text{ and } \gamma_{\bar{x}z} \in \Gamma_{\bar{x},z}^A.$$

From (3.2), it follows that (3.16) holds. \square

Similar to the proof for [Theorem 3.3](#) but using [Corollary 3.1](#) instead of [Theorem 3.2](#), we have the following corollary.

Corollary 3.3. *Let $A \subset M$ be a nonempty compact, weakly convex set and let V be a bounded vector field on A . Then there exists $\bar{r}_A > 0$ such that (3.16) holds for each $r \in (0, \bar{r}_A]$ and each $\bar{x} \in A$.*

Proof. Set $r(A) = \min_{y \in A} r(y)$ and $b_V = \sup_{y \in A} \|V(y)\|$. Define

$$\bar{r}_A = \frac{\min\{r(A), \hat{r}_A\}}{b_V},$$

where \hat{r}_A is given in [Theorem 3.2](#). The remainder of the proof is almost the same as that for [Theorem 3.3](#) and so omitted here. \square

Remark 3.1. In the case when M is a Hadamard manifold, $r, \hat{r}_{\bar{x}}$ and $\bar{r}_{\bar{x}}$ appeared respectively in [Proposition 3.3](#), [Theorems 3.2](#) and [3.3](#) can be taken to be $+\infty$.

4. Existence and uniqueness results

As in the previous sections, we assume that $A \subseteq M$ is a nonempty closed subset. This section is devoted to the study of the existence of solutions of the variational inequality problem (3.1). We begin with the following simple example which shows that, even in the case when A is totally convex and V is continuous, the problem (3.1) does not have a solution, in general.

Example 4.1. Consider the 1-dimensional unit sphere $M = S^1 := \{(x_1, x_2) : x_1^2 + x_2^2 = 1, x_1, x_2 \in \mathbb{R}\}$ endowed with the standard metric. Let $A = M = S^1$. Then A is totally convex, compact subset of M . Define the vector field V by

$$V(x) = (-x_2, x_1) \quad \text{for each } x = (x_1, x_2) \in S^1.$$

Clearly V is a continuous vector field on A . By definition, it is easy to verify that the problem (3.1) has no solution.

To study the existence problem of the solutions of the problem (3.1), we recall from [9, p.110] (see also [19, Remark V.4.2]) that a point $o \in M$ is called a *pole* of M if $\exp_o : T_oM \rightarrow M$ is a global diffeomorphism, which implies that for each point $x \in M$, the geodesic of M from o to x is unique. Motivated by this notion, we introduce the notion of the weak pole of A .

Definition 4.1. A point $o \in A$ is called a *weak pole* of A if for each $x \in A$, the minimal geodesic of M from o to x is unique and lies in A .

Note that if A has a weak pole, then A is connected. For the remainder of this paper we always assume that A has a weak pole, which is denoted by o . Let $x \in A$. we use $\tilde{\gamma}_{ox}$ to denote the unique geodesic from o to x . We set

$$\tilde{A} = \{\dot{\tilde{\gamma}}_{ox}(0) \in T_oM : x \in A\}.$$

Then, \exp_o is a continuous bijection from \tilde{A} to A .

Proposition 4.1. *Suppose that A is compact. Then \tilde{A} is compact in T_oM and \exp_o is a homeomorphism from \tilde{A} to A .*

Proof. It is easy to verify that \tilde{A} is bounded in T_oM . Thus it suffices to show that \tilde{A} is closed in T_oM . For this purpose, let $\{u_n\} \subseteq \tilde{A}$ and $u_0 \in T_oM$ be such that $u_n \rightarrow u_0$. Write $x_n = \exp_o u_n$ for each n . Then $\{x_n\} \subseteq A$ and $x_n \rightarrow x_0 := \exp_o u_0$. It follows from the compactness of A that $x_0 \in A$ and so $u_0 \in \tilde{A}$. The proof is complete. \square

The following proposition, taken from [19, pp.169-171], will play a key role in our study. Recall from [17,19,18] that a submanifold N of M is totally geodesic if for any curve c in N , c is a geodesic in N if and only if it is also a geodesic in M . Thus if N is totally geodesic and $x \in N$, \exp_x is a local diffeomorphism from T_xN to N at 0 . For a subset C of M , define

$$\epsilon(p) = \sup\{r > 0 : C \cap \mathbf{B}(p, r) \text{ is strongly convex}\} \quad \text{for each } p \in C.$$

By definition, if C is locally convex, then $\epsilon(p) > 0$ for each $p \in C$.

Proposition 4.2. *Let $C \subset M$ be a connected, locally convex closed set. Then there exists a connected (embedded) k -dimensional totally geodesic submanifold N of M such that $C = \bar{N}$ and $\gamma_{xy}([0, 1]) \subset N$ for any $p \in C, x \in \mathbf{B}(p, \epsilon(p)) \cap N, y \in \mathbf{B}(p, \epsilon(p)) \cap C$ and $\gamma_{xy} \in \Gamma_{x,y}$. Further, if $y \notin N$, then $\gamma_{xy}(t) \notin C$ for each $t \in (1, t_0]$, where $t_0 > 1$ is such that $\gamma_{xy}([0, t_0]) \subset \mathbf{B}(p, \epsilon(p))$.*

Following [19], N and $C \setminus N$ are called the interior and the boundary of C , respectively, which are denoted, respectively, by $\text{int } C$ and ∂C .

Proposition 4.3. *Let $A \subset M$ be a closed, locally convex set. Then, for each $x \in A, \tilde{\gamma}_{ox}([0, 1]) \subseteq \text{int } A$.*

Proof. Suppose there are some points in $\bar{\gamma}_{ox}([0, 1])$ which are in ∂A . Let $p \in \bar{\gamma}_{ox}([0, 1])$ be such a point that is the nearest one from o . Let $s_0 \in (0, 1)$ be such that $\bar{\gamma}_{ox}(s_0) = p$. Then, $\mathbf{B}_{\epsilon(p)}(p) \cap \bar{\gamma}_{ox}([0, s_0]) \subset \mathbf{B}_{\epsilon(p)}(p) \cap \text{int } A$. Take $z \in \mathbf{B}_{\epsilon(p)}(p) \cap \bar{\gamma}_{ox}([0, s_0])$ such that $d(z, p) < \epsilon(p)$ and $y = p$. Assume that $z = \bar{\gamma}_{ox}(s_1)$ for some $s_1 \in [0, s_0]$. Then define $\gamma_{zy} \in \Gamma_{z,y}$ by

$$\gamma_{zy}(t) = \bar{\gamma}_{ox}((1-t)s_1 + ts_0) \quad \text{for each } t \in [0, 1].$$

Then there exists $t_0 > 1$ such that $\gamma_{zy}([0, t_0]) \subset \mathbf{B}_{\epsilon(p)}(p)$. Since $y \notin \text{int } A$, it follows from Proposition 4.2 that $\gamma_{zy}(t) \notin A$ for each $t \in (1, t_0]$. This contradicts that $\bar{\gamma}_{ox} \subset A$ and the proof completes. \square

For simplicity, we write N for $\text{int } A$, that is, $N = \text{int } A$. Let $x \in A$. Set

$$C_{(x)} = \{v \in T_x M \setminus \{0\} : \exp_x tv / \|v\| \in N \text{ for some } t \in (0, \epsilon(x))\} \cup \{0\}$$

and let $\widehat{C}_{(x)}$ denote the subspace of $T_x M$ spanned by $C_{(x)}$. Then, by [19, p. 171],

$$x \in N \iff C_{(x)} = \widehat{C}_{(x)} = T_x N. \tag{4.1}$$

Moreover, since N is totally geodesic, the mapping \exp_x is a local diffeomorphism at 0 from $T_x N$ to N if $x \in N$ (cf. [19,18]). This means that

$$x \in \text{int } A \iff \exp_x(\mathbf{B}_{T_x N}(0, \delta)) \subseteq \text{int } A \text{ for some } \delta > 0. \tag{4.2}$$

Consequently, if A is closed, ∂A is also closed in M .

For the following key result, we set $E := T_o N$. Let $\text{int } \tilde{A}$ and $\partial \tilde{A}$ denote the interior and the boundary of \tilde{A} in the space E , respectively.

Proposition 4.4. *Let $A \subset M$ be a closed, locally convex set and suppose that the weak pole $o \in \text{int } A$. Then,*

$$\exp_o^{-1}(\text{int } A) \subseteq \text{int } \tilde{A} \subseteq E, \tag{4.3}$$

Proof. Since $o \in \text{int } A$ is a weak pole of A , it is easy to verify by definition that $C_{(o)} = \text{cone } \tilde{A}$ and it follows from (4.1) and (4.2) that

$$\text{cone } \tilde{A} = C_{(o)} = E \quad \text{and} \quad 0 \in \text{int } \tilde{A}. \tag{4.4}$$

Let $x \in N$ and set $l_o(A) = \exp_o E$. Then

$$\epsilon := d(x, l_o(A) \setminus A) > 0. \tag{4.5}$$

In fact, otherwise, there exists a sequence $\{x_n\} \subseteq l_o(A) \setminus A$ such that $\lim_n d(x_n, x) = 0$. By Proposition 4.3, we may assume that, for each n , $x_n = \exp_o t_n u_n$ and $d(x_n, o) = \|t_n u_n\|$, where $t_n \geq 1$ and $u_n \in E$ such that $\exp_o u_n \in \partial A$. Hence $\{t_n u_n\}$ is bounded and so $\{u_n\}$ is bounded too because each $t_n \geq 1$. Since ∂A is closed, it follows that $\liminf_n \|u_n\| > 0$. This together with the boundedness of $\{t_n u_n\}$ implies that $\{t_n\}$ is bounded. Consequently, without loss of generality, we may assume that $t_n \rightarrow t_0$ and $u_n \rightarrow u_0$. Then $t_0 \geq 1$ and $\exp_o u_0 \in \partial A$. Hence $\exp_o t_0 u_0 \notin N$, which is a contradiction because $\exp_o(t_0 u_0) = x \in N$. Therefore, (4.5) is proved.

Since \exp_o is continuous on E , there exists $\delta > 0$ such that

$$d(\exp_o v, x) = d(\exp_o v, \exp_o u) < \frac{\epsilon}{2} \quad \text{for each } v \in \mathbf{B}_E(u, \delta).$$

It follows from (4.5) that $\exp_o(\mathbf{B}_E(u, \delta)) \subseteq A$ and so $\mathbf{B}_E(u, \delta) \subseteq \tilde{A}$. This implies that $\exp_o x = u \in \text{int } \tilde{A}$ and the proof is complete. \square

Recall that a topological space has the fixed point property if each continuous map between itself has a fixed point.

Theorem 4.1. *Let $A \subset M$ be a compact, and locally convex subset. Suppose that A has a weak pole $o \in \text{int } A$. Then A has the fixed point property.*

Proof. By Proposition 4.1, A is homeomorphic to \tilde{A} . Since the fixed point property is a topological property, it suffices to verify that \tilde{A} has the fixed point property. To do this, we shall show that $\tilde{A} \subset E (= T_o N)$ is homeomorphic to the closed unit ball \mathbf{B}_E of E . Granting this, the Brouwer fixed point Theorem is applicable to completing the proof.

Define the function $\phi : \tilde{A} \rightarrow \mathbb{R}^+$ by

$$\phi(u) = \begin{cases} \|\vec{0u} \cap \partial \tilde{A}\| & \text{if } u \neq 0, \\ 1 & \text{if } u = 0, \end{cases}$$

where $\vec{0u} := \{tu : t \geq 0\}$ is the ray-line from 0 through u . By Proposition 4.3, $\exp_o([0, u]) = \bar{\gamma}_{ox}([0, 1]) \subseteq N$ if $u \in \tilde{A}$, where $x = \exp_o u$. Hence Proposition 4.4 implies that

$$[0, u) \subseteq \text{int } \tilde{A} \quad \text{for each } u \in \tilde{A}, \tag{4.6}$$

where $[0, u) := \{tu : t \in [0, 1)\}$. This implies that $\vec{0u} \cap \partial\tilde{A}$ is a singleton and so ϕ is well-defined. Below we will show that ϕ is continuous at each point $u \neq 0$. Indeed, let $u_0 \in \tilde{A} \setminus \{0\}$ and denote $\bar{u} = \vec{0u} \cap \partial\tilde{A}$ for any $u \neq 0$. Then $\bar{u} = \frac{u}{\|u\|} \phi(u)$. Let $\{u_n\} \subset \tilde{A}$ be such that $u_n \rightarrow u_0$. Since ϕ is bounded, we can assume without loss of generality that $\phi(u_n) \rightarrow r \in \mathbb{R}^+$. Then,

$$\bar{u}_n \rightarrow \bar{v}_0 := \frac{u_0}{\|u_0\|} r \in \partial\tilde{A}.$$

We must prove that $r = \phi(u_0)$ to get the continuity of ϕ . To this end, write

$$\bar{v}_0 = \frac{u_0}{\|u_0\|} \phi(u_0) \frac{r}{\phi(u_0)} = \frac{r}{\phi(u_0)} \bar{u}_0. \tag{4.7}$$

If $r < \phi(u_0)$, then $\frac{r}{\phi(u_0)} < 1$. Thus, (4.6) and (4.7) imply that $\bar{v}_0 \in \text{int } \tilde{A}$ because $\bar{u}_0 \in \partial\tilde{A}$, which contradicts that $\bar{v}_0 \in \partial\tilde{A}$. Similarly, if $r > \phi(u_0)$, we get that $\bar{v}_0 \notin \tilde{A}$, which again contradicts that $\bar{v}_0 \in \partial\tilde{A}$. Therefore ϕ is continuous at each $u \neq 0$.

Now define the function $h : \tilde{A} \rightarrow \mathbf{B}_E$ by

$$h(u) = \frac{1}{\phi(u)} u, \quad \text{for each } u \in \tilde{A}.$$

Note that $\|h(u)\| = \frac{1}{\phi(u)} \|u\| \leq 1$. Thus h is well-defined and continuous, whose inverse function $h^{-1}(v) = \phi(v)v$ is continuous too. Hence, h is a homeomorphism from \tilde{A} to \mathbf{B}_E , completing the proof. \square

Theorem 4.2. *Let $A \subset M$ be a compact, and locally convex subset. Suppose that A has a weak pole $o \in \text{int } A$. Let V be a continuous vector field on A . Then the variational inequality problem (3.1) admits at least one solution \bar{x} .*

Proof. By Proposition 3.2, there exists open neighborhood U of A such that, P_A is single-valued and Lipschitz continuous on U . Since A is compact, there is $r_1 > 0$ such that

$$\{x \in M : d(x, A) < r_1\} \subseteq U. \tag{4.8}$$

Let $\hat{r}_A > 0$ be given by Corollary 3.2. Then $r := \min\{\hat{r}_A, r_1\} > 0$. Since V is continuous (and so bounded), it follows from Corollary 3.2 that, for each $\bar{x} \in A$,

$$\bar{x} \in P_A(\exp_{\bar{x}}(-rV(\bar{x}))) \implies \bar{x} \text{ is a solution of (3.1)}. \tag{4.9}$$

Define the map $f : A \rightarrow A$ by

$$f(x) = P_A(\exp_x(-rV(x))) \quad \text{for each } x \in A.$$

Then, by (4.8) and Proposition 3.2, f is well-defined and continuous. Thus Theorem 4.1 is applicable and f has a fixed point $\bar{x} \in A$, that is, $\bar{x} = P_A(\exp_{\bar{x}}(-rV(\bar{x})))$. This together with (4.9) implies that \bar{x} is a solution of the problem (3.1) and the proof completes. \square

Note that if M is a Hadamard Manifold, then each point of A is a pole. Hence the following corollary, which was proved in [6], now is a direct consequence of Theorem 4.2.

Corollary 4.1. *Let $A \subset M$ be a compact, weakly convex set. Suppose that M is a Hadamard manifold and V is a continuous vector field on A . Then the variational inequality problem (3.1) admits at least one solution \bar{x} .*

We now give an example to illustrate the application of Theorem 4.2.

Example 4.2. Let $n \geq 3$ and let $M = S^n$ be the n -dimensional unit sphere, that is,

$$S^n := \left\{ (x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1 \right\},$$

endowed with the standard metric. For any two points x and y in S^n , all smooth curves connecting x and y and containing in the great circle through x and y are the geodesics, and the tangent spaces are

$$T_x M = \{v \in \mathbb{R}^{n+1} : \langle v, x \rangle = 0\} \quad \text{for each } x \in M.$$

Let $A = \{(x_1, x_2, \dots, x_{n+1}) \in M : x_1 \geq 0\}$. Then A is a compact, locally convex subset of M with weak pole $o = (1, 0, \dots, 0) \in \text{int } A$. Define

$$V(x) = (-x_2, x_1, -x_4, x_3, 0, \dots, 0) \quad \text{for each } x = (x_1, x_2, \dots, x_{n+1}) \in A.$$

Then $V(x)$ is a continuous vector field on A . Hence by Theorem 4.2, the variational inequality problem (3.1) admits at least one solution \bar{x} .

The remainder of the present paper is devoted to the study of the existence and uniqueness of solutions of the variational inequality problem (3.1) for the case when A is not necessarily bounded. Let $R > 0, o \in A$ and write $A_R := A \cap \mathbf{B}(o, R)$. Consider the following variational inequality problem: find $x_R \in A_R$ such that

$$\langle V(x_R), \dot{\gamma}_{x_R y}(0) \rangle \geq 0 \quad \text{for each } y \in A_R \text{ and each } \gamma_{x_R y} \in \Gamma_{x_R, y}^{A_R}. \tag{4.10}$$

The following theorem establishes the relationship between the problems (4.10) and (3.1), which is an extension of the corresponding result in [6] from Hadamard manifolds to Riemannian manifolds.

Theorem 4.3. *Let $A \subset M$ be a closed set and V be a vector field on A . Then the problem (3.1) admits a solution if and only if there exist $R > 0$ and a solution $x_R \in A_R$ of problem (4.10) such that*

$$d(o, x_R) < R, \tag{4.11}$$

for some point $o \in A$.

Proof. Suppose that \bar{x} is a solution to problem (3.1). Then, for each $R > d(o, \bar{x})$ and $o \in A, x_R := \bar{x} \in A_R$ is a solution of (4.10) satisfying (4.11).

Conversely, suppose there exist $o \in A$ and $R > 0$ such that the problem (4.10) has a solution $x_R \in A_R$ satisfying (4.11). Below we will show that $\bar{x} := x_R$ is a solution of the problem (3.1). In fact, let $y \in A$ and $\gamma_{\bar{x}y} \in \Gamma_{\bar{x}, y}^A$. Then there exists $\delta > 0$ such that

$$\|\delta \dot{\gamma}_{\bar{x}y}(0)\| < \min\{r(\bar{x}), R - d(o, x_R)\},$$

where $r(\bar{x})$ is the convexity radius at \bar{x} . Set $z = \exp_{\bar{x}}(\delta \dot{\gamma}_{\bar{x}y}(0))$. Then $z \in A_R$ and there exists a unique geodesic $\gamma_{\bar{x}, z} \in \Gamma_{\bar{x}, z}$, which is contained in A_R and satisfies that $\dot{\gamma}_{\bar{x}z}(0) = \delta \dot{\gamma}_{\bar{x}y}(0)$. Since $\bar{x} = x_R$ is a solution of the problem (4.10), it follows that

$$\langle V(\bar{x}), \delta \dot{\gamma}_{\bar{x}y}(0) \rangle = \langle V(x_R), \delta \dot{\gamma}_{x_R y}(0) \rangle \geq 0. \tag{4.12}$$

Consequently, $\langle V(\bar{x}), \dot{\gamma}_{\bar{x}y}(0) \rangle \geq 0$ and \bar{x} is a solution of the problem (3.1) because $y \in A$ and $\gamma_{\bar{x}y} \in \Gamma_{\bar{x}, y}^A$ are arbitrary. \square

For the following existence result, we need to introduce the coerciveness condition. Recall that $\bar{\gamma}_{ox}$ denotes the unique minimal geodesic from o to x when o is a weak pole of A and $x \in A$.

Definition 4.2. Let $A \subset M$ be subset with a weak pole $o \in A$. Let V be a vector field on A . V is said to satisfy the coerciveness condition if

$$\frac{\langle V(x), \dot{\gamma}_{ox}(1) \rangle - \langle V(o), \dot{\gamma}_{ox}(0) \rangle}{d(o, x)} \rightarrow -\infty \quad \text{as } d(o, x) \rightarrow +\infty \text{ for } x \in A. \tag{4.13}$$

Theorem 4.4. *Let $A \subset M$ be a closed locally convex set with a weak pole $o \in \text{int } A$ and let V be a continuous vector field on A which satisfies the coerciveness condition. Suppose that, for any $R > 0$, there exists a locally convex compact subset of M containing $\mathbf{B}(o, R)$. Then the variational inequality problem (3.1) admits at least a solution.*

Proof. Taking $H > \|V(o)\|$, by the assumed coerciveness condition, there is $R > 0$ such that

$$\langle V(x), \dot{\gamma}_{ox}(1) \rangle - \langle V(o), \dot{\gamma}_{ox}(0) \rangle \leq -Hd(o, x) \quad \text{for each } x \in A \text{ with } d(o, x) \geq R. \tag{4.14}$$

Hence, for each $x \in A$ with $d(o, x) \geq R$, one has that

$$\langle V(x), \dot{\gamma}_{ox}(1) \rangle \leq -Hd(o, x) + \|V(o)\| \cdot \|\dot{\gamma}_{ox}(0)\| = (\|V(o)\| - H)d(o, x) < 0. \tag{4.15}$$

By assumptions, there exists a locally convex compact subset K_R of M containing $\mathbf{B}(o, R)$. Then $\hat{A}_R := A \cap K_R \subseteq M$ is a locally convex compact set. Moreover, it is easy to see that $o \in \text{int } \hat{A}_R$ is a weak pole of \hat{A}_R . Thus Theorem 4.2 is applied to \hat{A}_R in place of A to get that there exists $x_R \in \hat{A}_R$ such that

$$\langle V(x_R), \dot{\gamma}_{x_R y}(0) \rangle \geq 0 \quad \text{for each } y \in \hat{A}_R \text{ and each } \gamma_{x_R y} \in \Gamma_{x_R, y}^{\hat{A}_R}.$$

Since $A_R \subseteq \hat{A}_R$, it follows that x_R is also a solution of the problem (4.10). Furthermore, by (4.15), we have that $d(o, x_R) < R$. It follows from Theorem 4.3 that $\bar{x} := x_R$ is a solution of the problem (3.1) and the proof is complete. \square

The following proposition provides two family of manifolds where for each $R > 0$ there exists a locally convex compact subset of M containing $\mathbf{B}(o, R)$, one of which is the family of Hadamard manifolds.

Proposition 4.5. *Suppose that the sectional curvature of M is nonnegative everywhere or nonpositive everywhere. Let $o \in M$. Then for each $R > 0$ there exists a totally convex compact subset of M containing $\mathbf{B}(o, R)$.*

Proof. If the sectional curvature of M is nonpositive everywhere, that is, M is a Hadamard manifold, then the function $x \mapsto d(o, x)$ is convex on M (cf. [19, P.222, Proposition 4.3]). Hence each closed metric ball $\overline{\mathbf{B}(o, R)}$ is convex and compact, and the conclusion holds. Now suppose that the sectional curvature of M is nonnegative everywhere. Without loss of generality, we may assume that M is noncompact. Let $\gamma_o : [0, +\infty) \rightarrow M$ be a ray through o , that is, $\gamma_o(0) = o$ and $d(\gamma_o(t_1), \gamma_o(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \geq 0$. Let F_{γ_o} be the Busemann function for the ray γ_o which is defined by

$$F_{\gamma_o}(x) := \lim_{t \rightarrow +\infty} (t - d(x, \gamma_o(t))) \quad \text{for each } x \in M.$$

Define

$$F(x) := \sup_{\gamma_o} F_{\gamma_o}(x) \quad \text{for each } x \in M,$$

where the supremum is taken over all rays through o . It is known in [19, P.212, Proposition 3.1] that F is convex on M . Furthermore, $F^{-1}(R)$ is compact and contains $\mathbf{B}(o, R)$. This shows the conclusion and completes the proof. \square

By Proposition 4.5 and Theorem 4.4, the following corollary is immediate.

Corollary 4.2. *Let $A \subset M$ be a closed locally convex set with a weak pole $o \in \text{int } A$ and let V be a continuous vector field on A which satisfies the coerciveness condition. Suppose that the sectional curvature of M is nonnegative everywhere or nonpositive everywhere. Then the variational inequality problem (3.1) admits at least one solution.*

The following example shows an application of Corollary 4.2.

Example 4.3. Let $M = \{(x_1, x_2, z) : z = x_1^2 + x_2^2, x_1, x_2, z \in \mathbb{R}\}$ be the paraboloid in \mathbb{R}^3 and endowed with the standard metric, see for example [9, P.111]. The sectional curvature of M is positive everywhere and the point $o = (0, 0, 0)$ is a pole. The geodesics through o are exactly the intersections of M with the planes containing the z -axis. Let $A := \{(x_1, x_2, z) : x_1 = 0, z = x_2^2\}$. Then A is a geodesic of M through o and so is closed locally convex. Clearly A is unbound and $o \in \text{int } A$ is a weak pole of A . Note that the tangent space of M at $x \in M$ is

$$T_x M = \{v \in \mathbb{R}^3 : \langle v, u \rangle = 0 \text{ with } u = (2x_1, 2x_2, -1)\} \quad \text{for each } x = (x_1, x_2, z) \in M.$$

Define

$$V(x) = (0, -x_2^4, -2x_2^5) \quad \text{for each } x = (x_1, x_2, z) \in A. \tag{4.16}$$

Then, $V(x)$ is a continuous vector field on A . Let $x = (x_1, x_2, z) \in A$ and let $\bar{\gamma}_{ox} \in \Gamma_{ox}$ be the geodesic. Then, $\dot{\bar{\gamma}}_{ox}(0) = (0, 1, 0)$, $\dot{\bar{\gamma}}_{ox}(1) = (0, 1, 2x_2)$ and

$$d(o, x) = \frac{1}{4} \left(|2x_2| \sqrt{4x_2^2 + 1} + \ln \left(|2x_2| + \sqrt{4x_2^2 + 1} \right) \right).$$

By (4.16),

$$\frac{\langle V(x), \dot{\bar{\gamma}}_{ox}(1) \rangle - \langle V(o), \dot{\bar{\gamma}}_{ox}(0) \rangle}{d(o, x)} = \frac{-16x_2^6 - 4x_2^2}{|2x_2| \sqrt{4x_2^2 + 1} + \ln(|2x_2| + \sqrt{4x_2^2 + 1})};$$

hence

$$\frac{\langle V(x), \dot{\bar{\gamma}}_{ox}(1) \rangle - \langle V(o), \dot{\bar{\gamma}}_{ox}(0) \rangle}{d(o, x)} \rightarrow -\infty \quad \text{as } d(o, x) \rightarrow +\infty.$$

This shows that V satisfies the coerciveness condition and so Corollary 4.2 is applicable to concluding that the variational inequality problem (3.1) admits at least one solution.

The notions of monotonicity in the following definition were given and studied extensively in [23–31,33], which are extensions of the corresponding ones in Hilbert spaces.

Definition 4.3. Let $A \subset M$ be a weakly convex set. A vector field V on A is said to be

(1) monotone if

$$\langle V(x), \dot{\gamma}_{xy}(0) \rangle - \langle V(y), \dot{\gamma}_{xy}(1) \rangle \leq 0 \quad \text{for all } x, y \in A \text{ and each } \gamma_{xy} \in \Gamma_{x,y}^A; \quad (4.17)$$

(2) strictly monotone if

$$\langle V(x), \dot{\gamma}_{xy}(0) \rangle - \langle V(y), \dot{\gamma}_{xy}(1) \rangle < 0 \quad \text{for all } x, y \in A \text{ and each } \gamma_{xy} \in \Gamma_{x,y}^A; \quad (4.18)$$

(3) strongly monotone if there exists $\alpha > 0$ such that

$$\langle V(x), \dot{\gamma}_{xy}(0) \rangle - \langle V(y), \dot{\gamma}_{xy}(1) \rangle \leq -\alpha d^2(x, y) \quad \text{for all } x, y \in A \text{ and each } \gamma_{xy} \in \Gamma_{x,y}^A. \quad (4.19)$$

Now we have the following uniqueness result on the solution of the problem (3.1).

Theorem 4.5. Let $A \subset M$ be a weakly convex set. Suppose that V is strictly monotone vector field on A . Then the variational inequality problem (3.1) admits at most one solution.

Proof. Suppose on the contrary that the problem (3.1) admits two distinct solutions \bar{x} and \bar{y} . Since A is weakly convex, there exists a minimal geodesic $\gamma_{\bar{x}\bar{y}} \in \Gamma_{\bar{x},\bar{y}}^A$. Then

$$\langle V(\bar{x}), \dot{\gamma}_{\bar{x}\bar{y}}(0) \rangle \geq 0 \quad \text{and} \quad \langle V(\bar{y}), -\dot{\gamma}_{\bar{x}\bar{y}}(1) \rangle \geq 0. \quad (4.20)$$

It follows that

$$\langle V(\bar{x}), \dot{\gamma}_{\bar{x}\bar{y}}(0) \rangle - \langle V(\bar{y}), \dot{\gamma}_{\bar{x}\bar{y}}(1) \rangle \geq 0.$$

This contradicts (4.19) and the proof completes. \square

It is a routine to verify that if V is strongly monotone, then it satisfies the coerciveness condition. Therefore, the following corollary is straightforward.

Corollary 4.3. Let $A \subset M$ be a closed weakly convex set with a weak pole $o \in \text{int } A$ and let V be a continuous, strongly monotone vector field on A . Suppose that the sectional curvature of M is nonnegative everywhere or nonpositive everywhere. Then the variational inequality problem (3.1) admits a unique solution.

Acknowledgement

We thank Dr. S. Z. Németh for his helpful comments.

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