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Monotone vector fields and the proximal point algorithm on Hadamard manifolds

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Abstract

The maximal monotonicity notion in Banach spaces is extended to Riemannian manifolds of nonpositive sectional curvature, Hadamard manifolds, and proved to be equivalent to the upper semicontinuity. We consider the problem of finding a singularity of a multivalued vector field in a Hadamard manifold and present a general proximal point method to solve that problem, which extends the known proximal point algorithm in Euclidean spaces. We prove that the sequence generated by our method is well defined and converges to a singularity of a maximal monotone vector field, whenever it exists. Applications in minimization problems with constraints, minimax problems and variational inequality problems, within the framework of Hadamard manifolds, are presented.

1. *Introduction*

A valuable tool in the study of gradient and subdifferential mappings and other mappings that appear in many problems, such as optimization, equilibrium or in variational inequality problems, is the concept of monotonicity (see [**38**]). Given a Banach space E with dual space E^*, recall that a multivalued mapping $A : E \to 2^{E^*}$ is said to be a monotone operator provided that

$$(x^* - y^*, x - y) \geqslant 0 \quad \forall x, y \in \mathcal{D}(A) \text{ and } x^* \in A(x), \;\; y^* \in A(y),$$

where $\mathcal{D}(A)$ denotes the domain of A defined by $\mathcal{D}(A) := \{x \in E : A(x) \neq \emptyset\}$.

When dealing with monotone operators there are basically two problems: to give conditions for the existence of zeros that is, solutions of the inclusion $0 \in A(x)$, and to design algorithms for approximating zeros, whenever those exist.

Regarding the problem of the existence of zeros, it is essential to consider the concept of maximal monotone operators and its relationship with the notion of upper-semicontinuity (see [**4**]).

In the case of Hilbert spaces, the problem of finding zeros of monotone operators has been investigated by many authors, for example, Bruck, Brezis, Lions, Reich, Martinet, Rockafellar, Kamimura, Takahashi and others (cf. [**3**, **6**, **7**, **16**, **17**, **32**]). Bearing in mind the fact that zeros of a maximal monotone operator are fixed points of its resolvent, which is a nonexpansive mapping, Rockafellar, in a seminal work [**32**], defined the proximal point algorithm for monotone operators by means of the following iterative scheme:

$$0 \in A(x_{n+1}) + \lambda_n (x_{n+1} - x_n), \quad n = 0, 1, 2, \ldots,$$

where $\{\lambda_n\}$ is a sequence of real positive numbers and x_0 is an initial point. The study of the convergence problem of this algorithm has been very fruitful and has engaged researchers from

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different areas, such as variational inequalities, optimization and metric fixed-point theory. The key tool is the corresponding resolvent.

The extension to convex abstract spaces and Riemannian manifolds of the concepts and techniques that fit in Euclidean spaces is natural and nontrivial. Actually, in recent years some algorithms defined to solve nonlinear equations, variational inequalities and minimization problems, which involve monotone operators, have been extended from the Hilbert space framework to the more general setting of Riemannian manifolds and the Hilbert unit ball (see, for example, [**9**, **11**, **12**, **15**, **23**, **28**, **30**, **36**]). In particular, the notion of monotone vector fields on Hadamard manifolds was introduced by Németh [**20**] and extended by Da Cruz Neto, Ferreira and Lucambio Pérez [**8**] to the case of multivalued vector fields. One of the most important examples of monotone vector fields is the gradient of a convex function on manifolds (cf. [**9**, **20**]).

The purpose of this paper is to study maximal monotone multivalued vector fields as well as the proximal point algorithm on Hadamard manifolds. In this case, unlike the Hilbert space setting, the resolvent is not known to be well defined and this makes our task more complicated. In Section 2, we present some basic facts on Hadamard manifolds that are essential for our study. Section 3 contains the main result, where we prove that, as in the linear case, the class of maximal monotone vector fields coincides with the class of vector fields that are upper-semicontinuous and have a closed convex image. This is the main result that is used in proving all subsequent assertions.

Section 4 deals with the problem of approximating singularities (solutions of the inclusion $0 \in A(x)$) of a maximal monotone vector field A on a Hadamard manifold M. We follow the iteration scheme of Rockafellar on Hilbert spaces to define the following proximal point type algorithm.

Given $x_0 \in M$ and $\{\lambda_n\} \subset (0, 1)$, define x_{n+1} such that

$$0 \in A(x_{n+1}) - \lambda_n \exp^{-1}_{x_{n+1}} x_n, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where exp is the exponential map (see Section 2 for the details). This algorithm is a generalization of the proximal point algorithm on Euclidean spaces. In order to prove that the algorithm is well defined, we establish the existence result of singularities for strongly monotone vector fields. In the study of the asymptotic behaviour of the iterates $\{x_n\}$, the assumptions on the parameters ensuring their convergence are similar to the corresponding conditions in the Hilbert space setting.

The last section is devoted to different examples of how to apply our results to different problems. The first application corresponds to minimization problems. Given a proper convex lower semicontinuous function, we prove that its subdifferential is a monotone and upper Kuratowski semicontinuous vector field, and, as a consequence, we obtain results for the convergence of algorithms for constrained minimization problems. Let M_1 and M_2 be Hadamard manifolds and $L : M_1 \times M_2 \to [-\infty, +\infty]$ be a mapping that is concave on the first variable and convex on the second one. Following Rockafellar's ideas for the linear case in [**31**], we introduce a multivalued mapping $A_L : M_1 \times M_2 \to 2^{TM_1} \times 2^{TM_1}$, the singularities of which coincide with the global saddle points of L. We further verify that the vector field A_L is maximal monotone and then our results can be applied. Finally, an application to variational inequality problems is proved for vector fields on a closed convex subset of a Hadamard manifold M.

2. *Preliminaries*

In this section, we introduce some fundamental definitions, properties and notation of Riemannian manifolds, which can be found in any textbook on Riemannian geometry, for example, [**10**, **27**, **33**].

Let M be a complete and connected m-dimensional manifold. We assume that M can be endowed with a Riemannian metric $\langle\cdot,\cdot\rangle$, with the corresponding norm denoted by $\|\cdot\|$, to become a *Riemannian* manifold. Let $p \in M$. The tangent space of M at p is denoted by T_pM and the tangent bundle of M by $TM = \bigcup_{p\in M} T_pM$, which is naturally a manifold. Given a piecewise smooth curve $\gamma : [a, b] \to M$ joining p to q (that is, $\gamma(a) = p$ and $\gamma(b) = q$), we can define the length of γ by using the metric as $L(\gamma) = \int_a^b \|\gamma'(t)\|\, dt$. Then the Riemannian distance $\mathrm{d}(p, q)$ is defined by minimizing this length over the set of all such curves joining p to q, which induces the original topology on M.

Let ∇ be the Levi–Civita connection associated to $(M, \langle,\rangle)$. Let γ be a smooth curve in M. A vector field X is said to be parallel along γ if $\nabla_{\gamma'} X = 0$. If γ' itself is parallel along γ, then we say that γ is a *geodesic*, and in this case $\|c'\|$ is constant. When $\|c'\| = 1$, then c is called normalized. A geodesic joining p to q in M is said to be minimal if its length equals $\mathrm{d}(p, q)$. By the Hopf–Rinow theorem, we know that if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover, (M, d) is a complete metric space, and bounded closed subsets are compact. We use $P_{\gamma,\cdot,\cdot}$ to denote the parallel transport on the tangent bundle TM along γ with respect to ∇, which is defined by

$$P_{\gamma,\gamma(b),\gamma(a)}(v) = V(\gamma(b)) \quad \forall a, b \in \mathbb{R} \text{ and } v \in T_{\gamma(a)}M,$$

where V is the unique vector field satisfying $\nabla_{\gamma'(t)} V = 0$ for all t and $V(\gamma(a)) = v$. Then, for any $a, b \in \mathbb{R}$, the parallel transport $P_{\gamma,\gamma(b),\gamma(a)}$ is an isometry from $T_{\gamma(a)}M$ to $T_{\gamma(b)}M$. Note that, for any a, b, b_1, $b_2 \in \mathbb{R}$, we have

$$P_{\gamma,\gamma(b_2),\gamma(b_1)} \circ P_{\gamma,\gamma(b_1),\gamma(a)} = P_{\gamma,\gamma(b_2),\gamma(a)} \quad \text{and} \quad P^{-1}_{\gamma,\gamma(b),\gamma(a)} = P_{\gamma,\gamma(a),\gamma(b)}.$$

We will write $P_{q,p}$ instead of $P_{\gamma,q,p}$ in the case when γ is a minimal geodesic joining p to q and no confusion arises.

Assuming that M is complete, the *exponential map* $\exp_p : T_pM \to M$ at $p \in M$ is defined by $\exp_p v = \gamma_v(1, x)$, where $\gamma(\cdot) = \gamma_v(\cdot, p)$ is the geodesic starting at p with velocity v. Then, for any value of t, we have $\exp_p tv = \gamma_v(t, p)$. Note that the map $\exp_p$ is differentiable on T_pM for any $p \in M$.

A complete, simply connected Riemannian manifold of nonpositive sectional curvature is called a *Hadamard manifold*. Throughout the remainder of this paper, we will always assume that M is an m-dimensional Hadamard manifold. The following result is well known (see, for example, [**33**, Theorem 4.1, p. 221]).

Proposition 2.1. *Let $p \in M$. Then $\exp_p : T_pM \to M$ is a diffeomorphism, and for any two points $p, q \in M$ there exists a unique normalized geodesic joining p to q, which is, in fact, a minimal geodesic.*

This proposition shows that M is diffeomorphic to the Euclidean space $\mathbb{R}^m$. Thus, we see that M has the same topology and differential structure as $\mathbb{R}^m$. Moreover, Hadamard manifolds and Euclidean spaces have some similar geometrical properties. One of the most important properties is described in the following proposition, which is taken from [**33**, Proposition 4.5, p. 223] and will be useful in our study. Recall that a geodesic triangle $\Delta(p_1p_2p_3)$ of a Riemannian manifold is a set consisting of three points p_1, p_2 and p_3, and three minimal geodesics joining these points.

Proposition 2.2 (Comparison theorem for triangles). *Let $\Delta(p_1p_2p_3)$ be a geodesic triangle. Denote, for each $i = 1, 2, 3 \pmod 3$, by $\gamma_i : [0, l_i] \to M$ the geodesic joining p_i to*

p_{i+1}, *and set* $l_i := L(\gamma_i)$ *and* $\alpha_i := \angle(\gamma_i'(0), -\gamma_{i-1}'(l_{i-1}))$. *Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \leqslant \pi, \tag{2.1}$$

$$l_i^2 + l_{i+1}^2 - 2l_i l_{i+1} \cos\alpha_{i+1} \leqslant l_{i-1}^2. \tag{2.2}$$

In terms of the distance and the exponential map, the inequality (2.2) can be rewritten as

$$\mathrm{d}^2(p_i, p_{i+1}) + \mathrm{d}^2(p_{i+1}, p_{i+2}) - 2\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2}\rangle \leqslant \mathrm{d}^2(p_{i-1}, p_i) \tag{2.3}$$

since

$$\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2}\rangle = \mathrm{d}(p_i, p_{i+1})\mathrm{d}(p_{i+1}, p_{i+2}) \cos\alpha_{i+1}.$$

A subset $K \subseteq M$ is said to be convex if, for any two points p and q in K, the geodesic joining p to q is contained in K, that is, if $\gamma : [a, b] \to M$ is a geodesic such that $p = \gamma(a)$ and $q = \gamma(b)$, then $\gamma((1-t)a + tb) \in K$ for all $t \in [0, 1]$. From now on, K will denote a nonempty closed convex set in M, unless explicitly stated otherwise.

Let $f : M \to (-\infty, +\infty]$ be a proper extended real-valued function. The domain of the function f is denoted by $\mathcal{D}(f)$ and defined by $\mathcal{D}(f) := \{x \in M : f(x) \neq +\infty\}$. The function f is said to be convex if, for any geodesic γ in M, the composition function $f \circ \gamma : \mathbb{R} \to (-\infty, +\infty]$ is convex, that is,

$$(f \circ \gamma)(ta + (1-t)b) \leqslant t(f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b)$$

for any $a, b \in \mathbb{R}$ and $0 \leqslant t \leqslant 1$. The following proposition describes the convexity property of the distance function (cf. [**33**, Proposition 4.3, p. 222]).

PROPOSITION 2.3. *Let* $\mathrm{d} : M \times M \to \mathbb{R}$ *be the distance function. Then* $\mathrm{d}(\cdot, \cdot)$ *is a convex function with respect to the product Riemannian metric, that is, given any pair of geodesics* $\gamma_1 : [0, 1] \to M$ *and* $\gamma_2 : [0, 1] \to M$, *the following inequality holds for all* $t \in [0, 1]$:

$$\mathrm{d}(\gamma_1(t), \gamma_2(t)) \leqslant (1-t)\mathrm{d}(\gamma_1(0), \gamma_2(0)) + t\mathrm{d}(\gamma_1(1), \gamma_2(1)).$$

In particular, for each $p \in M$, *the function* $\mathrm{d}(\cdot, p) : M \to \mathbb{R}$ *is a convex function on* M.

Using the properties of the parallel transport and the exponential map, we obtain the following lemma that will be used frequently in the next sections.

LEMMA 2.4. *Let* $x_0 \in M$ *and* $\{x_n\} \subset M$ *be such that* $x_n \to x_0$. *Then the following assertions hold.*

(i) *For any* $y \in M$, *we have*

$$\exp_{x_n}^{-1} y \longrightarrow \exp_{x_0}^{-1} y \quad \textit{and} \quad \exp_y^{-1} x_n \longrightarrow \exp_y^{-1} x_0.$$

(ii) *If* $v_n \in T_{x_n}M$ *and* $v_n \to v_0$, *then* $v_0 \in T_{x_0}M$.

(iii) *Given* $u_n, v_n \in T_{x_n}M$ *and* $u_0, v_0 \in T_{x_0}M$, *if* $u_n \to u_0$ *and* $v_n \to v_0$, *then*

$$\langle u_n, v_n\rangle \longrightarrow \langle u_0, v_0\rangle.$$

(iv) *For any* $u \in T_{x_0}M$, *the function* $F : M \to TM$ *defined by* $F(x) = P_{x,x_0}u$ *for each* $x \in M$ *is continuous on* M.

Proof. Assertions (i)–(iii) are direct consequences of definitions. Below, we just prove assertion (iv). Let $\varphi := \exp_{x_0}^{-1} : M \to T_{x_0}M$. Since M is a Hadamard manifold, φ is a diffeomorphism from M onto $T_{x_0}M$. Let $x \in X$ and $u \in T_{x_0}M$. Then the geodesic joining x_0

to x can be expressed as $\gamma_x(t) = \exp_{x_0} t\varphi(x)$ for each $t \in [0,1]$. By the definition of the parallel transport, there exists a unique C^1 vector field Y_x such that $Y_x(x_0) = u$, $Y_x(x) = P_{x,x_0}u$ and

$$\nabla_{\gamma'(t)} Y_x(\gamma(t)) = 0 \quad \forall t \in [0,1]. \tag{2.4}$$

Since φ is a diffeomorphism, we have that (M,φ) is a chart for each point of M. For each $j = 1,2,\ldots,m$, define $y^j : M \to \mathbb{R}$ by $y^j = \pi^j \circ \varphi$, where $\pi^j : T_{x_0}M \to \mathbb{R}$ is the projection defined by

$$\pi(a_1,\ldots,a_j,\ldots,a_m) = a_j \quad \forall (a_1,\ldots,a_j,\ldots,a_m) \in T_{x_0}M.$$

Then the sequence $\{y^1, y^2, \ldots, y^m\}$ is a global coordinate of M. For simplicity, we put $y_x^j := y^j \circ \gamma_x(\cdot)$ for each $j = 1,2,\ldots,m$. Furthermore, we assume that

$$Y_x(\gamma_x(t)) = \sum_j Y_x^j(t) \frac{\partial}{\partial y^j}\Big|_{\gamma_x(t)} \quad \forall t \in [0,1],$$

where each $Y_x^j : [0,1] \to \mathbb{R}$ is a C^1-function. Then (2.4) is equivalent to (cf. [**10**, p. 52]) the following equation:

$$\frac{dY_x^k}{dt} + \sum_{i,j} \Gamma_{i,j}^k(\gamma_x) \frac{dy_x^i}{dt} Y_x^j = 0, \quad k = 1,2,\ldots,m, \tag{2.5}$$

where $\Gamma_{i,j}^k$ is the Christoffel symbols of the connection ∇. For each pair (k,j) with $k,j = 1,2,\ldots,m$, define the functions $a_{k,j} : [0,1] \times M \to \mathbb{R}$ by

$$a_{k,j}(t,x) = \sum_{i=1}^m \Gamma_{i,j}^k(\gamma_x(t)) \frac{dy_x^i(t)}{dt} \quad \forall (t,x) \in [0,1] \times M.$$

Then each $a_{k,j}$ is continuous on $[0,1] \times M$ and $\{Y_x^k\}$ is the unique solution of the following parameter linear differential equations:

$$\begin{aligned} &\frac{dY_x^k}{dt} + \sum_{j=1}^m a_{k,j} Y_x^j = 0, \quad k = 1,2,\ldots,m, \\ &\sum_j Y_x^j(0) \frac{\partial}{\partial y^j}\Big|_{x_0} = u. \end{aligned} \tag{2.6}$$

Thus, by the well-known result about the continuity on parameters for linear differential equations (see, for example, [**1**]), the solution $\{Y_\cdot^k(\cdot)\}$ is continuous on $[0,1] \times M$; or, equivalently, $Y_\cdot(\gamma_\cdot(\cdot))$ is continuous on $[0,1] \times M$. Since

$$F(x) = Y_x(x) = Y_x(\gamma_x(1)) \quad \forall x \in M,$$

it follows that F is continuous on M and the proof is complete. □

3. *Monotone vector fields on Hadamard manifolds*

Let $\mathcal{X}(M)$ denote the set of all multivalued vector fields $A : M \to 2^{TM}$ such that $A(x) \subseteq T_xM$ for each $x \in M$ and the domain $\mathcal{D}(A)$ of A is closed and convex, where the domain $\mathcal{D}(A)$ of A is defined by

$$\mathcal{D}(A) = \{x \in M : A(x) \neq \emptyset\}.$$

In the spirit of the corresponding notions in Hilbert spaces (cf. [**4**, **18**, **38**], for example), Definition 3.1 extends some notions of the monotonicity to multivalued vector fields on Hadamard manifolds. In particular, concepts (a), (b) and (c) were introduced and studied in [**20**] for the single-valued case and in [**8**] for the multivalued case.

Definition 3.1. Let $A \in \mathcal{X}(M)$. Then A is said to be

(i) *monotone* if the following condition holds for any $x, y \in \mathcal{D}(A)$:

$$\langle u, \exp_x^{-1} y\rangle \leqslant \langle v, -\exp_y^{-1} x\rangle \quad \forall u \in A(x) \text{ and } \forall v \in A(y); \tag{3.1}$$

(ii) *strictly monotone* if (3.1) holds with strict inequality for any $x, y \in \mathcal{D}(A)$ with $x \neq y$, that is,

$$\langle u, \exp_x^{-1} y\rangle < \langle v, -\exp_y^{-1} x\rangle \quad \forall u \in A(x) \text{ and } \forall v \in A(y); \tag{3.2}$$

(iii) *strongly monotone* if there exists $\rho > 0$ such that, for any $x, y \in \mathcal{D}(A)$, we have

$$\langle u, \exp_x^{-1} y\rangle - \langle v, -\exp_y^{-1} x\rangle \leqslant -\rho \mathrm{d}^2(x, y) \quad \forall u \in A(x) \text{ and } \forall v \in A(y); \tag{3.3}$$

(iv) *maximal monotone* if it is monotone and the following implication holds for any $x \in \mathcal{D}(A)$ and $u \in T_x M$:

$$\langle u, \exp_x^{-1} y\rangle \leqslant \langle v, -\exp_y^{-1} x\rangle \quad \forall y \in \mathcal{D}(A) \text{ and } v \in A(y) \Longrightarrow u \in A(x). \tag{3.4}$$

As we pointed out in Section 1, the gradient of a convex function is a monotone vector field. For examples of monotone vector fields that are not of gradient type, see [**21**–**24**].

An alternative definition of monoticity, in terms of the derivative of the distance function between geodesics, was introduced by Iwamiya and Okochi [**15**]. We shall prove in a subsequent paper that both definitions are equivalent. The relationship between monoticity and scalar derivatives has been deeply studied by Isac and Németh in their recent book [**14**, Chapter 5].

Remark 3.2. By definition, if A is a monotone vector field and $x \in \operatorname{int} \mathcal{D}(A)$, then, for each $v \in T_x M$, there exists $\mu > 0$ such that $\langle u, v\rangle \leqslant \mu$ for all $u \in A(x)$. This means that $A(x)$ is bounded for each $x \in \operatorname{int} \mathcal{D}(A)$.

The notions of upper semicontinuity and Kuratowski semicontinuity (cf. [**34**, p. 55]) as well as the local boundedness for operators in Banach spaces are extended in the following definition to the setting of Hadamard manifolds.

Definition 3.3. Let $A \in \mathcal{X}(M)$ and $x_0 \in \mathcal{D}(A)$. Then A is said to be

(i) *upper semicontinuous* at x_0 if, for any open set V satisfying $A(x_0) \subseteq V \subseteq T_{x_0} M$, there exists an open neighbourhood $U(x_0)$ of x_0 such that $P_{x_0,x} A(x) \subseteq V$ for any $x \in U(x_0)$;

(ii) *upper Kuratowski semicontinuous* at x_0 if, for any sequences $\{x_k\} \subseteq \mathcal{D}(A)$ and $\{u_k\} \subset TM$ with each $u_k \in A(x_k)$, the relations $\lim_{k\to\infty} x_k = x_0$ and $\lim_{k\to\infty} u_k = u_0$ imply that $u_0 \in A(x_0)$;

(iii) *locally bounded* at x_0 if there exists an open neighbourhood $U(x_0)$ of x_0 such that the set $\bigcup_{x\in U(x_0)} A(x)$ is bounded;

(iv) upper semicontinuous (resp. upper Kuratowski semicontinuous, locally bounded) on M if it is upper semicontinuous (resp. upper Kuratowski semicontinuous, locally bounded) at each point $x_0 \in \mathcal{D}(A)$.

Remark 3.4. Clearly, upper semicontinuity implies upper Kuratowski semicontinuity. The converse is also true if A is locally bounded on M.

The following proposition shows that maximality implies upper Kuratowski semicontinuity. Let $x_0 \in \mathcal{D}(A)$ and let $T : T_{x_0}M \to 2^{T_{x_0}M}$ be the mapping defined by

$$T(u) = P_{x_0,\exp_{x_0} u} A(\exp_{x_0} u) \quad \forall u \in T_{x_0}M. \tag{3.5}$$

PROPOSITION 3.5. *Let $A \in \mathcal{X}(M)$. Consider the following assertions.*

(i) *The vector field A is maximal monotone.*

(ii) *For each $x_0 \in \mathcal{D}(A)$, the mapping $T : T_{x_0}M \to 2^{T_{x_0}M}$ defined by (3.5) is upper Kuratowski semicontinuous on $T_{x_0}M$.*

(iii) *The vector field A is upper Kuratowski semicontinuous on M.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. (i) $\Rightarrow$ (ii): Suppose that (i) holds. Let $x_0 \in M$ and $u_0 \in T_{x_0}M$. Let $\{u_n\} \subset T_{x_0}M$ and $\{v_n\} \subset T_{x_0}M$ with each $v_n \in T(u_n)$ such that $u_n \to u_0$ and $v_n \to v_0$ for some $v_0 \in T_{x_0}M$. We have to verify that $v_0 \in T(u_0)$. To this end, set $x_n = \exp_{x_0} u_n$ and $\bar{v}_n = P_{x_n,x_0} v_n$ for each n. Then, by Lemma 2.4, we have $x_n \to \bar{x} := \exp_{x_0} u_0$ and $\bar{v}_n \in A(x_n)$ for each n. Furthermore, we have that $\bar{v}_n \to P_{\bar{x},x_0} v_0$ because

$$P_{x_n,x_0}(v_n - v_0) \longrightarrow 0, \quad P_{x_n,x_0} v_0 \longrightarrow P_{x_0,\bar{x}} v_0$$

and

$$\bar{v}_n = P_{x_n,x_0}(v_n - v_0) + P_{x_n,x_0} v_0.$$

On the other hand, by monotonicity, we have

$$\langle \bar{v}_n, \exp_{x_n}^{-1} y\rangle + \langle v, \exp_y^{-1} x_n\rangle \leqslant 0 \quad \forall y \in M \text{ and } v \in A(y).$$

Taking the limit as $n \to \infty$ yields that

$$\langle P_{\bar{x},x_0} v_0, \exp_{\bar{x}}^{-1} y\rangle + \langle v, \exp_y^{-1} \bar{x}\rangle \leqslant 0 \quad \forall y \in M \text{ and } v \in A(y).$$

By (i), it follows that A is maximal monotone. Hence $P_{\bar{x},x_0} v_0 \in A(\bar{x})$. By the definition of T and the fact that $\bar{x} = \exp_{x_0} u_0$, one has that $v_0 \in T(u_0)$ and completes the proof of the implication.

(ii) $\Rightarrow$ (iii): Suppose that (ii) holds and let $x_0 \in M$. It suffices to prove that A is upper Kuratowski semicontinuous at x_0. By (ii), the mapping $T : T_{x_0}M \to 2^{T_{x_0}M}$ defined by (3.5) is upper Kuratowski semicontinuous on $T_{x_0}M$. Since $\exp_{x_0}^{-1} : M \to T_{x_0}M$ is a diffeomorphism, it follows that the composition $T \circ \exp_{x_0}^{-1}$ is upper Kuratowski semicontinuous on M. Since

$$A(x) = P_{x,x_0}(T \circ \exp_{x_0}^{-1})(x) \quad \forall x \in M,$$

one sees that A is upper Kuratowski semicontinuous on M as P_{x,x_0} is an isometry and so A is upper Kuratowski semicontinuous at x_0. This completes the proof. □

Recall the well-known result that maximal monotonicity and upper semicontinuity are equivalent for a multivalued operator with closed and convex values in a Hilbert space (cf. [**26**]). To extend this result to multivalued operators on Hadamard manifolds, we first need to prove the following lemma.

LEMMA 3.6. *Suppose that $A \in \mathcal{X}(M)$ is maximal monotone and that $\mathcal{D}(A) = M$. Then A is locally bounded on M.*

Proof. Let $x_0 \in M$. Suppose, on the contrary, that A is not locally bounded at x_0. Then there exist sequences $\{x_n\} \subset \mathcal{D}(A)$ and $\{v_n\} \subset TM$ with each $v_n \in A(x_n)$ such that $x_n \to x_0$

but $\|v_n\| \to \infty$. Note that $A(x_0)$ is bounded by Remark 3.2. Hence

$$C := \sup\{\|u\| : u \in A(x_0)\} < \infty.$$

Taking $v_0 \in A(x_0)$, we define

$$u_n = (1-t_n)P_{x_n,x_0}v_0 + t_n v_n \quad \forall n = 1,2,\dots,$$

where $\{t_n\} \subset [0,1]$ such that $\|u_n\| = C+1$ for each $n = 1,2,\dots$. This means that $\{u_n\}$ is bounded and $t_n \to 0$. Without loss of generality, assume that $u_n \to u_0$ for some $u_0 \in T_{x_0}M$. Then $\|u_0\| = C+1$ and $u_0 \notin A(x_0)$. On the other hand, for any $y \in \mathcal{D}(A)$ and $v \in A(y)$, one has, for each $n = 1,2,\dots$, that

$$\begin{aligned}\langle u_n, \exp_{x_n}^{-1} y\rangle + \langle v, \exp_y^{-1} x_n\rangle &= (1-t_n)\big(\langle P_{x_n,x_0}v_0, \exp_{x_n}^{-1} y\rangle + \langle v, \exp_y^{-1} x_n\rangle\big)\\ &\quad + t_n\big(\langle v_n, \exp_{x_n}^{-1} y\rangle + \langle v, \exp_y^{-1} x_n\rangle\big)\\ &\leqslant (1-t_n)\big(\langle P_{x_n,x_0}v_0, \exp_{x_n}^{-1} y\rangle + \langle v, \exp_y^{-1} x_n\rangle\big),\end{aligned}$$

where the last inequality holds because $t_n \geqslant 0$ and $\langle v_n, \exp_{x_n}^{-1} y\rangle + \langle v, \exp_y^{-1} x_n\rangle \leqslant 0$ due to the monotonicity of A. Now, letting $n \to \infty$, we obtain that

$$\langle u_0, \exp_{x_0}^{-1} y\rangle + \langle v, \exp_y^{-1} x_0\rangle \leqslant \langle v_0, \exp_{x_0}^{-1} y\rangle + \langle v, \exp_{x_0}^{-1} y\rangle \leqslant 0$$

by Lemma 2.4 and the monotonicity of A. Since A is maximal monotone, it follows that $u_0 \in A(x_0)$, which is a contradiction. This completes the proof. □

Now we are ready to prove the main theorem of this section.

THEOREM 3.7. *Suppose that $A \in \mathcal{X}(M)$ is monotone and that $\mathcal{D}(A) = M$. Then the following statements are equivalent.*

(i) *The vector field A is maximal monotone.*

(ii) *For any $x_0 \in M$, the mapping $T : T_{x_0}M \to 2^{T_{x_0}M}$ defined by (3.5) is upper semicontinuous on $T_{x_0}M$, and $T(u)$ is closed and convex for each $u \in T_{x_0}M$.*

(iii) *The vector field A is upper semicontinuous on M, and $A(x)$ is closed and convex for each $x \in M$.*

Proof. (i) ⇒ (ii): Suppose that (i) holds. Then it follows that A is locally bounded by Lemma 3.6. Let $x_0 \in M$ and let T be defined by (3.5). Then T is locally bounded because the mapping $\exp_{x_0} : T_{x_0}M \to M$ is a diffeomorphism and the parallel transport $P_{x_0,\exp_{x_0}\cdot}$ is an isometry. Furthermore, T is upper Kuratowski semicontinuous on $T_{x_0}M$ by Proposition 3.5. Thus, bearing in mind Remark 3.4, we conclude that T is upper semicontinuous on $T_{x_0}M$.

It remains to prove that $T(u)$ is closed and convex for each $u \in T_{x_0}M$. To this end, let $u \in T_{x_0}M$ and write $x = \exp_{x_0} u$. For the sake of simplicity, we use $G(A)$ to denote the graph of A defined by

$$G(A) := \{(y,v) \in M \times TM :\ v \in A(y)\}.$$

By the maximality of A, we see that

$$T(u) = P_{x_0,x}A(x) = P_{x_0,x}\bigcap_{(y,v)\in G(A)}\big\{w \in T_{x_0}M : \langle w, \exp_{x_0}^{-1} y\rangle + \langle v, \exp_y^{-1} x_0\rangle \leqslant 0\big\}.$$

Therefore, we conclude that $T(u)$ is closed and convex. This completes the proof of the implication.

(ii) ⇒ (iii): Suppose that (ii) holds and let $x_0 \in M$. It suffices to prove that A is upper semicontinuous at x_0 because $A(x_0) = T(0)$ is convex and closed as assumed in (ii), where

$T : T_{x_0}M \to 2^{T_{x_0}M}$ is defined by (3.5). For this purpose, consider the multivalued mapping $S : M \to 2^{T_{x_0}M}$ defined by

$$S(x) = P_{x_0,x}A(x) \quad \forall x \in M.$$

It is clear that A is upper semicontinuous at x_0 if and only if S is also. Since the mapping T is upper semicontinuous on $T_{x_0}M$ by (ii) and since $\exp_{x_0}^{-1} : M \to T_{x_0}M$ is a diffeomorphism, it follows that the composition $T \circ \exp_{x_0}^{-1}$ is upper semicontinuous on M. Noting that

$$S(x) = P_{x_0,x}A(x) = (T \circ \exp_{x_0}^{-1})(x) \quad \forall x \in M,$$

one sees that S is upper semicontinuous on M and so at x_0.

(iii) $\Rightarrow$ (i): Suppose that (iii) holds but A is not maximal. Then there exist $x_0 \in M$ and $u_0 \in T_{x_0}M \setminus A(x_0)$ such that

$$\langle u_0, \exp_{x_0}^{-1} y\rangle \leqslant \langle v, -\exp_y^{-1} x_0\rangle \quad \forall y \in M \text{ and } \forall v \in A(y). \tag{3.6}$$

Note that $A(x_0)$ is a convex closed set by (iii), so that the well-known separation theorem is applicable and there exists $h \in T_{x_0}M$ such that

$$\langle u_0, h\rangle > \alpha = \sup_{u \in A(x_0)} \langle u, h\rangle.$$

Define $V := \{u \in T_{x_0}M : \langle u, h\rangle < \langle u_0, h\rangle\}$. Then V is an open set containing $A(x_0)$. By (iii), it follows that A is upper semicontinuous at x_0, and so there exists a neighbourhood $U(x_0)$ of x_0 such that $P_{x_0,x}A(x) \subseteq V$ for each $x \in U(x_0)$. Now, set $x_t := \exp_{x_0} th$ for each $t > 0$. Then $x_t \to x_0$ as $t \to 0$. Hence $x_t \in U(x_0)$ and $P_{x_0,x_t}A(x_t) \subseteq V$ for all $t > 0$ small enough. This means that we can take some $t > 0$ such that

$$\langle P_{x_0,x_t}v, h\rangle < \langle u_0, h\rangle \quad \forall v \in A(x_t).$$

Since $th = \exp_{x_0}^{-1} x_t$, the previous inequality turns into

$$\langle P_{x_0,x_t}v, \exp_{x_0}^{-1} x_t\rangle < \langle u_0, \exp_{x_0}^{-1} x_t\rangle \quad \forall v \in A(x_t).$$

Therefore, for each $v \in A(x_t)$, we have

$$\langle v, -\exp_{x_t}^{-1} x_0\rangle = \langle v, P_{x_t,x_0} \exp_{x_0}^{-1} x_t\rangle = \langle P_{x_0,x_t}v, \exp_{x_0}^{-1} x_t\rangle < \langle u_0, \exp_{x_0}^{-1} x_t\rangle,$$

which contradicts (3.6). Hence the implication (iii) $\Rightarrow$ (i) is proved. □

4. *Proximal point algorithm for monotone vector fields*

Recall that $A \in \mathcal{X}(M)$ is a multivalued vector field with closed and convex domain $\mathcal{D}(A)$. We say that $x \in \mathcal{D}(A)$ is a *singularity* of A if $0 \in A(x)$. The set of all singularities of A is denoted by $A^{-1}(0)$, that is,

$$A^{-1}(0) := \{x \in \mathcal{D}(A) : 0 \in A(x)\}.$$

The following proposition on the uniqueness of the singularity is a direct consequence of strict monotonicity.

PROPOSITION 4.1. *Let $A \in \mathcal{X}(M)$ be strictly monotone. Then A has at most one singularity.*

For the existence of singularities of A, we need the following known result (see, for example, [**26**, p. 115]).

Proposition 4.2. *Let X be a finite-dimensional space and $T : X \to 2^X$ be an upper semicontinuous multivalued operator. Suppose that $\mathcal{D}(T) = X$ and that T satisfies the following coercivity condition:*

$$\lim_{\|z\|\to\infty} \frac{\langle w, z\rangle}{\|z\|} = +\infty \quad \forall w \in T(z). \tag{4.1}$$

Then there exists $x \in X$ such that $0 \in T(x)$.

Theorem 4.3. *Let $A \in \mathcal{X}(M)$ be a maximal, strongly monotone vector field with the domain $\mathcal{D}(A) = M$. Then there exists a unique singularity of A.*

Proof. Since strong monotonicity implies strict monotonicity, the uniqueness of the singularity follows from Proposition 4.1. Now we will prove the existence of a singularity. To this end, let $x_0 \in M$ and let $T : T_{x_0}M \to 2^{T_{x_0}M}$ be defined by (3.5), that is,

$$T(u) = P_{x_0,\exp_{x_0} u} A(\exp_{x_0} u) \quad \forall u \in T_{x_0}M.$$

Then T is upper semicontinuous by Theorem 3.7. Moreover, by the strong monotonicity, there exists $\rho > 0$ such that, for any $x \in \mathcal{D}(A)$, we have

$$\langle u, \exp_{x_0}^{-1} x\rangle - \langle v, -\exp_x^{-1} x_0\rangle \leqslant -\rho \mathrm{d}^2(x_0, x) \quad \forall u \in A(x_0) \text{ and } \forall v \in A(x),$$

which is equivalent to

$$\langle P_{x_0,x}v - u, \exp_{x_0}^{-1} x\rangle \geqslant \rho \mathrm{d}^2(x_0, x) \quad \forall u \in A(x_0) \text{ and } \forall v \in A(x). \tag{4.2}$$

Letting $u \in T_{x_0}M$ and $w \in T(u)$, we set $x = \exp_{x_0} u$ and assume that $w = P_{x_0,x}v$ for some $v \in A(x)$. If $v_0 \in A(x_0)$, then, by (4.2), we obtain that

$$\begin{aligned}\langle w, u\rangle &= \langle P_{x_0,x}v - v_0, \exp_{x_0}^{-1} x\rangle + \langle v_0, \exp_{x_0}^{-1} x\rangle \\ &\geqslant \rho \mathrm{d}^2(x_0, x) + \langle v_0, \exp_{x_0}^{-1} x\rangle \\ &\geqslant \rho\|u\|^2 - \|v_0\|\|u\|.\end{aligned}$$

Therefore,

$$\lim_{\|u\|\to\infty} \frac{\langle w, u\rangle}{\|u\|} = +\infty \quad \forall w \in T(u).$$

This shows that T satisfies the coercivity condition (4.1). Consequently, Proposition 4.2 is applicable, with the conclusion that there exists a point $u \in T_{x_0}M$ such that $0 \in T(u)$. Then $x := \exp_{x_0} u \in M$ is a singularity of A and the proof is complete. □

Now we are going to present an iterative method to approximate a singularity of A, which is motivated by the proximal point algorithm on $\mathbb{R}^m$ introduced and studied by Moreau [**19**], Martinet [**17**] and Rockafellar [**32**]. Let $x_0 \in \mathcal{D}(A)$ and $\{\lambda_n\} \subset (0, 1)$. Letting $n = 0, 1, 2, \ldots$ and having x_n, define x_{n+1} such that

$$0 \in A(x_{n+1}) - \lambda_n \exp_{x_{n+1}}^{-1} x_n. \tag{4.3}$$

Remark 4.4. Note that the algorithm (4.3) is an implicit method. Thus one basic problem is to determine when this algorithm is well defined. For each n, define $B_n \in \mathcal{X}(M)$ by

$$B_n(x) := A(x) - \lambda_n \exp_x^{-1} x_n \quad \forall x \in \mathcal{D}(A).$$

Then, in the case when $A \in \mathcal{X}(M)$ is monotone, each B_n is strongly monotone. Thus, in view of Proposition 4.1 and Theorem 4.3, the following assertions hold when A is monotone.

(i) The algorithm (4.3) is well defined if and only if $B_n^{-1}(0) \neq \emptyset$ for each $n = 0, 1, 2, \ldots$.
(ii) If $\mathcal{D}(A) = M$ and A is maximal monotone, then the algorithm (4.3) is well defined.

The main result of this section is Theorem 4.7, which shows the convergence of the algorithm for upper Kuratowski semicontinuous monotone vector fields provided that the algorithm is well defined. We first recall the notion of Fejér convergence and the following related result (see [**12**]).

DEFINITION 4.5. Let X be a complete metric space and $K \subseteq X$ be a nonempty set. A sequence $\{x_n\} \subset X$ is called *Fejér convergent* to K if

$$\mathrm{d}(x_{n+1}, y) \leqslant \mathrm{d}(x_n, y) \quad \forall y \in K \text{ and } n = 0, 1, 2, \ldots.$$

LEMMA 4.6. *Let X be a complete metric space and let $K \subseteq X$ be a nonempty set. Let $\{x_n\} \subset X$ be Fejér convergent to K and suppose that any cluster point of $\{x_n\}$ belongs to K. Then $\{x_n\}$ converges to a point of K.*

THEOREM 4.7. *Let $A \in \mathcal{X}(M)$ be such that $A^{-1}(0) \neq \emptyset$. Suppose that A is monotone and upper Kuratowski semicontinuous. Let $\{\lambda_n\} \subset (0, 1)$ satisfy*

$$\sup\{\lambda_n : n \geqslant 0\} < \infty. \tag{4.4}$$

Let $x_0 \in \mathcal{D}(A)$ and suppose that the sequence $\{x_n\}$ generated by the algorithm (4.3) is well defined. Then $\{x_n\}$ converges to a singularity of A.

Proof. We first verify that $\{x_n\}$ is Fejér convergent to $A^{-1}(0)$ and $\lim_{n\to\infty} \mathrm{d}(x_{n+1}, x_n) = 0$. For this purpose, let $x \in A^{-1}(0)$ and $n \geqslant 0$. Then $0 \in A(x)$ and $\lambda_n \exp^{-1}_{x_{n+1}} x_n \in A(x_{n+1})$ by (4.3). This together with the monotonicity of A implies that

$$\langle \lambda_n \exp^{-1}_{x_{n+1}} x_n, \exp^{-1}_{x_{n+1}} x \rangle \leqslant \langle 0, -\exp^{-1}_x x_{n+1} \rangle = 0. \tag{4.5}$$

Consider the geodesic triangle $\Delta(x_n x_{n+1} x)$. By inequality (2.3) of the comparison theorem for triangles, we have that

$$\mathrm{d}^2(x_{n+1}, x) + \mathrm{d}^2(x_{n+1}, x_n) - 2\langle \exp^{-1}_{x_{n+1}} x_n, \exp^{-1}_{x_{n+1}} x \rangle \leqslant \mathrm{d}^2(x_n, x).$$

It follows from (4.5) that

$$\mathrm{d}^2(x_{n+1}, x) + \mathrm{d}^2(x_{n+1}, x_n) \leqslant \mathrm{d}^2(x_n, x). \tag{4.6}$$

This clearly implies that $\mathrm{d}^2(x_{n+1}, x) \leqslant \mathrm{d}^2(x_n, x)$, and so $\{x_n\}$ is Fejér convergent to $A^{-1}(0)$ as $n \geqslant 0$ is arbitrary. Furthermore, by (4.6), we obtain

$$\mathrm{d}^2(x_{n+1}, x_n) \leqslant \mathrm{d}^2(x_n, x) - \mathrm{d}^2(x_{n+1}, x). \tag{4.7}$$

Since the sequence $\{\mathrm{d}(x_n, x)\}$ is bounded and monotone, it is also convergent. Hence $\lim_{n\to\infty} \mathrm{d}^2(x_{n+1}, x_n) = 0$ by (4.7).

Thus, by Lemma 4.6, to complete the proof, we only need to prove that any cluster point of $\{x_n\}$ belongs to $A^{-1}(0)$. Let x be a cluster point of $\{x_n\}$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \to x$. Hence $\mathrm{d}(x_{n_k}, x_{n_k+1}) \to 0$ by the assertion just proved, and so $x_{n_k+1} \to x$. It follows that

$$u_{n_k+1} := \lambda_{n_k} \exp^{-1}_{x_{n_k+1}} x_{n_k} \longrightarrow 0 \tag{4.8}$$

since $\{\lambda_n\}$ is bounded by assumption (4.4). By the algorithm (4.3), we obtain that $u_{n_k+1} \in A(x_{n_k+1})$ for each k. Combining this with (4.8) implies that $0 \in A(x)$ because A is upper Kuratowski semicontinuous at x, that is, $x \in A^{-1}(0)$. □

In the case when $\mathcal{D}(A) = M$, maximal monotonicity is equivalent to upper semicontinuity by Theorem 3.7. Moreover, the maximality implies that x_n generated by (4.3) is well defined by Remark 4.4. Therefore the following corollary is direct.

COROLLARY 4.8. *Let $A \in \mathcal{X}(M)$ be such that $A^{-1}(0) \neq \emptyset$ and $\mathcal{D}(A) = M$. Suppose that A is maximal monotone. Let $\{\lambda_n\} \subset (0,1)$ satisfy (4.4) and let $x_0 \in M$. Then the sequence $\{x_n\}$ generated by the algorithm (4.3) is well defined and converges to a singularity of A.*

This corollary is an extension to Hadamard manifolds of the corresponding convergence theorem for the proximal point algorithm in Hilbert spaces (see [**32**]).

5. *Applications*

This section is devoted to three applications of the convergence result obtained for the proximal point algorithm (4.3). The first one is on constrained minimization problems, followed by the application on a minimax problem consisting of finding a saddle point. The last one is on variational inequality problems.

5.1. *Constrained optimization problems*

Some nonconvex constrained optimization problems can be solved after being written as convex problems in Riemann manifolds (see, for example, [**9**, **11**, **28**, **36**]). More specifically, an algorithm that can be used for solving any constrained problem in $\mathbb{R}^n$ with a convex objective function and constraint set being a constant curvature Hadamard manifold is considered in [**11**]. Regarding the problem of how to determine the sectional curvature of a manifold given in an implicit form, very good results have been recently obtained by Rapcsk [**29**], where sectional curvature is calculated. In this section, we apply our results to a constrained minimization problem.

Recall that M is a Hadamard manifold. Let $f : M \to (-\infty, +\infty]$ be a proper lower semicontinuous convex function. For simplicity, we write $K = D(f)$, the domain of f. Then K is a closed convex subset of M. Let $x \in M$ and let

$$T_xK = \{u \in T_xM : \exp_x tu \in K \text{ for some } t > 0\}.$$

Then T_xK is a convex cone. Define the directional derivative of f at x in the direction $u \in T_xM$ by

$$f'(x, u) = \lim_{t \to 0+} \frac{f(\exp_x tu) - f(x)}{t}.$$

It can be proved that $f'(x, \cdot)$ is subadditive and positively homogeneous on T_xK. Moreover, we have, by definition, that $D(f'(x, \cdot)) = T_xK$.

The *subdifferential* of f at x is defined by

$$\partial f(x) = \{u \in T_xM : \langle u, \exp_x^{-1} y\rangle \leqslant f(y) - f(x) \ \forall y \in M\}.$$

Then $\partial f(x)$ is a closed convex (possible empty) set, and

$$u \in \partial f(x) \Longleftrightarrow \langle u, h\rangle \leqslant f'(x, h) \quad \text{for each } h \in T_xM. \tag{5.1}$$

The proofs of the above assertions can be found in [**36**]. Furthermore, it is clear from the definition that

$$\partial f(x) = \partial(f'(x, \cdot))(0). \tag{5.2}$$

Throughout the whole section, we always assume that $\mathcal{D}(\partial f) \neq \emptyset$.

THEOREM 5.1. *Let f be a proper, lower semicontinuous convex function on M. The subdifferential $\partial f(\cdot)$ is a monotone and upper Kuratowski semicontinuous multivalued vector field. Furthermore, if, in addition, $D(f) = M$, then the subdifferential ∂f of f is a maximal monotone vector field.*

Proof. The monotonicity of ∂f is a consequence of the definition of the subdifferential, which was already pointed out in [**8**]. Indeed, for any $x, y \in D(f)$, $u \in \partial f(x)$ and $v \in \partial f(y)$, we have that

$$\langle u, \exp_x^{-1} y\rangle \leqslant f(y) - f(x) \leqslant \langle v, -\exp_y^{-1} x\rangle.$$

To verify the upper Kuratowski semicontinuity, let $x_0 \in D(f)$, $\{x_n\} \subset D(f)$ and $\{v_n\} \subset TM$ with each $v_n \in \partial f(x_n)$ be such that $x_n \to x_0$ and $v_n \to v_0$ for some $v_0 \in TM$. Then, by definition,

$$\langle v_n, \exp_{x_n}^{-1} y\rangle \leqslant f(y) - f(x_n) \quad \forall y \in M.$$

Taking lower limits in the previous inequality, we obtain that

$$\langle v_0, \exp_{x_0}^{-1} y\rangle \leqslant f(y) - f(x_0) \quad \forall y \in M \tag{5.3}$$

because $\lim_{n\to\infty}\langle v_n, \exp_{x_n}^{-1} y\rangle = \langle v_0, \exp_{x_0}^{-1} y\rangle$ by Lemma 2.4 and $f(x_0) \leqslant \liminf_{n\to\infty} f(x_n)$ by the lower semicontinuity of f. Thus (5.3) means that $v_0 \in \partial f(x_0)$ and proves the upper Kuratowski semicontinuity of ∂f.

Finally, we assume additionally that $D(f) = M$. By Theorem 3.7 and Remark 3.4, it suffices to prove that ∂f is locally bounded. To do this, let $x_0 \in M$. From [**13**], we know that f is local Lipschitz. Therefore, there exist $\epsilon > 0$ and $L > 0$ such that

$$\|f(y) - f(x)\| \leqslant L\mathrm{d}(x, y) \quad \forall x, y \in \mathbf{U}(x_0, \epsilon), \tag{5.4}$$

where $\mathbf{U}(x_0, \epsilon)$ denotes the open metric ball with centre x_0 and radius ϵ. For each $x \in \mathbf{U}(x_0, \epsilon)$, there exists $r > 0$ such that $\mathbf{U}(x, r) \subset \mathbf{U}(x_0, \epsilon)$. Hence, due to (5.4) and the definition of the subdifferential, for each $x \in \mathbf{U}(x_0, \epsilon)$ and each $v \in \partial f(x)$, we obtain

$$\langle v, \exp_x^{-1} y\rangle \leqslant \|f(y) - f(x)\| \leqslant L\mathrm{d}(x, y) \leqslant Lr \quad \forall y \in \mathbf{U}(x, r).$$

This implies that $\|v\| \leqslant L$, and so ∂f is locally bounded because $x \in \mathbf{U}(x_0, \epsilon)$ and $v \in \partial f(x)$ are arbitrary. The proof is complete. □

Recall that $f : M \to (-\infty, +\infty]$ is a proper, lower semicontinuous convex function. Consider the optimization problem

$$\min_{x\in M} f(x). \tag{5.5}$$

We use S_f to denote the solution set of (5.5), that is,

$$S_f := \{x \in M : f(x) \leqslant f(y) \ \forall y \in M\}.$$

It is easy to check that

$$x \in S_f \Longleftrightarrow 0 \in \partial f(x). \tag{5.6}$$

Applying the algorithm (4.3) to the multivalued vector field ∂f, we obtain the following proximal point algorithm for the optimization problem (5.5):

$$0 \in \partial f(x_{n+1}) - \lambda_n \exp_{x_{n+1}}^{-1} x_n \quad \forall n \geqslant 0. \tag{5.7}$$

REMARK 5.2. Let $y \in M$ and $\lambda > 0$. We define a real-valued convex function $\phi_{\lambda,y}$ by

$$\phi_{\lambda,y}(x) = \frac{\lambda}{2}\mathrm{d}(y,x)^2 \quad \forall x \in M.$$

Consider the following algorithm for finding a solution of the optimization problem (5.5):

$$x_{n+1} \in S_{f+\phi_{\lambda_n,x_n}} \quad \forall n \geqslant 0, \tag{5.8}$$

which was presented and studied by Ferreira and Oliveira [**12**] in the special case when f is a real-valued convex function on M. From [**37**], the derivative of $\phi_{\lambda,y}$ is given by

$$\phi'_{\lambda,y}(x) = -\lambda \exp_x^{-1} y \quad \forall x \in M.$$

Using (5.1), it is routine to verify that

$$\partial(f + \phi_{\lambda,y})(x) = \partial f(x) - \frac{\lambda}{2}\exp_x^{-1} y \quad \forall x \in D(f). \tag{5.9}$$

Hence, by (5.6), the proximal point algorithms (5.7) and (5.8) are equivalent.

The following theorem on the convergence of the proximal point algorithm (5.7) is a consequence of Theorem 4.7 (cf. [**12**, Theorem 6.1]).

THEOREM 5.3. *Let $f : M \to (-\infty, +\infty]$ be a proper, lower semicontinuous convex function with the solution set $S_f \neq \emptyset$. Let $x_0 \in M$ and $\{\lambda_n\} \subset (0,1)$ satisfy (4.4). Then the sequence $\{x_n\}$ generated by the algorithm (5.7) is well defined and converges to a point $x \in S_f$, a minimizer of f in M.*

Proof. By Theorem 5.1, the differential ∂f is monotone and upper Kuratowski semicontinuous. Thus, it suffices to show that the algorithm (5.7) is well defined. Since $S_f \neq \emptyset$, it follows that $\inf_{x\in M} f(x) > -\infty$. We claim that $S_{f+\phi_{\lambda,y}} \neq \emptyset$ for each $\lambda > 0$ and each $y \in M$. In fact, let $\{x_n\} \subset M$ be such that

$$\lim_{n\to\infty}(f(x_n) + \phi_{\lambda,y}(x_n)) = \inf_{x\in M}(f(x) + \phi_{\lambda,y}(x)).$$

Then $\{x_n\}$ is bounded because, otherwise, $\limsup_{n\to\infty}(f(x_n) + \phi_{\lambda,y}(x_n)) = +\infty$. Hence, without loss of generality, we may assume that $\{x_n\}$ converges to a point x. Then $x \in S_{f+\phi_{\lambda,y}}$ and $S_{f+\phi_{\lambda,y}} \neq \emptyset$. This together with Remark 5.2 implies that the algorithm (5.7) is well defined and completes the proof. □

Now let $f : M \to \mathbb{R}$ be a convex function and K a closed and convex subset of M. Consider the following optimization problem with constraints:

$$\min_{x\in K} f(x). \tag{5.10}$$

Define $f_K := f + \delta_K$, where δ_K is the indicate function defined by $\delta_K(x) = 0$ if $x \in K$ and $\delta_K(x) = +\infty$ otherwise. Then a point $x \in K$ is a solution of problem (5.10) if and only if it is a solution of problem (5.5) with f replaced by f_K. Let $N_K(x)$ denote the normal cone of the set K at $x \in K$:

$$N_K(x) := \{u \in T_xM : \langle u, \exp_x^{-1} y\rangle \leqslant 0 \ \forall y \in K\}.$$

Then

$$N_K(x) = N_{T_xK}(0) = \partial\delta_K(x) \quad \forall x \in K. \tag{5.11}$$

To apply Theorem 5.3, we need to establish the following clear fact on the subdifferential of f_K.

Proposition 5.4. *Let $f : M \to \mathbb{R}$ be a convex function and K a closed and convex subset of M. Then*

$$\partial f_K(x) = \partial f(x) + N_K(x) \quad \forall x \in K. \tag{5.12}$$

Proof. By definition, it is obvious that

$$f'_K(x,u) = f'(x,u) + \delta_{T_x K}(u) \quad \forall u \in T_x M.$$

Applying (5.2), we obtain that

$$\partial f_K(x) = \partial f'_K(x,\cdot)|_{u=0} \quad \text{and} \quad \partial f(x) = \partial f'(x,\cdot)|_{u=0}. \tag{5.13}$$

Since $f'(x,\cdot)$ is a continuous convex function on $T_x M$, it follows from the well-known subdifferential sum rule (see, for example, [**2**]) that

$$\partial f'_K(x,\cdot)|_{u=0} = \partial f'(x,\cdot)|_{u=0} + \partial \delta_{T_x K}(0). \tag{5.14}$$

According to (5.11), (5.13) and (5.14), we obtain (5.12) and the proof is complete. □

Consider the following algorithm with initial point $x_0 \in K$:

$$0 \in \partial f(x_{n+1}) + N_K(x_{n+1}) - \lambda_n \exp^{-1}_{x_{n+1}} x_n \quad \forall n \geqslant 0. \tag{5.15}$$

Then the theorem below is a direct consequence of Corollary 4.8.

Theorem 5.5. *Let $f : M \to \mathbb{R}$ be a convex function and K be a closed convex set of M such that the solution set of the optimization problem* (5.10) *is nonempty. Let $x_0 \in M$ and $\{\lambda_n\}$ satisfy* (4.4)*. Then the sequence $\{x_n\}$ generated by the algorithm* (5.15) *is well defined and converges to a solution of the optimization problem* (5.10).

5.2. *Saddle points in a minimax problem*

In the spirit of the works by Rockafellar [**31**, **32**] on the convergence of the proximal point algorithm in terms of an associated maximal monotone operator for saddle functions on the product Hilbert space $H_1 \times H_2$, this section is focused on the study of the convergence of the proximal point algorithm for saddle functions on Hadamard manifolds.

Let M_1 and M_2 be Hadamard manifolds. A function $L : M_1 \times M_2 \to \mathbb{R}$ is called a *saddle function* if $L(x,\cdot)$ is convex on M_2 for each $x \in M_1$ and $L(\cdot,y)$ is concave, that is, $-L(\cdot,y)$ is convex on M_1 for each $y \in M_2$. A point $\bar{z} = (\bar{x},\bar{y}) \in M_1 \times M_2$ is called a *saddle point* of L if

$$L(x,\bar{y}) \leqslant L(\bar{x},\bar{y}) \leqslant L(\bar{x},y) \quad \forall z = (x,y) \in M_1 \times M_2. \tag{5.16}$$

Associated with the saddle function L, define the multivalued vector field $A_L : M_1 \times M_2 \to 2^{TM_1} \times 2^{TM_2}$ by

$$A_L(x,y) = \partial\big(-L(\cdot,y)\big)(x) \times \partial\big(L(x,\cdot)\big)(y) \quad \forall (x,y) \in M_1 \times M_2. \tag{5.17}$$

By [**33**, Problem 10, p. 239], the product space $M = M_1 \times M_2$ is a Hadamard manifold and the tangent space of M at $z = (x,y)$ is $T_z M = T_x M_1 \times T_y M_2$. The corresponding metric is given by

$$\langle w, w' \rangle = \langle u, u' \rangle + \langle v, v' \rangle \quad \forall w = (u,v), w' = (u',v') \in T_z M.$$

Note also that a geodesic in the product manifold M is the product of two geodesics in M_1 and M_2. Then, for any two points $z = (x,y)$ and $z' = (x',y')$ in M, we have

$$\exp^{-1}_z z' = \exp^{-1}_{(x,y)}(x',y') = (\exp^{-1}_x x', \exp^{-1}_y y').$$

Therefore, in view of the definition of monotonicity, the multivalued vector field $A : M_1 \times M_2 \to 2^{TM_1} \times 2^{TM_2}$ is monotone if and only if, for any $z = (x, y)$, $z' = (x', y')$, $w = (u, v) \in A(z)$ and $w' = (u', v') \in A(z')$, we have

$$\langle u, \exp_x^{-1} x' \rangle + \langle v, \exp_y^{-1} y' \rangle \leqslant \langle u', -\exp_{x'}^{-1} x \rangle + \langle v', -\exp_{y'}^{-1} y \rangle. \tag{5.18}$$

THEOREM 5.6. *Let L be a saddle function on $M = M_1 \times M_2$ and A_L the multivalued vector field defined by (5.17). Then A_L is maximal monotone.*

Proof. Consider two points $z = (x, y)$ and $z' = (x', y')$ in M, and $w = (u, v) \in A_L(z)$ and $w' = (u', v') \in A_L(z')$. Since $\partial\big(-L(\cdot, y)\big)$ and $\partial\big(L(x, \cdot)\big)$ are monotone by Theorem 5.1, it follows from the definition of A_L that

$$\langle u, \exp_x^{-1} x' \rangle \leqslant \langle u', -\exp_{x'}^{-1} x \rangle \quad \text{and} \quad \langle v, \exp_y^{-1} y' \rangle \leqslant \langle v', -\exp_{y'}^{-1} y \rangle.$$

Hence (5.18) holds and A_L is monotone because $z, z' \in M$ and $w \in A_L(z)$ and $w' \in A_L(z')$ are arbitrary.

To verify the maximality, let $z = (x, y) \in M_1 \times M_2$ and $w = (u, v) \in T_xM_1 \times T_yM_2$ such that

$$\langle u, \exp_x^{-1} x' \rangle + \langle v, \exp_y^{-1} y' \rangle \leqslant \langle u', -\exp_{x'}^{-1} x \rangle + \langle v', -\exp_{y'}^{-1} y \rangle \tag{5.19}$$

for any $z' = (x', y') \in M_1 \times M_2$ and $w' = (u', v') \in A_L(z')$. We have to prove that $w \in A_L(z)$, that is, $u \in \partial\big(-L(\cdot, y)\big)(x)$ and $v \in \partial\big(L(x, \cdot)\big)(y)$. Taking $y' = y$ in (5.19), we obtain

$$\langle u, \exp_x^{-1} x' \rangle \leqslant \langle u', -\exp_{x'}^{-1} x \rangle \quad \forall x' \in M_1 \text{ and } u' \in \partial\big(-L(\cdot, y)\big)(x'). \tag{5.20}$$

Note that $\partial\big(-L(\cdot, y)\big)$ is maximal by Theorem 5.1, and hence (5.20) implies that $u \in \partial\big(-L(\cdot, y)\big)(x)$. Similarly, taking $x' = x$ in (5.19), we obtain that $v \in \partial\big(-L(x, \cdot)\big)(y)$, as desired. □

It is straightforward to check that a point $\bar{z} = (\bar{x}, \bar{y}) \in M$ is a saddle point of L if and only if it is a singularity of A_L. Consider the following algorithm for A_L:

$$0 \in A_L(z_{n+1}) - \lambda_n \exp_{z_{n+1}}^{-1} z_n, \tag{5.21}$$

where $z_0 \in M_1 \times M_2$ and $\{\lambda_n\} \subset (0, 1)$. Thus, by applying Corollary 4.8 to the vector field A_L associated with the saddle function L, we immediately obtain the following theorem.

THEOREM 5.7. *Let $L : M = M_1 \times M_2 \to \mathbb{R}$ be a saddle function and $A_L : M_1 \times M_2 \to 2^{TM_1} \times 2^{TM_2}$ be the associated maximal monotone vector field. Suppose that L has a saddle point. Let $z_0 \in M$ and let $\{\lambda_n\} \subset (0, 1)$ satisfy (4.4). Then the sequence $\{z_n\}$ generated by the algorithm (5.21) is well defined and converges to a saddle point of L.*

5.3. *Variational inequalities*

Let K be a convex subset of M and $V : K \to TM$ a univalued vector field, that is, $V(x) \in T_xM$ for each $x \in K$. Following [**25**], the problem of finding $x \in K$ such that

$$\langle V(x),\ \exp_x^{-1} y \rangle \geqslant 0 \quad \forall y \in K \tag{5.22}$$

is called a *variational inequality* on K. A point $x \in K$ satisfying (5.22) is called a solution of the variational inequality (5.22). Clearly, a point $x \in K$ is a solution of the variational inequality (5.22) if and only if x satisfies

$$0 \in V(x) + N_K(x),$$

that is, the point x is a singularity of the multivalued vector field $A := V + N_K$. Applying the algorithm (4.3) to A, we obtain the following proximal point algorithm with initial point x_0 for finding solutions of the variational inequality (5.22):

$$0 \in V(x_{n+1}) + N_K(x_{n+1}) - \lambda_n \exp_{x_{n+1}}^{-1} x_n \quad \forall n \geqslant 0. \tag{5.23}$$

The remainder of this section is directed toward the study of the convergence of algorithm (5.23). To apply Theorem 4.7, one needs to prove that the algorithm (5.23) is well defined. For this purpose and for the sake of completeness, we first include some lemmas. One of them (Lemma 5.11) is an extension to Hadamard manifolds of the well-known Brouwer fixed point theorem. This fact could be deduced from more general results, but this would imply the introduction of some abstract concepts such as the Lefschetz number on acyclic spaces (see [**5**]) and the NPC, metric spaces with non-positive curvature (see [**35**]). Németh [**25**] gave a similar but incomplete proof to the one we present below. The following proposition is a direct consequence of [**33**, Theorem 5.5 and Lemma 5.4, p. 170] (noting that M is a Hadamard manifold).

PROPOSITION 5.8. *Let K be a convex compact subset of M. Then there exists a totally geodesic submanifold $N \subseteq K$ such that $K = \overline{N}$, the closure of N, and the following condition holds: for any $q \in K\backslash N$ and $p \in N$, we have $\exp_p t(\exp_p^{-1} q) \in N$ for all $t \in (0, 1)$ and $\exp_p t(\exp_p^{-1} q) \notin K$ for any $t \in (1, +\infty)$.*

REMARK 5.9. Following [**33**], $\operatorname{int} K := N$ is called the interior of K and $\operatorname{bd} K := K\backslash N$ the boundary of K. Moreover, if K is a compact convex set, then $\operatorname{bd} K \neq \emptyset$.

LEMMA 5.10. *Let K be a convex compact subset of M and let $p_0 \in \operatorname{int} K$. Then*

$$\exp_{p_0}^{-1}(\operatorname{bd} K) = \operatorname{bd}(\exp_{p_0}^{-1} K) \quad \textit{and} \quad \exp_{p_0}^{-1}(\operatorname{int} K) = \operatorname{int}(\exp_{p_0}^{-1} K). \tag{5.24}$$

Proof. Let $p_0 \in \operatorname{int} K$. Since $\exp_{p_0}^{-1}$ is a bijection from K to $\exp_{p_0}^{-1} K$ and $(\operatorname{bd} K) \cap (\operatorname{int} K) = \emptyset$, it follows that

$$\exp_{p_0}^{-1}(K) = \exp_{p_0}^{-1}(\operatorname{bd} K) \cup \exp_{p_0}^{-1}(\operatorname{int} K).$$

Thus, to complete the proof, it suffices to prove that

$$\exp_{p_0}^{-1}(\operatorname{bd} K) \subseteq \operatorname{bd}(\exp_{p_0}^{-1} K) \quad \text{and} \quad \exp_{p_0}^{-1}(\operatorname{int} K) \subseteq \operatorname{int}(\exp_{p_0}^{-1} K). \tag{5.25}$$

To show the first inclusion, let $q \in \operatorname{bd} K$. Then $\exp_{p_0}^{-1} q \in \exp_{p_0}^{-1} K$. By Proposition 5.8, we see that $\exp_{p_0} t(\exp_{p_0}^{-1} q) \notin K$ for all $t > 1$. Hence $t(\exp_{p_0}^{-1} q) \notin \exp_{p_0}^{-1} K$ for any $t > 1$. Therefore, $\exp_{p_0}^{-1} q \in \operatorname{bd}(\exp_{p_0}^{-1} K)$ and the inclusion $\exp_{p_0}^{-1}(\operatorname{bd} K) \subseteq \operatorname{bd}(\exp_{p_0}^{-1} K)$ is proved.

Below, we show the inclusion $\exp_{p_0}^{-1}(\operatorname{int} K) \subseteq \operatorname{int}(\exp_{p_0}^{-1} K)$. For simplicity, we use $\mathbf{U}_E(x, \epsilon)$ to denote the open ball at $x \in E$ with radius ϵ in a metric space E. Note that $\operatorname{int} K = N$ is the totally geodesic submanifold given by Proposition 5.8. Then, from [**33**, p. 171] (noting that M is a Hadamard manifold), for any $q \in \operatorname{int} K$, we have

$$T_q N = \{v \in T_q M\backslash\{0\} : \exp_q tv/\|v\| \in N \text{ for some } t > 0\} \cup \{0\} \tag{5.26}$$

and $\exp_q$ is a local diffeomorphism at 0 from $T_q N$ to N (cf. [**27**, **33**]), that is, there exists an open ball $\mathbf{U}_{T_q N}(0, \epsilon) \subseteq T_q N$ at 0 such that $\exp_q(\mathbf{U}_{T_q N}(0, \epsilon)) \subseteq N$. This means that

$$q \in \operatorname{int} K \Longleftrightarrow \exp_q(\mathbf{U}_{T_q N}(0, \delta)) \subseteq \operatorname{int} K \quad \text{for some } \delta > 0. \tag{5.27}$$

Consequently, the boundary $\operatorname{bd} K$ is closed in M because K is closed. Let $q \in \operatorname{int} K$ and set $M_0 = \exp_{p_0}(T_{p_0} N)$. Then

$$\epsilon := \mathrm{d}(q, M_0\backslash K) > 0. \tag{5.28}$$

In fact, there exists a sequence $\{q_k\} \subseteq M_0 \backslash K$ such that $\lim_k \mathrm{d}(q_k, q) = 0$. By Proposition 5.8 and equation (5.26) (noting that $p_0 \in N$), we may assume, for each k, that $q_k = \exp_{p_0} t_k u_k$, where $t_k \geqslant 1$ and $u_k \in T_{p_0}N$ is such that $\exp_{p_0} u_n \in \mathrm{bd}\, K$. Since $\{t_k u_k\}$ is bounded and each $t_k \geqslant 1$, it follows that $\{u_k\}$ is bounded too. Since $\mathrm{bd}\, K$ is closed, it follows that $\liminf_k \|u_k\| > 0$. This together with the boundedness of $\{t_k u_k\}$ implies that $\{t_k\}$ is bounded. Without loss of generality, we may assume that $t_k \to t_0$ and $u_k \to u_0$. Then $t_0 \geqslant 1$ and $\exp_{p_0} u_0 \in \mathrm{bd}\, K$. Hence $\exp_{p_0} t_0 u_0 \notin N$ by Proposition 5.8, which is a contradiction because $\exp_{p_0}(t_0 u_0) = x \in N$. Therefore, we conclude that (5.28) is proved.

Since $\exp_{p_0}$ is continuous on $T_{p_0}N$, there exists $\delta > 0$ such that

$$\mathrm{d}(\exp_{p_0} v, q) = \mathrm{d}(\exp_{p_0} v, \exp_{p_0}^{-1} q) < \frac{\epsilon}{2} \quad \text{for each } v \in \mathbf{U}_{T_{p_0}N}(\exp_{p_0}^{-1} q, \delta).$$

It follows from (5.28) that $\exp_{p_0}(\mathbf{U}_{T_{p_0}N}(\exp_{p_0}^{-1} q, \delta)) \subseteq K$. This implies that $\exp_{p_0} q \in \mathrm{int}\,(\exp_{p_0}^{-1} K)$ and the proof is complete. □

Lemma 5.11. *Let K be a compact convex subset of M. Let $F : K \to K$ be a continuous map. Then F has a fixed point in K.*

Proof. Note that the fixed point property is a topological property, that is, if X and Y are homeomorphic topological spaces and any continuous mapping on X has fixed points, then so does any continuous mapping on Y. Let $p_0 \in \mathrm{int}\, K$ and, for simplicity, write $\tilde{K} = \exp_{p_0}^{-1} K$. Then K is homeomorphic to $\tilde{K}$. By the Brouwer fixed point theorem, it suffices to prove that $\tilde{K} \subset T_{p_0}N$ is homeomorphic to the closed unit ball of $T_{p_0}N$, denoted by $\mathbf{B}$. We define the function $\phi : \tilde{K} \to \mathbb{R}^+$ by

$$\phi(x) = \begin{cases} \|\widehat{0x} \cap \mathrm{bd}\tilde{K}\| & \text{if } \ x \neq 0, \\ 1 & \text{if } \ x = 0, \end{cases}$$

where $\widehat{0x} := \{tx : t \geqslant 0\}$ is the straight half-line joining 0 to x. By Proposition 5.8, the geodesic joining p_0 and $\exp_{p_0} x$ intersects $\mathrm{bd}\, K$ at just one point, and so, by Lemma 5.10, $\widehat{0x}$ intersects $\mathrm{bd}\, \tilde{K}$ at just one point. This implies that ϕ is well defined and continuous at each point $x \neq 0$. Indeed, let $x_0 \in \tilde{K} \backslash \{0\}$ and write $\overline{x} = \widehat{0x} \cap \mathrm{bd}\, \tilde{K}$ for any $x \neq 0$. Then $\overline{x} = (x/\|x\|)\phi(x)$. Let $\{x_n\} \subset \tilde{K}$ be such that $x_n \to x_0$. Since ϕ is bounded, we can assume, without lost of generality, that $\phi(x_n) \to r \in \mathbb{R}^+$. Then

$$\overline{x_n} \longrightarrow \overline{y_0} = \frac{x_0}{\|x_0\|} r \in \mathrm{bd}\, \tilde{K}.$$

We must prove that $r = \phi(x_0)$ to obtain the continuity of ϕ. To this end, we can write

$$\overline{y_0} = \frac{x_0}{\|x_0\|} \phi(x_0) \frac{r}{\phi(x_0)} = \frac{r}{\phi(x_0)} \overline{x}. \tag{5.29}$$

If $r < \phi(x_0)$, then $r/\phi(x_0) < 1$. Thus, by Proposition 5.8 and Lemma 5.10, equation (5.29) implies that $\overline{y_0} \in \mathrm{int}\, \tilde{K}$ because $\overline{x} \in \mathrm{bd}\, K$, which contradicts that $\overline{y_0} \in \mathrm{bd}\, \tilde{K}$. Similarly, if $r > \phi(x_0)$, we obtain that $\overline{y_0} \notin \tilde{K}$, which again contradicts that $\overline{y_0} \in \mathrm{bd}\, \tilde{K}$. Therefore ϕ is continuous at each $x \neq 0$.

Now define the function $h : \tilde{K} \to \mathbf{B}$ by

$$h(x) = \frac{1}{\phi(x)} x \quad \forall x \in \tilde{K}.$$

Note that $\|h(x)\| = (1/\|\phi(x)\|)\|x\| \leqslant 1$. Thus h is well defined and continuous, whose inverse function $h^{-1}(y) = \phi(y)y$ is continuous too. Hence, h is a homeomorphism from $\tilde{K}$ to $\mathbf{B}$ and the proof is complete. □

THEOREM 5.12. *Let K be a closed convex subset of M and $V : K \to TM$ a univalued continuous monotone vector field. Let $x_0 \in K$ and $\{\lambda_n\} \subset (0,1)$ satisfy (4.4). Suppose that the variational inequality (5.22) has a solution. Then the sequence $\{x_n\}$ generated by the algorithm (5.23) is well defined and converges to a solution of the variational inequality (5.22).*

Proof. By Theorem 4.7 and Remark 4.4, we only need to prove that the sequence $\{x_n\}$ generated by the algorithm (5.23) is well defined. Let $\lambda > 0$ and $y_0 \in K$. Consider the following variational inequality:

$$\langle V(x) - \lambda \exp_x^{-1} y_0,\ \exp_x^{-1} y\rangle \geqslant 0 \quad \forall y \in K. \tag{5.30}$$

For fixed n, note that x_{n+1} satisfies (5.23) if and only if x_{n+1} is a solution of the variational inequality (5.30) with $\lambda = \lambda_n$ and $y_0 = x_n$. Thus, it suffices to prove that the variational inequality (5.30) has a solution. The proof is standard (see [**25**]). However, we present the proof here for completeness. Let $R > 0$ be such that $\|V(y_0)\| - 2R\lambda < 0$ and set

$$K_R = \{x \in K : \mathrm{d}(x, y_0) \leqslant R\}.$$

Then K_R is a compact convex subset of M. Let $P_{K_R} : M \to K_R$ be the projection to K_R. Then, by [**25**, **37**], the projection P_{K_R} is Lipschitz continuous and characterized by

$$\langle \exp^{-1}_{P_{K_R}x} x, \exp^{-1}_{P_{K_R}x} y\rangle \leqslant 0 \quad \forall x \in M \ \text{ and } \ y \in K_R. \tag{5.31}$$

Consider the continuous map $F : K_R \to K_R$ defined by

$$F(x) := P_{K_R}(\exp_x(-V(x) + \lambda \exp_x^{-1} y_0)) \quad \forall x \in K_R.$$

By Lemma 5.11, the map F has a fixed point x_R. This together with (5.31) implies that

$$\langle V(x_R) - \lambda \exp^{-1}_{x_R} y_0, \exp^{-1}_{x_R} y\rangle \geqslant 0 \tag{5.32}$$

holds for any $y \in K_R$. Since $\langle V(x), \exp_x^{-1} y_0\rangle \leqslant \langle -V(y_0), \exp^{-1}_{y_0} x\rangle$ by monotonicity and $\langle \exp_x^{-1} y_0, \exp_x^{-1} y_0\rangle = \langle \exp^{-1}_{y_0} x, \exp^{-1}_{y_0} x\rangle = \mathrm{d}(x, y_0)^2$, it follows that, if $\mathrm{d}(x, y_0) = R$, then

$$\begin{aligned}\langle V(x) - \lambda \exp_x^{-1} y_0, \exp_x^{-1} y_0\rangle &\leqslant \langle V(y_0), \exp^{-1}_{y_0} x\rangle - 2\lambda \mathrm{d}(x, y_0)^2 \\ &\leqslant (\|V(y_0)\| - 2R\lambda)R \\ &< 0.\end{aligned}$$

This means that $\mathrm{d}(x_R, y_0) < R$. We shall now show that (5.32) holds for any $y \in K$ and this will complete the proof. Indeed, given $y \in K$ and $y_t = \exp_{x_R} t(\exp^{-1}_{x_R} y) \in K_R$ for $t > 0$ sufficiently small. Consequently,

$$t\langle V(x_R) - \lambda \exp^{-1}_{x_R} y_0,\ \exp^{-1}_{x_R} y\rangle = \langle V(x_R) - \lambda \exp^{-1}_{x_R} y_0, \exp^{-1}_{x_R} y_t\rangle \geqslant 0.$$

Thus, we conclude that (5.32) holds for $y \in K$. □

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