It’s very difficult to visualize a function \( f \) of three variables by its graph, since that would lie in a four-dimensional space. However, we do gain some insight into \( f \) by examining its **level surfaces**, which are the surfaces with equations \( f(x, y, z) = k \), where \( k \) is a constant. If the point \((x, y, z)\) moves along a level surface, the value of \( f(x, y, z) \) remains fixed.

**EXAMPLE 12** Find the level surfaces of the function

\[
f(x, y, z) = x^2 + y^2 + z^2
\]

**SOLUTION** The level surfaces are \( x^2 + y^2 + z^2 = k \), where \( k \geq 0 \). These form a family of concentric spheres with radius \( \sqrt{k} \). (See Figure 13.) Thus, as \((x, y, z)\) varies over any sphere with center \( O \), the value of \( f(x, y, z) \) remains fixed.

Functions of any number of variables can be considered. A **function of \( n \) variables** is a rule that assigns a number \( z = f(x_1, x_2, \ldots, x_n) \) to an \( n \)-tuple \((x_1, x_2, \ldots, x_n)\) of real numbers. We denote by \( \mathbb{R}^n \) the set of all such \( n \)-tuples. For example, if a company uses \( n \) different ingredients in making a food product, \( c_i \) is the cost per unit of the \( i \)th ingredient, and \( x_i \) units of the \( i \)th ingredient are used, then the total cost \( C \) of the ingredients is a function of the \( n \) variables \( x_1, x_2, \ldots, x_n \):

\[
C = f(x_1, x_2, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n
\]

The function \( f \) is a real-valued function whose domain is a subset of \( \mathbb{R}^n \). Sometimes we will use vector notation in order to write such functions more compactly: If \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \), we often write \( f(\mathbf{x}) \) in place of \( f(x_1, x_2, \ldots, x_n) \). With this notation we can rewrite the function defined in Equation 3 as

\[
f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}
\]

where \( \mathbf{c} = (c_1, c_2, \ldots, c_n) \) and \( \mathbf{c} \cdot \mathbf{x} \) denotes the dot product of the vectors \( \mathbf{c} \) and \( \mathbf{x} \) in \( V_n \).

In view of the one-to-one correspondence between points \((x_1, x_2, \ldots, x_n)\) in \( \mathbb{R}^n \) and their position vectors \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \) in \( V_n \), we have three ways of looking at a function \( f \) defined on a subset of \( \mathbb{R}^n \):

1. As a function of \( n \) real variables \( x_1, x_2, \ldots, x_n \)
2. As a function of a single point variable \((x_1, x_2, \ldots, x_n)\)
3. As a function of a single vector variable \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \)

We will see that all three points of view are useful.

### Exercises

1. In Example 1 we considered the function \( I = f(T, v) \), where \( I \) is the wind-chill index, \( T \) is the actual temperature, and \( v \) is the wind speed. A numerical representation is given in Table 1.
   (a) What is the value of \( f(8, 60) \)? What is its meaning?
   (b) Describe in words the meaning of the question “For what value of \( v \) is \( f(-12, v) = -26 \)” Then answer the question.
   (c) Describe in words the meaning of the question “For what value of \( T \) is \( f(T, 80) = -14 \)” Then answer the question.
   (d) What is the meaning of the function \( I = f(-4, v) \)? Describe the behavior of this function.
   (e) What is the meaning of the function \( I = f(T, 50) \)? Describe the behavior of this function.
2. The temperature-humidity index $I$ (or humidex, for short) is the perceived air temperature when the actual temperature is $T$ and the relative humidity is $h$, so we can write $I = f(T, h)$. The following table of values of $I$ is an excerpt from a table compiled by the National Oceanic and Atmospheric Administration.

<table>
<thead>
<tr>
<th>$T$ (°F)</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
</tr>
</thead>
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<tr>
<td>80</td>
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<td>99</td>
<td>104</td>
<td>110</td>
<td>120</td>
<td>132</td>
<td>144</td>
</tr>
</tbody>
</table>

TABLE 3 Apparent temperature as a function of temperature and humidity

(a) What is the value of $f(95, 70)$? What is its meaning?
(b) For what value of $h$ is $f(90, h) = 100$?
(c) For what value of $T$ is $f(T, 50) = 88$?
(d) What are the meanings of the functions $I = f(80, h)$ and $I = f(100, h)$? Compare the behavior of these two functions of $h$.

3. Verify for the Cobb-Douglas production function

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

discussed in Example 2 that the production will be doubled if both the amount of labor and the amount of capital are doubled. Is this also true for the general production function $P(L, K) = bL^nK^m$?

4. The temperature-humidity index $I$ discussed in Exercise 2 has been modeled by the following fourth-degree polynomial:

$$I(T, h) = -42.379 + 2.04901523T + 10.1433127h - 0.2247554171Th - 0.00683783T^2 - 0.05481717h^2 + 0.00122874T^2h + 0.00085282Th^2 - 0.00000199T^2h^2$$

Check to see how closely this model agrees with the values in Table 3 for a few values of $T$ and $h$. Do you prefer the numerical or algebraic representation of this function?

5. Find and sketch the domain of the function $f(x, y) = \ln(9 - x^2 - 9y^2)$.

6. Find and sketch the domain of the function $f(x, y) = \sqrt{1 + x - y^2}$. What is the range of $f$?

7. Let $f(x, y, z) = e^{(x+y+z)}$.
(a) Evaluate $f(2, -1, 6)$.
(b) Find the domain of $f$.
(c) Find the range of $f$.

8. Let $g(x, y, z) = \ln(25 - x^2 - y^2 - z^2)$
(a) Evaluate $g(2, -2, 4)$.
(b) Find the domain of $g$.
(c) Find the range of $g$.

9. A contour map for a function $f$ is shown. Use it to estimate the values of $f(-3, 3)$ and $f(3, -2)$. What can you say about the shape of the graph?

10. Two contour maps are shown. One is for a function $f$ whose graph is a cone. The other is for a function $g$ whose graph is a paraboloid. Which is which, and why?

11. Locate the points $A$ and $B$ in the map of Lonesome Mountain (Figure 5). How would you describe the terrain near $A$? Near $B$?

12. Make a rough sketch of a contour map for the function whose graph is shown.
13–14 ■ A contour map of a function is shown. Use it to make a rough sketch of the graph of 

13. A thin metal plate, located in the $xy$-plane, has temperature $T(x, y)$ at the point $(x, y)$. The level curves of $T$ are called isothermals because at all points on an isothermal the temperature is the same. Sketch some isothermals if the temperature function is given by

$$T(x, y) = 100/(1 + x^2 + 2y^2)$$

26. If $V(x, y)$ is the electric potential at a point $(x, y)$ in the $xy$-plane, then the level curves of $V$ are called equipotential curves because at all points on such a curve the electric potential is the same. Sketch some equipotential curves if $V(x, y) = c/\sqrt{x^2 - x^2 - y^2}$, where $c$ is a positive constant.

27–30 ■ Use a computer to graph the function using various domains and viewpoints. Get a printout of one that, in your opinion, gives a good view. If your software also produces level curves, then plot some contour lines of the same function and compare with the graph.

27. $f(x, y) = e^x \cos y$

28. $f(x, y) = (1 - 3x^2 + y^2)e^{-x^2-y^2}$

29. $f(x, y) = xy^2 - x^3$ (monkey saddle)

30. $f(x, y) = xy^3 - yx^3$ (dog saddle)

31–36 ■ Match the function (a) with its graph (labeled A–F on page 759) and (b) with its contour map (labeled I–VI). Give reasons for your choices.

31. $z = \sin \sqrt{x^2 + y^2}$

32. $z = x^2y^2e^{-x^2-y^2}$

33. $z = \frac{1}{x^2 + 4y^2}$

34. $z = x^2 - 3xy^2$

35. $z = \sin x \sin y$

36. $z = \sin^2x + \frac{1}{2}y^2$

37–40 ■ Describe the level surfaces of the function.

37. $f(x, y, z) = x + 3y + 5z$

38. $f(x, y, z) = x^2 + 3y^2 + 5z^2$

39. $f(x, y, z) = x^2 - y^2 + z^2$

40. $f(x, y, z) = x^2 - y^2$

41–42 ■ Describe how the graph of $g$ is obtained from the graph of $f$.

41. (a) $g(x, y) = f(x, y) + 2$

(b) $g(x, y) = 2f(x, y)$

(c) $g(x, y) = -f(x, y)$

(d) $g(x, y) = 2 - f(x, y)$

42. (a) $g(x, y) = f(x - 2, y)$

(b) $g(x, y) = f(x, y + 2)$

(c) $g(x, y) = f(x + 3, y - 4)$

43. Use a computer to investigate the family of functions $f(x, y) = e^{x^2+y^2}$. How does the shape of the graph depend on $c$?

44. Graph the functions

$$f(x, y) = \sqrt{x^2 + y^2} \\
 f(x, y) = \ln(x^2 + y^2) \\
 f(x, y) = \sin(\sqrt{x^2 + y^2})$$

and

$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$$

In general, if $g$ is a function of one variable, how is the graph of $f(x, y) = g(\sqrt{x^2 + y^2})$ obtained from the graph of $g$?

45. (a) Show that, by taking logarithms, the general Cobb-Douglas function $P = bK^{\alpha - u}$ can be expressed as

$$\ln \frac{P}{K} = \ln b + \alpha \ln \frac{L}{K}$$

(b) If we let $x = \ln(L/K)$ and $y = \ln(P/K)$, the equation in part (a) becomes the linear equation $y = \alpha x + \ln b$. Use Table 2 (in Example 2) to make a table of values of $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. Then use a graphing calculator or computer to find the least squares regression line through the points $(\ln(L/K), \ln(P/K))$.

(c) Deduce that the Cobb-Douglas production function is $P = 1.01L^{0.75}K^{0.25}$. 
Every thing that we have done in this section can be extended to functions of three or more variables. The notation

\[
\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = L
\]

means that the values of \( f(x, y, z) \) approach the number \( L \) as the point \( (x, y, z) \) approaches the point \( (a, b, c) \) along any path in the domain of \( f \). The function \( f \) is continuous at \( (a, b, c) \) if

\[
\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = f(a, b, c)
\]

For instance, the function

\[
f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}
\]

is a rational function of three variables and so is continuous at every point in \( \mathbb{R}^3 \) except where \( x^2 + y^2 + z^2 = 1 \). In other words, it is discontinuous on the sphere with center the origin and radius 1.

**Exercises**

1. Suppose that \( \lim_{(x, y) \to (3, 1)} f(x, y) = 6 \). What can you say about the value of \( f(3, 1) \)? What if \( f \) is continuous?

2. Explain why each function is continuous or discontinuous.
   (a) The outdoor temperature as a function of longitude, latitude, and time
   (b) Elevation (height above sea level) as a function of longitude, latitude, and time
   (c) The cost of a taxi ride as a function of distance traveled and time

3–4 ■ Use a table of numerical values of \( f(x, y) \) for \( (x, y) \) near the origin to make a conjecture about the value of the limit of \( f(x, y) \) as \( (x, y) \to (0, 0) \). Then explain why your guess is correct.

3. \( f(x, y) = \frac{x^4y^3 + x^2y^2 - 5}{2 - xy} \)

4. \( f(x, y) = \frac{2xy}{x^2 + 2y^2} \)

5–18 ■ Find the limit, if it exists, or show that the limit does not exist.

5. \( \lim_{(x, y) \to (5, -2)} (x^3 + 4x^2y - 5xy^2) \)

6. \( \lim_{(x, y) \to (0, 3)} xy \cos(x - 2y) \)

7. \( \lim_{(x, y) \to (0, 0)} \frac{x^2}{x^2 + y^2} \)

8. \( \lim_{(x, y) \to (0, 0)} \frac{x + y}{x^2 + y^2} \)

9. \( \lim_{(x, y) \to (0, 0)} \frac{8x^2y^2}{x^3 + y^3} \)

10. \( \lim_{(x, y) \to (0, 0)} \frac{x^3 + 2xy^2}{x^3 + y^3} \)

11. \( \lim_{(x, y) \to (0, 0)} \frac{xy}{\sqrt{x^2 + y^2}} \)

12. \( \lim_{(x, y) \to (0, 0)} \frac{x^2\sin^2y}{x^2 + 2y^2} \)

13. \( \lim_{(x, y) \to (0, 0)} \frac{2x^2y}{x^4 + y^2} \)

14. \( \lim_{(x, y) \to (2, 0)} \frac{xy - 2y}{x^2 + y^2 - 4x + 4} \)

15. \( \lim_{(x, y) \to (0, 0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2} + 1 - 1} \)

16. \( \lim_{(x, y, z) \to (3, 2, 0)} e^{x/z} \cos(y + z) \)

17. \( \lim_{(x, y, z) \to (0, 0, 0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^2} \)

18. \( \lim_{(x, y, z) \to (0, 0, 0)} \frac{x^2 + 2y^2 + 3z^2}{x^2 + y^2 + z^2} \)

19–20 ■ Use a computer graph of the function to explain why the limit does not exist.

19. \( \lim_{(x, y) \to (0, 0)} \frac{2x^2 + 3xy + 4y^2}{3x^2 + 5y^2} \)

20. \( \lim_{(x, y) \to (0, 0)} \frac{xy^3}{x^2 + y^6} \)

21–22 ■ Find \( h(x, y) = g(f(x, y)) \) and the set on which \( h \) is continuous.

21. \( g(t) = t^2 + \sqrt{t} \) \quad \( f(x, y) = 2x + 3y - 6 \)

22. \( g(z) = \sin z \) \quad \( f(x, y) = y \ln x \)

23–24 ■ Graph the function and observe where it is discontinuous. Then use the formula to explain what you have observed.

23. \( f(x, y) = e^{x/y^2} \)

24. \( f(x, y) = \frac{1}{1 - x^2 - y^2} \)
25–32 Determine the largest set on which the function is continuous.

25. \( F(x, y) = \frac{1}{x^2 - y} \)  
26. \( F(x, y) = \frac{x - y}{1 + x^2 + y^2} \)

27. \( F(x, y) = \arctan(x + \sqrt{y}) \)

28. \( G(x, y) = \sin^{-1}(x^2 + y^2) \)

29. \( f(x, y, z) = \frac{xyz}{x^2 + y^2 - z} \)

30. \( f(x, y, z) = \sqrt{x + y + z} \)

31. \( f(x, y) = \begin{cases} \frac{x^2 y}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases} \)

32. \( f(x, y) = \begin{cases} \frac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \)

33–34 Use polar coordinates to find the limit. [If \((r, \theta)\) are polar coordinates of the point \((x, y)\) with \(r \geq 0\), note that \(r \to 0^+\) as \((x, y) \to (0, 0)\).]

33. \( \lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{x^2 + y^2} \)

34. \( \lim_{(x,y) \to (0,0)} (x^2 + y^2) \ln(x^2 + y^2) \)

35. Use spherical coordinates to find

\[ \lim_{(x,y,z) \to (0,0,0)} \frac{xyz}{x^2 + y^2 + z^2} \]

36. At the beginning of this section we considered the function

\[ f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \]

and guessed that \( f(x, y) \to 1 \) as \( (x, y) \to (0, 0) \) on the basis of numerical evidence. Use polar coordinates to confirm the value of the limit. Then graph the function.

---

**Partial Derivatives**

On a hot day, extreme humidity makes us think the temperature is higher than it really is, whereas in very dry air we perceive the temperature to be lower than the thermometer indicates. The National Weather Service has devised the *heat index* (also called the temperature-humidity index, or humidex) to describe the combined effects of temperature and humidity. The heat index \( I \) is the perceived air temperature when the actual temperature is \( T \) and the relative humidity is \( H \). So \( I \) is a function of \( T \) and \( H \) and we can write \( I = f(T, H) \). The following table of values of \( I \) is an excerpt from a table compiled by the National Weather Service.

<table>
<thead>
<tr>
<th>Actual temperature (°F)</th>
<th>50</th>
<th>55</th>
<th>60</th>
<th>65</th>
<th>70</th>
<th>75</th>
<th>80</th>
<th>85</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>96</td>
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<td>100</td>
<td>103</td>
<td>106</td>
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<td>147</td>
<td>154</td>
<td>161</td>
<td>168</td>
</tr>
</tbody>
</table>

If we concentrate on the highlighted column of the table, which corresponds to a relative humidity of \( H = 70\% \), we are considering the heat index as a function of the
where $b$ is a constant that is independent of both $L$ and $K$. Assumption (i) shows that 
\[ \alpha > 0 \text{ and } \beta > 0. \]

Notice from Equation 8 that if labor and capital are both increased by a factor $m$, then
\[ P(mL, mK) = b(mL)\alpha(mK)\beta = m^{\alpha+\beta}L^\alpha K^\beta = m^{\alpha+\beta}P(L, K) \]

If $\alpha + \beta = 1$, then $P(mL, mK) = mP(L, K)$, which means that production is also increased by a factor of $m$. That is why Cobb and Douglas assumed that $\alpha + \beta = 1$ and therefore
\[ P(L, K) = bL^\alpha K^{1-\alpha} \]

This is the Cobb-Douglas production function that we discussed in Section 11.1.

### Exercises

1. The temperature $T$ at a location in the Northern Hemisphere depends on the longitude $x$, latitude $y$, and time $t$, so we can write $T = f(x, y, t)$. Let’s measure time in hours from the beginning of January.
   (a) What are the meanings of the partial derivatives $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial t$?
   (b) Honolulu has longitude 158°W and latitude 21°N. Suppose that at 9:00 a.m. on January 1 the wind is blowing hot air to the northeast, so the air to the west and south is warm and the air to the north and east is cooler. Would you expect $f_x(158, 21, 9)$, $f_y(158, 21, 9)$, and $f_t(158, 21, 9)$ to be positive or negative? Explain.

2. At the beginning of this section we discussed the function $I = f(T, H)$, where $I$ is the heat index, $T$ is the temperature, and $H$ is the relative humidity. Use Table 1 to estimate $f_t(92, 60)$ and $f_y(92, 60)$. What are the practical interpretations of these values?

3. The wind-chill index $I$ is the perceived temperature when the actual temperature is $T$ and the wind speed is $v$, so we can write $I = f(T, v)$. Table 2 (at the bottom of the page) is an excerpt from a table of values of $I$ compiled by the National Atmospheric and Oceanic Administration.
   (a) Estimate the values of $f_x(12, 20)$ and $f_y(12, 20)$. What are the practical interpretations of these values?

(b) In general, what can you say about the signs of $\partial I/\partial T$ and $\partial I/\partial v$?
(c) What appears to be the value of the following limit?
\[ \lim_{v \to 0} \frac{\partial I}{\partial v} \]

4. The wave heights $h$ in the open sea depend on the speed $v$ of the wind and the length of time $t$ that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table.

<table>
<thead>
<tr>
<th>Wind speed (knots)</th>
<th>Duration (hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>20</td>
<td>15</td>
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<tr>
<td>50</td>
<td>40</td>
</tr>
<tr>
<td>60</td>
<td>50</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Wind speed (km/h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actual temperature (°C)</td>
</tr>
<tr>
<td>T</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>16</td>
</tr>
<tr>
<td>12</td>
</tr>
<tr>
<td>8</td>
</tr>
</tbody>
</table>
(a) What are the meanings of the partial derivatives \( \frac{\partial h}{\partial v} \) and \( \frac{\partial h}{\partial w} \)?

(b) Estimate the values of \( f_1(40, 15) \) and \( f_1(40, 15) \). What are the practical interpretations of these values?

(c) What appears to be the value of the following limit?

\[
\lim_{v \to 0} \frac{\partial h}{\partial t}
\]

5–6 Determine the signs of the partial derivatives for the function \( f \) whose graph is shown.

5. (a) \( f_1(1, 2) \) (b) \( f_2(1, 2) \)

6. (a) \( f_1(-1, 2) \) (b) \( f_2(-1, 2) \)
   (c) \( f_3(-1, 2) \) (d) \( f_4(-1, 2) \)

7. The following surfaces, labeled \( a, b, \) and \( c \), are graphs of a function \( f \) and its partial derivatives \( f_1 \) and \( f_2 \). Identify each surface and give reasons for your choices.

8. A contour map is given for a function \( f \). Use it to estimate \( f_1(2, 1) \) and \( f_2(2, 1) \).

9. If \( f(x, y) = 16 - 4x^2 - y^2 \), find \( f_1(1, 2) \) and \( f_2(1, 2) \) and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

10. If \( f(x, y) = \sqrt{4 - x^2 - y^2} \), find \( f_1(1, 0) \) and \( f_2(1, 0) \) and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

11–12 Find \( f_1 \) and \( f_2 \) and graph \( f_1, f_2, \) and \( f \) with domains and viewpoints that enable you to see the relationships between them.

11. \( f(x, y) = x^2 + y^2 + x^2 y \)

12. \( f(x, y) = xe^{-x-y} \)

13–34 Find the first partial derivatives of the function.

13. \( f(x, y) = 3x - 2y^4 \)

14. \( f(x, y) = x^5 + 3x^3y^2 + 3xy^4 \)

15. \( z = xe^{xy} \)

16. \( z = y \ln x \)

17. \( f(x, y) = \frac{x - y}{x + y} \)

18. \( f(x, y) = x^3 \)

19. \( w = \sin \alpha \cos \beta \)

20. \( f(x, t) = xt^3(x^2 + t^2) \)

21. \( f(u, v) = \tan^{-1}(u/v) \)

22. \( f(x, t) = e^{\sin(x/t)} \)

23. \( z = \ln(x + \sqrt{x^2 + y^2}) \)

24. \( f(x, y) = \int_y^t \cos(t^2) \, dt \)

25. \( f(x, y, z) = xy^2z^3 + 3yz \)

26. \( f(x, y, z) = xe^{yz} \)

27. \( w = \ln(x + 2y + 3z) \)

28. \( w = \sqrt{r^2 + s^2 + t^2} \)

29. \( u = xe^{-t} \sin \theta \)

30. \( u = x^{1/2} \)

31. \( f(x, y, z, t) = \frac{x - y}{z - t} \)

32. \( f(x, y, z, t) = xy^2z^4 + 3y^3z^2 \)

33. \( u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \)

34. \( u = \sin(x_1 + 2x_2 + \cdots + nx_n) \)

35–38 Find the indicated partial derivatives.

35. \( f(x, y) = \sqrt{x^2 + y^2}; \quad f_1(3, 4) \)

36. \( f(x, y) = \sin(2x + 3y); \quad f_2(-6, 4) \)
37. \( f(x, y, z) = x/\left(y + z\right) \); \( f_1(3, 2, 1) \)
38. \( f(u, v, w) = w \tan(\omega) \); \( f_2(2, 0, 3) \)

39–40 Use the definition of partial derivatives as limits (4) to find \( f_1(x, y) \) and \( f_2(x, y) \).

39. \( f(x, y) = x^2 - xy + 2y^2 \)
40. \( f(x, y) = \sqrt{3x - y} \)

41–44 Use implicit differentiation to find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \).

41. \( xy + yz = xz \)
42. \( xyz = \cos(x + y + z) \)
43. \( x^2 + y^2 - z^2 = 2x(y + z) \)
44. \( xy^2z^3 + x^3y^2z = x + y + z \)

45–46 Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \).

45. (a) \( z = f(x) + g(y) \)
(b) \( z = f(x + y) \)
46. (a) \( z = f(x)g(y) \)
(b) \( z = f(xy) \)
(c) \( z = f(x/y) \)

47–50 Find all the second partial derivatives.

47. \( f(x, y) = x^4 - 3x^2y^3 \)
48. \( f(x, y) = \ln(3x + 5y) \)
49. \( u = e^{-t} \sin t \)
50. \( z = y \tan 2x \)

51–52 Verify that the conclusion of Clairaut’s Theorem holds, that is, \( u_{xy} = u_{yx} \).

51. \( u = \ln \sqrt{x^2 + y^2} \)
52. \( u = xye^y \)

53–58 Find the indicated partial derivative.

53. \( f(x, y) = x^2y^3 - 2x^4y \); \( f_{xx} \)
54. \( f(x, y) = e^{-y^3} \); \( f_{xy} \)
55. \( f(x, y, z) = x^3 + x^2y^3 + yz^3 \); \( f_{yy} \)
56. \( f(x, y, z) = e^{x+y} \); \( f_{yy} \)
57. \( z = x \sin y \); \( \frac{\partial z}{\partial x} \)
58. \( u = x^2y^3z^6 ; \frac{\partial u}{\partial x^2y^3z^6} \)

59. Use the table of values of \( f(x, y) \) to estimate the values of \( f_1(3, 2), f_2(3, 2.2), \) and \( f_{xy}(3, 2) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 2.5 )</th>
<th>( 3.0 )</th>
<th>( 3.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1.8</td>
<td>2.0</td>
<td>2.2</td>
</tr>
<tr>
<td>( 2.5 )</td>
<td>12.5</td>
<td>10.2</td>
<td>9.3</td>
</tr>
<tr>
<td>( 3.0 )</td>
<td>18.1</td>
<td>17.5</td>
<td>15.9</td>
</tr>
<tr>
<td>( 3.5 )</td>
<td>20.0</td>
<td>22.4</td>
<td>26.1</td>
</tr>
</tbody>
</table>

60. Level curves are shown for a function \( f \). Determine whether the following partial derivatives are positive or negative at the point \( P \).
(a) \( f_x \)
(b) \( f_y \)
(c) \( f_{xy} \)
(d) \( f_{xx} \)
(e) \( f_{yy} \)

61. Verify that the function \( u = e^{-u^2/k} \) is a solution of the wave equation.

62. Determine whether each of the following functions is a solution of the wave equation.
(a) \( u = x^2 + y^2 \)
(b) \( u = x^2 - y^2 \)
(c) \( u = x^3 + 3xy^2 \)
(d) \( u = \ln \sqrt{x^2 + y^2} \)
(e) \( u = e^{-x^2} \cos y - e^{-y} \cos x \)

63. Verify that the function \( u = 1/\sqrt{x^2 + y^2} \) is a solution of the heat conduction equation.

64. Show that each of the following functions is a solution of the wave equation.
(a) \( u = \sin(kx) \sin(akt) \)
(b) \( u = t/(a^2t^2 - x^2) \)
(c) \( u = (x - at)^6 + (x + at)^6 \)
(d) \( u = \sin(x - at) + \ln(x + at) \)

65. If \( f \) and \( g \) are twice differentiable functions of a single variable, show that the function
\[ u(x, t) = f(x + at) + g(x - at) \]
is a solution of the wave equation given in Exercise 64.

66. Show that the Cobb-Douglas production function
\[ P = bL^aK^b \]
satisfies the equation
\[ \frac{dP}{dL} + L \frac{dP}{dK} = (\alpha + \beta)P \]

67. Show that the Cobb-Douglas production function satisfies
\[ P(L, K_0) = C(K_0) L^a \]
by solving the differential equation
\[ \frac{dP}{dL} = \frac{\alpha P}{L} \]
(See Equation 5.)

68. The temperature at a point \((x, y)\) on a flat metal plate is given by \( T(x, y) = 60/(1 + x^2 + y^2) \), where \( T \) is measured
in °C and x, y in meters. Find the rate of change of temperature with respect to distance at the point (2, 1) in (a) the x-direction and (b) the y-direction.

69. The total resistance \( R \) produced by three conductors with resistances \( R_1, R_2, R_3 \) connected in a parallel electrical circuit is given by the formula

\[
\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}
\]

Find \( \partial R / \partial R_1 \).

70. The gas law for a fixed mass \( m \) of an ideal gas at absolute temperature \( T \), pressure \( P \), and volume \( V \) is \( PV = mRT \), where \( R \) is the gas constant. Show that

\[
\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = -1
\]

71. The kinetic energy of a body with mass \( m \) and velocity \( v \) is \( K = \frac{1}{2}mv^2 \). Show that

\[
\frac{\partial K}{\partial m} \frac{\partial^2 K}{\partial v^2} = K
\]

72. If \( a, b, c \) are the sides of a triangle and \( A, B, C \) are the opposite angles, find \( \partial A / \partial a, \partial A / \partial b, \partial A / \partial c \) by implicit differentiation of the Law of Cosines.

73. You are told that there is a function \( f \) whose partial derivatives are \( f_x(x, y) = x + 4y \) and \( f_y(x, y) = 3x - y \) and whose second-order partial derivatives are continuous. Should you believe it?

74. The paraboloid \( z = 6 - x^2 - 2y^2 \) intersects the plane \( x = 1 \) in a parabola. Find parametric equations for the tangent line to this parabola at the point (1, 2, -4). Use a computer to graph the paraboloid, the parabola, and the tangent line on the same screen.

75. The ellipsoid \( 4x^2 + 2y^2 + z^2 = 16 \) intersects the plane \( y = 2 \) in an ellipse. Find parametric equations for the tangent line to this ellipse at the point (1, 2, 2).

76. In a study of frost penetration it was found that the temperature \( T \) at time \( t \) (measured in days) at a depth \( x \) (measured in feet) can be modeled by the function

\[
T(x, t) = T_0 + T_1e^{-ax} \sin(\omega t - \lambda x)
\]

where \( \omega = 2\pi/365 \) and \( \lambda \) is a positive constant.

(a) Find \( \partial T / \partial x \). What is its physical significance?

(b) Find \( \partial T / \partial t \). What is its physical significance?

(c) Show that \( T \) satisfies the heat equation \( T_t = kT_{xx} \) for a certain constant \( k \).

(d) If \( \lambda = 0.2, T_0 = 0 \), and \( T_1 = 10 \), use a computer to graph \( T(x, t) \).

(e) What is the physical significance of the term \(-\lambda x \) in the expression \( \sin(\omega t - \lambda x) \)?

77. If \( f(x, y) = x(x^2 + y^2)^{-3/2} \), find \( f_x(1, 0) \).

[Hint: Instead of finding \( f_x(x, y) \) first, note that it is easier to use Equation 1 or Equation 2.]

78. If \( f(x, y) = \sqrt{x^2 + y^2} \), find \( f_x(0, 0) \).

79. Let

\[
f(x, y) = \begin{cases} 
\frac{x^3 y - xy^3}{x^2 + y^2} & \text{if} \ (x, y) \neq (0, 0) \\
0 & \text{if} \ (x, y) = (0, 0)
\end{cases}
\]

(a) Use a computer to graph \( f \).

(b) Find \( f_x(x, y) \) and \( f_y(x, y) \) when \( (x, y) \neq (0, 0) \).

(c) Find \( f_x(0, 0) \) and \( f_y(0, 0) \) using Equations 2 and 3.

(d) Show that \( f_x(0, 0) = -1 \) and \( f_y(0, 0) = 1 \).

(e) Does the result of part (d) contradict Clairaut’s Theorem? Use graphs of \( f_x \) and \( f_y \) to illustrate your answer.

### 11.4 Tangent Planes and Linear Approximations

One of the most important ideas in single-variable calculus is that as we zoom in toward a point on the graph of a differentiable function the graph becomes indistinguishable from its tangent line and we can approximate the function by a linear function. (See Sections 2.9 and 3.8.) Here we develop similar ideas in three dimensions. As we zoom in toward a point on a surface that is the graph of a differentiable function of two variables, the surface looks more and more like a plane (its tangent plane) and we can approximate the function by a linear function of two variables. We also extend the idea of a differential to functions of two or more variables.

#### Tangent Planes

Suppose a surface \( S \) has equation \( z = f(x, y) \), where \( f \) has continuous first partial derivatives, and let \( P(x_0, y_0, z_0) \) be a point on \( S \). As in the preceding section, let \( C_1 \) and
Thus, a normal vector to the tangent plane is
\[ \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v \mathbf{i} - 4u \mathbf{j} + 4uv \mathbf{k} \]

Notice that the point (1, 1, 3) corresponds to the parameter values \( u = 1 \) and \( v = 1 \), so the normal vector there is
\[-2 \mathbf{i} - 4 \mathbf{j} + 4 \mathbf{k}\]

Therefore, an equation of the tangent plane at (1, 1, 3) is
\[-2(x - 1) - 4(y - 1) + 4(z - 3) = 0\]
or
\[x + 2y - 2z + 3 = 0\]

**Exercises**

1–4 Find an equation of the tangent plane to the given surface at the specified point.

1. \( z = 4x^2 - y^2 + 2y, \quad (-1, 2, 4) \)
2. \( z = e^{x+y}, \quad (1, -1, 1) \)
3. \( z = \sqrt[4]{4 - x^2 - 2y^2}, \quad (1, -1, 1) \)
4. \( z = y \ln x, \quad (1, 4, 0) \)

5–6 Graph the surface and the tangent plane at the given point. (Choose the domain and viewpoint so that you get a good view of both the surface and the tangent plane.) Then zoom in until the surface and the tangent plane become indistinguishable.

5. \( z = x^2 + xy + 3y^2, \quad (1, 1, 5) \)
6. \( z = \sqrt{x - y}, \quad (5, 1, 2) \)

7–8 Draw the graph of \( f \) and its tangent plane at the given point. (Use your computer algebra system both to compute the partial derivatives and to graph the surface and its tangent plane.) Then zoom in until the surface and the tangent plane become indistinguishable.

7. \( f(x, y) = e^{-x^2+y^2} \sin(x + \cos^2 y), \quad (2, 3, f(2, 3)) \)
8. \( f(x, y) = \sqrt{1 + 4x^2 + 4y^2} / (1 + x^2 + y^2), \quad (1, 1, 1) \)

9–12 Explain why the function is differentiable at the given point. Then find the linearization \( L(x, y) \) of the function at that point.

9. \( f(x, y) = x \sqrt{y}, \quad (1, 4) \)
10. \( f(x, y) = x/y, \quad (6, 3) \)
11. \( f(x, y) = \tan^{-1}(x + 2y), \quad (1, 0) \)
12. \( f(x, y) = \sin(2x + 3y), \quad (-3, 2) \)
13. Find the linear approximation of the function \( f(x, y) = \sqrt{20 - x^2 - 7y^2} \) at \( (2, 1) \) and use it to approximate \( f(1.95, 1.08) \).
14. Find the linear approximation of the function \( f(x, y) = \ln(x - 3y) \) at \( (7, 2) \) and use it to approximate \( f(6.9, 2.06) \). Illustrate by graphing \( f \) and the tangent plane.
15. Find the linear approximation of the function \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \) at \( (3, 2, 6) \) and use it to approximate the number \( \sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} \).
16. The wave heights \( h \) in the open sea depend on the speed \( v \) of the wind and the length of time \( t \) that the wind has been blowing at that speed. Values of the function \( h = f(v, t) \) are recorded in the following table.

<table>
<thead>
<tr>
<th>( v )</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>15</td>
<td>4</td>
</tr>
<tr>
<td>20</td>
<td>5</td>
</tr>
<tr>
<td>30</td>
<td>9</td>
</tr>
<tr>
<td>40</td>
<td>14</td>
</tr>
<tr>
<td>50</td>
<td>19</td>
</tr>
<tr>
<td>60</td>
<td>24</td>
</tr>
</tbody>
</table>

Use the table to find a linear approximation to the wave height function when \( v \) is near 40 knots and \( t \) is near 20 hours. Then estimate the wave heights when the wind has been blowing for 24 hours at 43 knots.
17. Use the table in Example 3 to find a linear approximation to the heat index function when the temperature is near 94 °F and the relative humidity is near 80%. Then estimate the heat index when the temperature is 95 °F and the relative humidity is 78%.

18. The wind-chill index $I$ is the perceived temperature when the actual temperature is $T$ and the wind speed is $v$, so we can write $I = f(T, v)$. The following table of values is an excerpt from a table compiled by the National Atmospheric and Oceanic Administration.

<table>
<thead>
<tr>
<th>Actual temperature (°C)</th>
<th>Wind speed (km/h)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
</tr>
<tr>
<td>20</td>
<td>18</td>
</tr>
<tr>
<td>16</td>
<td>14</td>
</tr>
<tr>
<td>12</td>
<td>9</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
</tr>
</tbody>
</table>

Use the table to find a linear approximation to the wind chill index function when $T$ is near 16 °C and $v$ is near 30 km/h. Then estimate the wind chill index when the temperature is 14 °C and the wind speed is 27 km/h.

19–22 ■ Find the differential of the function.

19. $u = e^x \sin y$
20. $v = y \cos x y$
21. $w = \sqrt{x^2 + y^2 + z^2}$
22. $u = r / (s + 2t)$

23. If $z = 5x^2 + y^2$ and $(x, y)$ changes from $(1, 2)$ to $(1.05, 2.1)$, compare the values of $\Delta z$ and $dz$.

24. If $z = x^2 - xy + 3y^2$ and $(x, y)$ changes from $(3, -1)$ to $(2.96, -0.95)$, compare the values of $\Delta z$ and $dz$.

25. The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of at most 0.1 cm in each. Use differentials to estimate the maximum error in calculating the area of the rectangle.

26. The dimensions of a closed rectangular box are measured as 80 cm, 60 cm, and 50 cm, respectively, with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.

27. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.

28. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

29. A boundary stripe 3 in. wide is painted around a rectangle whose dimensions are 100 ft by 200 ft. Use differentials to approximate the number of square feet of paint in the stripe.

30. The pressure, volume, and temperature of a mole of an ideal gas are related by the equation $\frac{PV}{T} = 8.317$, where $P$ is measured in kilopascals, $V$ in liters, and $T$ in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.

31. If $R$ is the total resistance of three resistors, connected in parallel, with resistances $R_1$, $R_2$, $R_3$, then

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

If the resistances are measured in ohms as $R_1 = 25 \Omega$, $R_2 = 40 \Omega$, and $R_3 = 50 \Omega$, with a possible error of 0.5% in each case, estimate the maximum error in the calculated value of $R$.

32. Four positive numbers, each less than 50, are rounded to the first decimal place and then multiplied together. Use differentials to estimate the maximum possible error in the computed product that might result from the rounding.

33–36 ■ Find an equation of the tangent plane to the given parametric surface at the specified point. Use a computer to graph the surface and the tangent plane.

33. $x = u + v$, $y = 3u^2$, $z = u - v$; $(2, 3, 0)$
34. $x = u^2$, $y = u - v^2$, $z = v^3$; $(1, 0, 1)$
35. $r(u, v) = u\cos v \hat{i} + u\sin v \hat{j} + v \hat{k}$; $(0, 0, 0)$
36. $r(u, v) = (u + v)\hat{i} + u\cos v \hat{j} + v \sin u \hat{k}$; $(1, 1, 0)$

37–38 ■ Show that the function is differentiable by finding values of $e_1$ and $e_2$ that satisfy Definition 7.

37. $f(x, y) = x^2 + y^2$
38. $f(x, y) = xy - 5y^2$

39. Prove that if $f$ is a function of two variables that is differentiable at $(a, b)$, then $f$ is continuous at $(a, b)$. [Hint: Show that $\lim_{(\Delta x, \Delta y) \to (0, 0)} f(a + \Delta x, b + \Delta y) = f(a, b)$.]

40. (a) The function

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

was graphed in Figure 4. Show that $f(0, 0)$ and $f(0, 0)$ both exist but $f$ is not differentiable at $(0, 0)$. [Hint: Use the result of Exercise 39.]

(b) Explain why $f_x$ and $f_y$ are not continuous at $(0, 0)$. 

Again, a version of the **Implicit Function Theorem** gives conditions under which our assumption is valid. If \( F \) is defined within a sphere containing \((a, b, c)\), where \( F(a, b, c) = 0, F_x(a, b, c) \neq 0, \) and \( F_y, F_z \) are continuous inside the sphere, then the equation \( F(x, y, z) = 0 \) defines \( z \) as a function of \( x \) and \( y \) near the point \((a, b, c)\) and the partial derivatives of this function are given by (7).

**EXAMPLE 9** Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) if \( x^3 + y^3 + z^3 + 6xyz = 1 \).

**SOLUTION** Let \( F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1 \). Then, from Equations 7, we have

\[
\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}
\]

\[
\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}
\]

\[\text{The solution to Example 9 should be compared to the one in Example 4 in Section 11.3.}\]

### Exercises

1–4 Use the Chain Rule to find \( dz/dt \) or \( dw/dt \).

1. \( z = \sin x \cos y, \quad x = \pi t, \quad y = \sqrt{t} \)
2. \( z = x \ln(x + 2y), \quad x = \sin t, \quad y = \cos t \)
3. \( w = xe^{x/y}, \quad x = t^2, \quad y = 1 - t, \quad z = 1 + 2t \)
4. \( w = xy + yz^2, \quad x = e^t, \quad y = e^t \sin t, \quad z = e^t \cos t \)

5–8 Use the Chain Rule to find \( \partial z/\partial s \), \( \partial z/\partial t \).

5. \( z = x^3 + xy + y^3, \quad x = s + t, \quad y = st \)
6. \( z = x/y, \quad x = se^t, \quad y = 1 + se^{-t} \)
7. \( z = e^t \cos \theta, \quad r = st, \quad \theta = \sqrt{s^2 + t^2} \)
8. \( z = \sin \alpha \tan \beta, \quad \alpha = 3s + t, \quad \beta = s - t \)

9. If \( z = f(x, y) \), where \( x = g(t), \quad y = h(t), \quad g(3) = 2, \quad g'(3) = 5, \quad h(3) = 7, \quad h'(3) = -4, \quad f_1(2, 7) = 6, \quad f_2(2, 7) = -8 \), find \( dz/dt \) when \( t = 3 \).

10. Let \( W(s, t) = F(u(s, t), v(s, t)) \), where \( u(1, 0) = 2, \quad u_1(1, 0) = -2, \quad u(1, 0) = 6, \quad v_1(1, 0) = 3, \quad v(1, 0) = 5, \quad v_1(1, 0) = 4, \quad F_u(2, 3) = -1, \quad \) and \( F_v(2, 3) = 10 \). Find \( W_1(1, 0) \) and \( W(1, 0) \).

11–14 Use a tree diagram to write out the Chain Rule for the given case. Assume all functions are differentiable.

11. \( u = f(x, y), \quad \) where \( x = x(r, s, t), \quad y = y(r, s, t) \)
12. \( w = f(x, y, z), \quad \) where \( x = x(t, u), \quad y = y(t, u), \quad z = z(t, u) \)
13. \( v = f(p, q, r), \quad \) where \( p = p(x, y, z), \quad q = q(x, y, z), \quad r = r(x, y, z) \)

14. \( u = f(s, t), \quad \) where \( s = s(w, x, y, z), \quad t = t(w, x, y, z) \)

15–19 Use the Chain Rule to find the indicated partial derivatives.

15. \( w = x^2 + y^2 + z^2, \quad x = st, \quad y = s \cos t, \quad z = s \sin t; \quad \frac{\partial w}{\partial s}, \quad \frac{\partial w}{\partial t} \) when \( s = 1, \quad t = 0 \)
16. \( u = xy + yz + zx, \quad x = st, \quad y = e^t, \quad z = t^2; \quad \frac{\partial u}{\partial s}, \quad \frac{\partial u}{\partial t} \) when \( s = 0, \quad t = 1 \)
17. \( z = y^3 \tan x, \quad x = t^2 uv, \quad y = u + v^2; \quad \frac{\partial z}{\partial u}, \quad \frac{\partial z}{\partial v}, \quad \frac{\partial z}{\partial t} \) when \( t = 2, \quad u = 1, \quad v = 0 \)
18. \( z = \frac{x}{y}, \quad x = re^t, \quad y = rse^t; \quad \frac{\partial z}{\partial r}, \quad \frac{\partial z}{\partial s}, \quad \frac{\partial z}{\partial t} \) when \( r = 1, \quad s = 2, \quad t = 0 \)
19. \( u = x + y, \quad y + z, \quad x = p + r + t, \quad y = p - r + t, \quad z = p + r - t; \quad \frac{\partial u}{\partial p}, \quad \frac{\partial u}{\partial q}, \quad \frac{\partial u}{\partial r} \)

20–22 Use Equation 6 to find \( dy/dx \).

20. \( y^3 + x^2 y^3 = 1 + ye^z \)
21. \( \cos(x - y) = xe^y \)
22. \( \sin x + \cos y = \sin x \cos y \)

23–26 Use Equations 7 to find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \).

23. \( xy^2 + yz^2 + zx^2 = 3 \)
24. \( xyz = \cos(x + y + z) \)
25. \( x^2 + yz + z^2 = 0 \)
26. \( \ln(x + yz) = 1 + xy^2 z^3 \)

27. The temperature at a point \((x, y)\) is \( T(x, y) \), measured in degrees Celsius. A bug crawls so that its position after \( t \) seconds is given by \( x = \sqrt{1 + t}, y = 2 + \frac{1}{t} \), where \( x \) and \( y \) are measured in centimeters. The temperature function satisfies \( T_x(2, 3) = 4 \) and \( T_y(2, 3) = 3 \). How fast is the temperature rising on the bug’s path after 3 seconds?

28. Wheat production in a given year, \( W \), depends on the average temperature \( T \) and the annual rainfall \( R \). Scientists estimate that the average temperature is rising at a rate of 0.15 °C/year and rainfall is decreasing at a rate of 0.1 cm/year. They also estimate that, at current production levels, \( \frac{\partial W}{\partial T} = -2 \) and \( \frac{\partial W}{\partial R} = 8 \).

(a) What is the significance of the signs of these partial derivatives?

(b) Estimate the current rate of change of wheat production, \( dW/dt \).

29. The speed of sound traveling through ocean water with salinity 35 parts per thousand has been modeled by the equation

\[
C = 1449.2 + 4.67 T - 0.0557 T^2 + 0.00029 T^3 + 0.016 D
\]

where \( C \) is the speed of sound (in meters per second), \( T \) is the temperature (in degrees Celsius), and \( D \) is the depth below the ocean surface (in meters). A scuba diver began a leisurely dive into the ocean water, the diver’s depth and surrounding water temperature over time are recorded in the following graphs. Estimate the rate of change (with respect to time) of the speed of sound through the ocean water experienced by the diver 20 minutes into the dive. What are the units?

30. The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in. and the height is 140 in.?

31. The length \( \ell \), width \( w \), and height \( h \) of a box change with time. At a certain instant the dimensions are \( \ell = 1 \) m and \( w = h = 2 \) m, and \( \ell \) and \( w \) are increasing at a rate of 2 m/s while \( h \) is decreasing at a rate of 3 m/s. At that instant find the rates at which the following quantities are changing.

(a) The volume

(b) The surface area

(c) The length of a diagonal

32. The voltage \( V \) in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance \( R \) is slowly increasing as the resistor heats up. Use Ohm’s Law, \( V = IR \), to find how the current \( I \) is changing at the moment when \( R = 400 \) \( \Omega \), \( I = 0.08 \) A, \( dV/dt = -0.01 \) V/s, and \( dR/dt = 0.03 \) \( \Omega/s \).

33. The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05 kPa/s and the temperature is increasing at a rate of 0.15 K/s. Use the equation in Example 2 to find the rate of change of the volume when the pressure is 20 kPa and the temperature is 320 K.

34. Car A is traveling north on Highway 16 and Car B is traveling west on Highway 83. Each car is approaching the intersection of these highways. At a certain moment, car A is 0.3 km from the intersection and traveling at 90 km/h while car B is 0.4 km from the intersection and traveling at 80 km/h. How fast is the distance between the cars changing at that moment?

35–38 Assume that all the given functions are differentiable.

35. If \( z = f(x, y) \), where \( x = r \cos \theta \) and \( y = r \sin \theta \), (a) find \( \frac{\partial z}{\partial r} \) and \( \frac{\partial z}{\partial \theta} \) and (b) show that

\[
\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 = \left( \frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial z}{\partial \theta} \right)^2
\]

36. If \( u = f(x, y) \), where \( x = e^s \cos t \) and \( y = e^s \sin t \), show that

\[
\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 = e^{2s} \left[ \left( \frac{\partial u}{\partial s} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 \right]
\]

37. If \( z = f(x - y) \), show that \( \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \).

38. If \( z = f(x, y) \), where \( x = s + t \) and \( y = s - t \), show that

\[
\left( \frac{\partial z}{\partial x} \right)^2 - \left( \frac{\partial z}{\partial y} \right)^2 = \frac{\partial z}{\partial x} \frac{\partial z}{\partial t}
\]

39–44 Assume that all the given functions have continuous second-order partial derivatives.

39. Show that any function of the form

\[
z = f(x + at) + g(x - at)
\]

is a solution of the wave equation

\[
\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}
\]

[Hint: Let \( u = x + at, v = x - at \).

40. Assume that all the given functions have continuous second-order partial derivatives.

41. Show that any function of the form

\[
z = f(x + at) + g(x - at)
\]

is a solution of the wave equation

\[
\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}
\]

[Hint: Let \( u = x + at, v = x - at \).

42. Assume that all the given functions have continuous second-order partial derivatives.

43. Show that any function of the form

\[
z = f(x + at) + g(x - at)
\]

is a solution of the wave equation

\[
\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}
\]

[Hint: Let \( u = x + at, v = x - at \).

44. Assume that all the given functions have continuous second-order partial derivatives.
Directional Derivatives and the Gradient Vector

The weather map in Figure 1 shows a contour map of the temperature function $T(x, y)$ for the states of California and Nevada at 3:00 p.m. on October 10, 1997. The level curves, or isothermals, join locations with the same temperature. The partial derivative $T_x$ at a location such as Reno is the rate of change of temperature with respect to distance if we travel east from Reno; $T_y$ is the rate of change of temperature if we travel north. But what if we want to know the rate of change of temperature when we travel southeast (toward Las Vegas), or in some other direction? In this section we introduce a type of derivative, called a directional derivative, that enables us to find the rate of change of a function of two or more variables in any direction.

**FIGURE 1**

Directional Derivatives

Recall that if $z = f(x, y)$, then the partial derivatives $f_x$ and $f_y$ are defined as

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$
Computer algebra systems have commands that plot sample gradient vectors. Each gradient vector $\nabla f(a, b)$ is plotted starting at the point $(a, b)$. Figure 13 shows such a plot (called a gradient vector field) for the function $f(x, y) = x^2 - y^2$ superimposed on a contour map of $f$. As expected, the gradient vectors point “uphill” and are perpendicular to the level curves.

**FIGURE 13**

### Exercises

1. A contour map of barometric pressure (in millibars) is shown for 7:00 A.M. on September 12, 1960, when Hurricane Donna was raging. Estimate the value of the directional derivative of the pressure function at Raleigh, North Carolina, in the direction of the eye of the hurricane. What are the units of the directional derivative?

2. The contour map shows the average annual snowfall (in inches) near Lake Michigan. Estimate the value of the directional derivative of this snowfall function at Muskegon, Michigan, in the direction of Ludington. What are the units?

3. A table of values for the wind chill index $I = f(T, v)$ is given in Exercise 3 on page 776. Use the table to estimate the value of $D_u f(16, 30)$, where $u = (i + j)/\sqrt{2}$.

4–6 Find the directional derivative of $f$ at the given point in the direction indicated by the angle $\theta$.

4. $f(x, y) = \sin(x + 2y)$, $(4, -2)$, $\theta = 3\pi/4$

5. $f(x, y) = \sqrt{5x - 4y}$, $(4, 1)$, $\theta = -\pi/6$

6. $f(x, y) = xe^{-y^2}$, $(5, 0)$, $\theta = \pi/2$
7–10
(a) Find the gradient of \( f \).
(b) Evaluate the rate of change of \( f \) at \( P \) in the direction of the vector \( \mathbf{u} \).
(c) Find the rate of change of \( f \) at \( P \) in the direction of the vector \( \mathbf{u} \).

7. \( f(x, y) = 5xy^2 - 4x^3y \), \( P(1, 2) \), \( \mathbf{u} = \left\langle \frac{1}{2}, 1 \right\rangle \)
8. \( f(x, y) = y \ln x \), \( P(1, -3) \), \( \mathbf{u} = \left\langle -\frac{1}{3}, \frac{1}{3} \right\rangle \)
9. \( f(x, y, z) = x^y + z \), \( P(1, -2, 1) \), \( \mathbf{u} = \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \)
10. \( f(x, y, z) = xy + yz^2 + xz^3 \), \( P(2, 0, 3) \), \( \mathbf{u} = \left\langle -\frac{2}{3}, -\frac{1}{3}, \frac{1}{3} \right\rangle \)

11–15
Find the directional derivative of the function at the given point in the direction of the vector \( \mathbf{v} \).
11. \( f(x, y) = 1 + 2x\sqrt{y} \), \( (3, 4) \), \( \mathbf{v} = \left\langle 4, -3 \right\rangle \)
12. \( g(r, \theta) = e^{-r} \sin \theta \), \( (0, \pi/3) \), \( \mathbf{v} = 3 \mathbf{i} - 2 \mathbf{j} \)
13. \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \), \( (1, 2, -2) \), \( \mathbf{v} = \left\langle -6, 6, -3 \right\rangle \)
14. \( f(x, y, z) = x/y + z \), \( (4, 1, 1) \), \( \mathbf{v} = \left\langle 1, 2, 3 \right\rangle \)
15. \( g(x, y, z) = x \tan^{-1}(y/z) \), \( (1, 2, -2) \), \( \mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k} \)

16. Use the figure to estimate \( D_\mathbf{u} f(2, 2) \).

17. Find the directional derivative of \( f(x, y) = \sqrt{xy} \) at \( P(2, 8) \) in the direction of \( Q(5, 4) \).
18. Find the directional derivative of \( f(x, y, z) = x^2 + y^2 + z^2 \) at \( P(2, 1, 3) \) in the direction of the origin.

19–22
Find the maximum rate of change of \( f \) at the given point and the direction in which it occurs.
19. \( f(x, y) = \sin(xy) \), \( (1, 0) \)
20. \( f(x, y) = \ln(x^2 + y^2) \), \( (1, 2) \)
21. \( f(x, y, z) = x + y/z \), \( (4, 3, -1) \)
22. \( f(x, y, z) = x^2 + y^3 + z^4 \), \( (1, 1, 1) \)

23. (a) Show that a differentiable function \( f \) decreases most rapidly at \( \mathbf{x} \) in the direction opposite to the gradient vector, that is, in the direction of \( -\nabla f(\mathbf{x}) \).
(b) Use the result of part (a) to find the direction in which the function \( f(x, y) = x^4y - x^3y^3 \) decreases fastest at the point \( (2, -3) \).

24. Find the directions in which the directional derivative of \( f(x, y) = x^2 + \sin xy \) at the point \( (1, 0) \) has the value 1.
25. Find all points at which the direction of fastest change of the function \( f(x, y) = x^2 + y^2 - 2x - 4y \) is \( \mathbf{i} + \mathbf{j} \).
26. Near a buoy, the depth of a lake at the point with coordinates \( (x, y) \) is \( z = 200 + 0.02x^2 - 0.001y^3 \), where \( x, y, \) and \( z \) are measured in meters. A fisherman in a small boat starts at the point \((80, 60)\) and moves toward the buoy, which is located at \((0, 0)\). Is the water under the boat getting deeper or shallower when he departs? Explain.

27. The temperature \( T \) in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point \((1, 2, 2)\) is \( 120^\circ \).
(a) Find the rate of change of \( T \) at \((1, 2, 2)\) in the direction toward the point \((2, 1, 3)\).
(b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points toward the origin.

28. The temperature at a point \((x, y, z)\) is given by
\[ T(x, y, z) = 200e^{-x^2-y^2-z^2} \]
where \( T \) is measured in \(^\circ\)C and \( x, y, z \) in meters.
(a) Find the rate of change of temperature at the point \( P(2, -1, 2) \) in the direction toward the point \((3, -3, 3)\).
(b) In which direction does the temperature increase fastest at \( P \)?
(c) Find the maximum rate of increase at \( P \).

29. Suppose that over a certain region of space the electrical potential \( V \) is given by
\[ V(x, y, z) = 5x^2 - 3xy + xyz \]
(a) Find the rate of change of the potential at \( P(3, 4, 5) \) in the direction of the vector \( \mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k} \).
(b) In which direction does \( V \) change most rapidly at \( P \)?
(c) What is the maximum rate of change at \( P \)?

30. Suppose that you are climbing a hill whose shape is given by the equation \( z = 1000 - 0.01x^2 - 0.02y^2 \) and you are standing at a point with coordinates \((60, 100, 764)\).
(a) In which direction should you proceed initially in order to reach the top of the hill fastest?
(b) If you climb in that direction, at what angle above the horizontal will you be climbing initially?

31. Let \( f \) be a function of two variables that has continuous partial derivatives and consider the points \( A(1, 3), B(3, 3), C(1, 7) \), and \( D(6, 15) \). The directional derivative of \( f \) at \( A \) in the direction of the vector \( AB \) is 3 and the directional derivative of \( f \) at \( A \) in the direction of \( AC \) is 26. Find the directional derivative of \( f \) at \( A \) in the direction of the vector \( AD \).
32. For the given contour map draw the curves of steepest ascent starting at \( P \) and at \( Q \).

![](image)

33. Show that the operation of taking the gradient of a function has the given property. Assume that \( u \) and \( v \) are differentiable functions of \( x \) and \( y \) and \( a, b \) are constants.

(a) \( \nabla (au + bv) = a \nabla u + b \nabla v \)

(b) \( \nabla (uv) = v \nabla u + u \nabla v \)

(c) \( \nabla \left( \frac{u}{v} \right) = \frac{v \nabla u - u \nabla v}{v^2} \)

(d) \( \nabla u^n = nu^{n-1} \nabla u \)

34. Sketch the gradient vector \( \nabla f(4, 6) \) for the function \( f \) whose level curves are shown. Explain how you chose the direction and length of this vector.

![](image)

35–38 Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

35. \( x^2 + 2y^2 + 3z^2 = 21 \), \((4, -1, 1)\)

36. \( x = y^3 + z^3 - 2 \), \((-1, 1, 0)\)

37. \( z + 1 = xe^z \cos z \), \((1, 0, 0)\)

38. \( xe^{y^2} = 1 \), \((1, 0, 5)\)

39–40 Use a computer to graph the surface, the tangent plane, and the normal line on the same screen. Choose the domain carefully so that you avoid extraneous vertical planes. Choose the viewpoint so that you get a good view of all three objects.

39. \( xy + yz + zx = 3 \), \((1, 1, 1)\)

40. \( xyz = 6 \), \((1, 2, 3)\)

41. If \( f(x, y) = x^2 + 4y^2 \), find the gradient vector \( \nabla f(2, 1) \) and use it to find the tangent line to the level curve \( f(x, y) = 8 \) at the point \((2, 1)\). Sketch the level curve, the tangent line, and the gradient vector.

42. If \( g(x, y) = x - y^2 \), find the gradient vector \( \nabla g(3, -1) \) and use it to find the tangent line to the level curve \( g(x, y) = 2 \) at the point \((3, -1)\). Sketch the level curve, the tangent line, and the gradient vector.

43. Show that the equation of the tangent plane to the ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \) at the point \((x_0, y_0, z_0)\) can be written as

\[
\frac{x-x_0}{a^2} = \frac{y-y_0}{b^2} = \frac{z-z_0}{c^2} = 1
\]

44. Find the points on the ellipsoid \( x^2 + 2y^2 + 3z^2 = 9 \) where the tangent plane is parallel to the plane \( 3x - y + 3z = 1 \).

45. Find the points on the hyperboloid \( x^2 - y^2 + 2z^2 = 1 \) where the normal line is parallel to the line that joins the points \((3, -1, 0)\) and \((5, 3, 6)\).

46. Show that the ellipsoid \( 3x^2 + 2y^2 + z^2 = 9 \) and the sphere \( x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0 \) are tangent to each other at the point \((1, 1, 2)\). (This means that they have a common tangent plane at the point.)

47. Show that the sum of the \( x \)-, \( y \)-, and \( z \)-intercepts of any tangent plane to the surface \( \sqrt{x} + \sqrt{y} + \sqrt{z} = c \) is a constant.

48. Show that every normal line to the sphere \( x^2 + y^2 + z^2 = r^2 \) passes through the center of the sphere.

49. Find parametric equations for the tangent plane to the curve of intersection of the paraboloid \( z = x^2 + y^2 \) and the ellipsoid \( 4x^2 + y^2 + z^2 = 9 \) at the point \((-1, 1, 2)\).

50. (a) The plane \( y + z = 3 \) intersects the cylinder \( x^2 + y^2 = 5 \) in an ellipse. Find parametric equations for the tangent line to this ellipse at the point \((1, 2, 1)\).

(b) Graph the cylinder, the plane, and the tangent line on the same screen.

51. (a) Two surfaces are called orthogonal at a point of intersection if their normal lines are perpendicular at that point. Show that surfaces with equations \( F(x, y, z) = 0 \) and \( G(x, y, z) = 0 \) are orthogonal at a point \( P \) where \( \nabla F \neq 0 \) and \( \nabla G \neq 0 \) if and only if

\[
F_x G_z + F_y G_y + F_z G_z = 0
\]

at \( P \).

(b) Use part (a) to show that the surfaces \( z^2 = x^2 + y^2 \) and \( x^2 + y^2 + z^2 = r^2 \) are orthogonal at every point of intersection. Can you see why this is true without using calculus?

52. (a) Show that the function \( f(x, y) = \sqrt{xy} \) is continuous and the partial derivatives \( f_x \) and \( f_y \) exist at the origin but the directional derivatives in all other directions do not exist.
53. Suppose that the directional derivatives of \( f(x, y) \) are known at a given point in two nonparallel directions given by unit vectors \( \mathbf{u} \) and \( \mathbf{v} \). Is it possible to find \( \nabla f \) at this point? If so, how would you do it?

54. Show that if \( z = f(x, y) \) is differentiable at \( x_0 = (x_0, y_0) \), then
\[
\lim_{x \to x_0} \frac{f(x) - f(x_0) - \nabla f(x_0) \cdot (x - x_0)}{|x - x_0|} = 0
\]

[Hint: Use Definition 11.4.7 directly.]

---

**Maximum and Minimum Values**

As we saw in Chapter 4, one of the main uses of ordinary derivatives is in finding maximum and minimum values. In this section we see how to use partial derivatives to locate maxima and minima of functions of two variables. In particular, in Example 6 we will see how to maximize the volume of a box without a lid if we have a fixed amount of cardboard to work with.

1. **Definition** A function of two variables has a local maximum at \((a, b)\) if \( f(x, y) \leq f(a, b) \) when \((x, y)\) is near \((a, b)\). [This means that \( f(x, y) \leq f(a, b) \) for all points \((x, y)\) in some disk with center \((a, b)\).] The number \( f(a, b) \) is called a local maximum value. If \( f(x, y) \geq f(a, b) \) when \((x, y)\) is near \((a, b)\), then \( f(a, b) \) is a local minimum value.

If the inequalities in Definition 1 hold for all points \((x, y)\) in the domain of \( f \), then \( f \) has an absolute maximum (or absolute minimum) at \((a, b)\).

The graph of a function with several maxima and minima is shown in Figure 1. You can think of the local maxima as mountain peaks and the local minima as valley bottoms.

2. **Theorem** If \( f \) has a local maximum or minimum at \((a, b)\) and the first-order partial derivatives of \( f \) exist there, then \( f_x(a, b) = 0 \) and \( f_y(a, b) = 0 \).

**Proof** Let \( g(x) = f(x, b) \). If \( f \) has a local maximum (or minimum) at \((a, b)\), then \( g \) has a local maximum (or minimum) at \( a \), so \( g'(a) = 0 \) by Fermat’s Theorem (see Theorem 4.2.4). But \( g'(a) = f_x(a, b) \) (see Equation 11.3.1) and so \( f_x(a, b) = 0 \).

Similarly, by applying Fermat’s Theorem to the function \( G(y) = f(a, y) \), we obtain \( f_y(a, b) = 0 \).

Notice that the conclusion of Theorem 2 can be stated in the notation of the gradient vector as \( \nabla f(a, b) = 0 \). If we put \( f_x(a, b) = 0 \) and \( f_y(a, b) = 0 \) in the equation of a tangent plane (Equation 11.4.2), we get \( z = z_0 \). Thus, the geometric interpretation of Theorem 2 is that if the graph of \( f \) has a tangent plane at a local maximum or minimum, then the tangent plane must be horizontal.

A point \((a, b)\) is called a critical point (or stationary point) of \( f \) if \( f_x(a, b) = 0 \) and \( f_y(a, b) = 0 \), or if one of these partial derivatives does not exist. Theorem 2 says that if \( f \) has a local maximum or minimum at \((a, b)\), then \((a, b)\) is a critical point of \( f \). However, as in single-variable calculus, not all critical points give rise to maxima or minima. At a critical point, a function could have a local maximum or a local minimum or neither.
This is a decreasing function of \( y \), so its maximum value is \( f(3, 0) = 9 \) and its minimum value is \( f(3, 2) = 1 \). On \( L_3 \) we have \( y = 2 \) and

\[
f(x, 2) = x^2 - 4x + 4 \quad 0 \leq x \leq 3
\]

By the methods of Chapter 4, or simply by observing that \( f(x, 2) = (x - 2)^2 \), we see that the minimum value of this function is \( f(2, 2) = 0 \) and the maximum value is \( f(0, 2) = 4 \). Finally, on \( L_4 \) we have \( x = 0 \) and

\[
f(0, y) = 2y \quad 0 \leq y \leq 2
\]

with maximum value \( f(0, 2) = 4 \) and minimum value \( f(0, 0) = 0 \). Thus, on the boundary, the minimum value of \( f \) is 0 and the maximum is 9.

In step 3 we compare these values with the value \( f(1, 1) = 1 \) at the critical point and conclude that the absolute maximum value of \( f \) on \( D \) is \( f(3, 0) = 9 \) and the absolute minimum value is \( f(0, 0) = f(2, 2) = 0 \). Figure 13 shows the graph of \( f \).

---

**Exercises**

1. Suppose \((1, 1)\) is a critical point of a function \( f \) with continuous second derivatives. In each case, what can you say about \( f \)?
   (a) \( f_x(1, 1) = 4 \), \( f_y(1, 1) = 1 \), \( f_{xy}(1, 1) = 2 \)
   (b) \( f_x(1, 1) = 4 \), \( f_y(1, 1) = 3 \), \( f_{xy}(1, 1) = 2 \)

2. Suppose \((0, 2)\) is a critical point of a function \( g \) with continuous second derivatives. In each case, what can you say about \( g \)?
   (a) \( g_x(0, 2) = -1 \), \( g_y(0, 2) = 6 \), \( g_{xy}(0, 2) = 1 \)
   (b) \( g_x(0, 2) = -1 \), \( g_y(0, 2) = 2 \), \( g_{xy}(0, 2) = -8 \)
   (c) \( g_x(0, 2) = 4 \), \( g_y(0, 2) = 6 \), \( g_{xy}(0, 2) = 9 \)

3-4 Use the level curves in the figure to predict the location of the critical points of \( f \) and whether \( f \) has a saddle point or a local maximum or minimum at each of those points. Explain your reasoning. Then use the Second Derivatives Test to confirm your predictions.

3. \( f(x, y) = 4 + x^3 + y^3 - 3xy \)

4. \( f(x, y) = 3x - x^3 - 2y^3 + y^4 \)

5-14 Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

5. \( f(x, y) = 9 - 2x + 4y - x^2 - 4y^2 \)
6. \( f(x, y) = x^3y + 12x^2 - 8y \)
7. \( f(x, y) = x^2 + y^2 + x^2y + 4 \)
8. \( f(x, y) = e^{x^2-y^2} \)
9. \( f(x, y) = xy - 2x - y \)
10. \( f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2 \)
11. \( f(x, y) = e^{x} \cos y \)
12. \( f(x, y) = x^2 + y^2 + \frac{1}{x^3 y^2} \)
13. \( f(x, y) = x \sin y \)
14. \( f(x, y) = (2x - x^2)(2y - y^2) \)

---

**Figure 13**

\( f(x, y) = x^3 - 2xy + 2y \)
15–18 Use a graph and/or level curves to estimate the local maximum and minimum values and saddle point(s) of the function. Then use calculus to find these values precisely.

15. \( f(x, y) = 3x^2y - y^3 - 3x^2 - 3y^2 + 2 \)

16. \( f(x, y) = xye^{-x^2-y^2} \)

17. \( f(x, y) = \sin x + \sin y + \sin(x + y), \quad 0 \leq x \leq 2\pi, \quad 0 \leq y \leq 2\pi \)

18. \( f(x, y) = \sin x + \sin y + \cos(x + y), \quad 0 \leq x \leq \pi/4, \quad 0 \leq y \leq \pi/4 \)

19–22 Use a graphing device as in Example 4 (or Newton’s method or a rootfinder) to find the critical points of correct to three decimal places. Then classify the critical points and find the highest or lowest points on the graph.

19. \( f(x, y) = x^2 - 5x^2 + y^2 + 3x + 2 \)

20. \( f(x, y) = 5 - 10xy - 4x^2 + 3y - y^4 \)

21. \( f(x, y) = 2x + 4x^2 - y^2 + 2xy^2 - x^4 - y^4 \)

22. \( f(x, y) = e^x + y^4 - x^3 + 4 \cos y \)

23–28 Find the absolute maximum and minimum values of \( f \) on the set \( D \).

23. \( f(x, y) = 1 + 4x - 5y, \quad D \) is the closed triangular region with vertices \((0, 0), (2, 0), \) and \((0, 3)\)

24. \( f(x, y) = 3xy - x - 2y, \quad D \) is the closed triangular region with vertices \((1, 0), (5, 0), \) and \((1, 4)\)

25. \( f(x, y) = x^2 + y^2 + x^2y + 4, \quad D = \{(x, y) \mid |x| \leq 1, |y| \leq 1\} \)

26. \( f(x, y) = 4x + 6y - x^2 - y^3, \quad D = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 5\} \)

27. \( f(x, y) = 1 + xy - x - y, \quad D \) is the region bounded by the parabola \( y = x^2 \) and the line \( y = 4 \)

28. \( f(x, y) = xy^2, \quad D = \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 3\} \)

29. For functions of one variable it is impossible to have two local maxima and no local minimum. But for functions of two variables such functions exist. Show that the function

\( f(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2 \)

has only two critical points, but has local maxima at both of them. Then use a computer to produce a graph with a carefully chosen domain and viewpoint to see how this is possible.

30. If a function of one variable is continuous on an interval and has only one critical number, then a local maximum has to be an absolute maximum. But this is not true for functions of two variables. Show that the function

\( f(x, y) = 3xe^y - x^3 - e^{by} \)

has exactly one critical point, and that \( f \) has a local maximum there that is not an absolute maximum. Then use a computer to produce a graph with a carefully chosen domain and viewpoint to see how this is possible.

31. Find the shortest distance from the point \((2, 1, -1)\) to the plane \( x + y - z = 1 \).

32. Find the point on the plane \( x - y + z = 4 \) that is closest to the point \((1, 2, 3)\).

33. Find the points on the surface \( z^2 = xy + 1 \) that are closest to the origin.

34. Find the points on the surface \( x^2 + y^2 = 1 \) that are closest to the origin.

35. Find three positive numbers whose sum is 100 and whose product is a maximum.

36. Find three positive numbers \( x, y, \) and \( z \) whose sum is 100 such that \( x^2y^2z^2 \) is a maximum.

37. Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid

\[ 9x^2 + 36y^2 + 4z^2 = 36 \]

38. Solve the problem in Exercise 37 for a general ellipsoid

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]

39. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane \( x + 2y + 3z = 6 \).

40. Find the dimensions of the rectangular box with largest volume if the total surface area is given as \( 64 \text{ cm}^2 \).

41. Find the dimensions of a rectangular box of maximum volume such that the sum of the lengths of its 12 edges is a constant \( c \).

42. The base of an aquarium with given volume \( V \) is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of the materials.

43. A cardboard box without a lid is to have a volume of 32,000 cm\(^3\). Find the dimensions that minimize the amount of cardboard used.

44. Three alleles (alternative versions of a gene) \( A, B, \) and \( O \) determine the four blood types \( A(AA \text{ or } AO), B(BB \text{ or } BO), O(OO), \) and \( AB \). The Hardy-Weinberg Law states that the proportion of individuals in a population who carry two different alleles is

\[ p^2 + 2pq + 2pr + 2pq + 2pq + 2pq = P \]

where \( p, q, \) and \( r \) are the proportions of \( A, B, \) and \( O \) in the population. Use the fact that \( p + q + r = 1 \) to show that \( p \) is at most \( \frac{1}{3} \).

45. Suppose that a scientist has reason to believe that two quantities \( x \) and \( y \) are related linearly, that is, \( y = mx + b, \) at
Thus, the line is found by solving these two equations in the two unknowns $m$ and $b$. (See Section 1.2 for a further discussion and applications of the method of least squares.)
EXAMPLE 5 Find the maximum value of the function \( f(x, y, z) = x + 2y + 3z \) on the curve of intersection of the plane \( x - y + z = 1 \) and the cylinder \( x^2 + y^2 = 1 \).

SOLUTION We maximize the function \( f(x, y, z) = x + 2y + 3z \) subject to the constraints \( g(x, y, z) = x - y + z = 1 \) and \( h(x, y, z) = x^2 + y^2 = 1 \). The Lagrange condition is \( \nabla f = \lambda \nabla g + \mu \nabla h \), so we solve the equations

\[
\begin{align*}
1 &= \lambda + 2x\mu \\
2 &= -\lambda + 2y\mu \\
3 &= \lambda \\
x - y + z &= 1 \\
x^2 + y^2 &= 1
\end{align*}
\]

Putting \( \lambda = 3 \) [from (19)] in (17), we get \( 2x\mu = -2 \), so \( x = -1/\mu \). Similarly, (18) gives \( y = 5/(2\mu) \). Substitution in (21) then gives

\[
\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1
\]

and so \( \mu^2 = \frac{29}{2} \), \( \mu = \pm\sqrt{29}/2 \). Then \( x = \pm 2/\sqrt{29} \), \( y = \pm 5/\sqrt{29} \), and, from (20), \( z = 1 - x + y = 1 \pm 7/\sqrt{29} \). The corresponding values of \( f \) are

\[
\pm \frac{2}{\sqrt{29}} + 2 \left( \pm \frac{5}{\sqrt{29}} \right) + 3 \left( 1 \pm \frac{7}{\sqrt{29}} \right) = 3 \pm \sqrt{29}
\]

Therefore, the maximum value of \( f \) on the given curve is \( 3 + \sqrt{29} \). \( \blacksquare \)

### Exercises

1. Pictured are a contour map of \( f \) and a curve with equation \( g(x, y) = 8 \). Estimate the maximum and minimum values of \( f \) subject to the constraint that \( g(x, y) = 8 \). Explain your reasoning.

2. (a) Use a graphing calculator or computer to graph the circle \( x^2 + y^2 = 1 \). On the same screen, graph several curves of the form \( x^2 + y^2 = c \) until you find two that just touch the circle. What is the significance of the values of \( c \) for these two curves?
   (b) Use Lagrange multipliers to find the extreme values of \( f(x, y) = x^2 + y \) subject to the constraint \( x^2 + y^2 = 1 \). Compare your answers with those in part (a).

3-17 Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

3. \( f(x, y) = x^2 - y^3 ; \quad x^2 + y^2 = 1 \)
4. \( f(x, y) = 4x + 6y ; \quad x^2 + y^2 = 13 \)
5. \( f(x, y) = x^2y ; \quad x^2 + 2y^2 = 6 \)
6. \( f(x, y) = x^2 + y^2 ; \quad x^4 + y^4 = 1 \)
7. \( f(x, y, z) = 2x + 6y + 10z; \quad x^2 + y^2 + z^2 = 35 \)
8. \( f(x, y, z) = 8x - 4z; \quad x^2 + 10y^2 + z^2 = 5 \)
9. \( f(x, y, z) = xyz; \quad x^2 + 2y^2 + 3z^2 = 6 \)
10. \( f(x, y, z) = x^2y^2z^2; \quad x^2 + y^2 + z^2 = 1 \)
11. \( f(x, y, z) = x^2 + y^2 + z^2; \quad x^2 + y^2 + z^2 = 1 \)

12. \( f(x, y, z) = x^4 + y^4 + z^4; \quad x^2 + y^2 + z^2 = 1 \)
13. \( f(x, y, z, t) = x + y + z + t; \quad x^2 + y^2 + z^2 + t^2 = 1 \)
14. \( f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n; \quad x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \)
15. \( f(x, y, z) = x + 2y; \quad x + y + z = 1, \quad y^2 + z^2 = 4 \)
16. \( f(x, y, z) = 3x - y - 3z; \quad x + y - z = 0, \quad x^2 + 2z^2 = 1 \)
17. \( f(x, y, z) = x^2 + 2x^2 = 1 \)
18–19 **Find the extreme values of \( f \) on the region described by the inequality.**
18. \( f(x, y) = 2x^2 + 3y^2 - 4x - 5; \quad x^2 + y^2 \leq 16 \)
19. \( f(x, y) = e^{-xy}; \quad x^2 + 2y^2 \leq 1 \)

**CAS 20.** (a) If your computer algebra system plots implicitly defined curves, use it to estimate the minimum and maximum values of \( f(x, y) = x^3 + y^3 + 3xy \) subject to the constraint \((x - 3)^2 + (y - 3)^2 = 9\) by graphical methods.
(b) Solve the problem in part (a) with the aid of Lagrange multipliers. Use your CAS to solve the equations. Compare your answers with those in part (a).

21. The total production \( P \) of a certain product depends on the amount \( L \) of labor used and the amount \( K \) of capital investment. In Sections 11.1 and 11.3 we discussed how the Cobb-Douglas model \( P = bL^\alpha K^{1-\alpha} \) follows from certain economic assumptions, where \( b \) and \( \alpha \) are positive constants and \( \alpha < 1 \). If the cost of a unit of labor is \( m \) and the cost of a unit of capital is \( n \), and the company can spend only \( p \) dollars as its total budget, then maximizing the production \( P \) is subject to the constraint \( mL + nK = p \). Show that the maximum production occurs when

\[
L = \frac{ap}{m} \quad \text{and} \quad K = \left(\frac{1 - \alpha}{\alpha}\right)p
\]

22. Referring to Exercise 21, we now suppose that the production is fixed at \( bL^\alpha K^{1-\alpha} = Q \), where \( Q \) is a constant. What values of \( L \) and \( K \) minimize the cost function \( C(L, K) = mL + nK \)?

23. Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter \( p \) is a square.

24. Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter \( p \) is equilateral. [**Hint:** Use Heron’s formula for the area: \( A = \sqrt{s(s-x)(s-y)(s-z)} \), where \( s = p/2 \) and \( x, y, z \) are the lengths of the sides.]

**25–35** Use Lagrange multipliers to give an alternate solution to the indicated exercise in Section 11.7.

25. Exercise 31
26. Exercise 32
27. Exercise 33
28. Exercise 34
29. Exercise 35
30. Exercise 36
31. Exercise 37
32. Exercise 38
33. Exercise 39
34. Exercise 40
35. Exercise 41

36. Find the maximum and minimum volumes of a rectangular box whose surface area is 1500 cm\(^2\) and whose total edge length is 200 cm.

37. The plane \( x + y + 2z = 2 \) intersects the paraboloid \( z = x^2 + y^2 \) in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

38. The plane \( 4x - 3y + 8z = 5 \) intersects the cone \( z^2 = x^2 + y^2 \) in an ellipse.

(a) Graph the cone, the plane, and the ellipse.
(b) Use Lagrange multipliers to find the highest and lowest points on the ellipse.

**CAS 39–40** Find the maximum and minimum values of \( f \) subject to the given constraints. Use a computer algebra system to solve the system of equations that arises in using Lagrange multipliers. (If your CAS finds only one solution, you may need to use additional commands.)

39. \( f(x, y, z) = ye^{-x}; \quad 9x^2 + 4y^2 + 36z^2 = 36, \quad xy + yz = 1 \)
40. \( f(x, y, z) = x + y + z; \quad x^2 - y^2 = z, \quad x^2 + z^2 = 4 \)

41. (a) Find the maximum value of

\[
f(x_1, x_2, \ldots, x_n) = \sqrt[3]{x_1x_2\cdots x_n}
\]
given that \( x_1, x_2, \ldots, x_n \) are positive numbers and \( x_1 + x_2 + \cdots + x_n = c \), where \( c \) is a constant.
(b) Deduce from part (a) that if \( x_1, x_2, \ldots, x_n \) are positive numbers, then

\[
\sqrt[3]{x_1x_2\cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}
\]

This inequality says that the geometric mean of \( n \) numbers is no larger than the arithmetic mean of the numbers. Under what circumstances are these two means equal to each other?
42. (a) Maximize $\sum_{i=1}^{n} x_i y_i$, subject to the constraints $\sum_{i=1}^{n} x_i^2 = 1$ and $\sum_{i=1}^{n} y_i^2 = 1$.

(b) Put

$$x_i = \frac{a_i}{\sqrt{\sum a_i^2}} \quad \text{and} \quad y_i = \frac{b_i}{\sqrt{\sum b_i^2}}$$

to show that

$$\sum a_i b_i \leq \sqrt{\sum a_i^2} \sqrt{\sum b_i^2}$$

for any numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$. This inequality is known as the Cauchy-Schwarz Inequality.

---

**Rocket Science**

Many rockets, such as the Pegasus XL currently used to launch satellites and the Saturn V that first put men on the Moon, are designed to use three stages in their ascent into space. A large first stage initially propels the rocket until its fuel is consumed, at which point the stage is jettisoned to reduce the mass of the rocket. The smaller second and third stages function similarly in order to place the rocket’s payload into orbit about Earth. (With this design, at least two stages are required in order to reach the necessary velocities, and using three stages has proven to be a good compromise between cost and performance.) Our goal here is to determine the individual masses of the three stages to be designed in such a way as to minimize the total mass of the rocket while enabling it to reach a desired velocity.

For a single-stage rocket consuming fuel at a constant rate, the change in velocity resulting from the acceleration of the rocket vehicle has been modeled by

$$\Delta V = -c \ln \left(1 - \frac{(1-S)M_r}{P + M_r}\right)$$

where $M_r$ is the mass of the rocket engine including initial fuel, $P$ is the mass of the payload, $S$ is a structural factor determined by the design of the rocket (specifically, it is the ratio of the mass of the rocket vehicle without fuel to the total mass of the rocket with payload), and $c$ is the (constant) speed of exhaust relative to the rocket.

Now consider a rocket with three stages and a payload of mass $A$. We will consider outside forces negligible and assume that $c$ and $S$ remain constant for each stage. If $M_i$ is the mass of the $i$th stage, we can initially consider the rocket engine to have mass $M_1$ and its payload to have mass $M_2 + M_3 + A$; the second and third stages can be handled similarly.

1. Show that the velocity attained after all three stages have been jettisoned is given by

$$v_f = c \left[ \ln \left( \frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A} \right) + \ln \left( \frac{M_2 + M_3 + A}{SM_2 + M_3 + A} \right) + \ln \left( \frac{M_3 + A}{SM_3 + A} \right) \right]$$

2. We wish to minimize the total mass $M = M_1 + M_2 + M_3$ of the rocket engine subject to the constraint that the desired velocity $v_f$ from Problem 1 is attained. The method of Lagrange multipliers is appropriate here, but difficult to implement using the current expressions. To simplify, we define variables $N_i$ so that the constraint equation may be expressed as $v_f = c(\ln N_1 + \ln N_2 + \ln N_3)$. Since $M$ is now difficult to express in terms of the $N_i’s$, we wish to use a simpler function that will be minimized at the same place. Show that

\[
\begin{align*}
\frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A} &= \frac{(1-S)N_1}{1 - SN_1} \\
\frac{M_2 + M_3 + A}{M_3 + A} &= \frac{(1-S)N_2}{1 - SN_2} \\
\frac{M_3 + A}{A} &= \frac{(1-S)N_3}{1 - SN_3}
\end{align*}
\]
and conclude that

\[
\frac{M + A}{A} = \frac{(1 - S)N_1N_2N_3}{(1 - SN_1)(1 - SN_2)(1 - SN_3)}
\]

3. Verify that \(\ln(M + A)/A\) is minimized at the same location as \(M\); use Lagrange multipliers and the results of Problem 2 to find expressions for the values of \(N_i\) where the minimum occurs subject to the constraint \(v_f = c(\ln N_1 + \ln N_2 + \ln N_3)\). \([Hint: Use properties of logarithms to help simplify the expressions.\]

4. Find an expression for the minimum value of \(M\) as a function of \(v_f\).

5. If we want to put a three-stage rocket into orbit 100 miles above Earth’s surface, a final velocity of approximately \(250\) \(\text{mi/h}\) is required. Suppose that each stage is built with a structural factor \(S = 0.2\) and an exhaust speed of \(c = 6000\) \(\text{mi/h}\).
   (a) Find the minimum total mass \(M\) of the rocket engines as a function of \(A\).
   (b) Find the mass of each individual stage as a function of \(A\). (They are not equally sized!)

6. The same rocket would require a final velocity of approximately \(24,700\) \(\text{mi/h}\) in order to escape Earth’s gravity. Find the mass of each individual stage that would minimize the total mass of the rocket engines and allow the rocket to propel a 500-pound probe into deep space.

**Applied Project**

**Hydro-Turbine Optimization**

The Great Northern Paper Company in Millinocket, Maine, operates a hydroelectric generating station on the Penobscot River. Water is piped from a dam to the power station. The rate at which the water flows through the pipe varies, depending on external conditions.

The power station has three different hydroelectric turbines, each with a known (and unique) power function that gives the amount of electric power generated as a function of the water flow arriving at the turbine. The incoming water can be apportioned in different volumes to each turbine, so the goal is to determine how to distribute water among the turbines to give the maximum total energy production for any rate of flow.

Using experimental evidence and Bernoulli’s equation, the following quadratic models were determined for the power output of each turbine, along with the allowable flows of operation:

\[
KW_1 = (-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5}Q_1^2)(170 - 1.6 \cdot 10^{-4}Q_1)
\]

\[
KW_2 = (-24.51 + 0.1358Q_2 - 4.69 \cdot 10^{-5}Q_2^2)(170 - 1.6 \cdot 10^{-4}Q_2)
\]

\[
KW_3 = (-27.02 + 0.1380Q_3 - 3.84 \cdot 10^{-5}Q_3^2)(170 - 1.6 \cdot 10^{-4}Q_3)
\]

\[
250 \leq Q_1 \leq 1110, \quad 250 \leq Q_2 \leq 1110, \quad 250 \leq Q_3 \leq 1225
\]

where

- \(Q_i\) = flow through turbine \(i\) in cubic feet per second
- \(KW_i\) = power generated by turbine \(i\) in kilowatts
- \(Q_T\) = total flow through the station in cubic feet per second

1. If all three turbines are being used, we wish to determine the flow \(Q_1\) to each turbine that will give the maximum total energy production. Our limitations are that the flows must
sum to the total incoming flow and the given domain restrictions must be observed. Consequently, use Lagrange multipliers to find the values for the individual flows (as functions of $Q_T$) that maximize the total energy production $KW_1 + KW_2 + KW_3$ subject to the constraints $Q_1 + Q_2 + Q_3 = Q_T$ and the domain restrictions on each $Q_i$.

2. For which values of $Q_T$ is your result valid?

3. For an incoming flow of 2500 ft$^3$/s, determine the distribution to the turbines and verify (by trying some nearby distributions) that your result is indeed a maximum.

4. Until now we assumed that all three turbines are operating; is it possible in some situations that more power could be produced by using only one turbine? Make a graph of the three power functions and use it to help decide if an incoming flow of 1000 ft$^3$/s should be distributed to all three turbines or routed to just one. (If you determine that only one turbine should be used, which one?) What if the flow is only 600 ft$^3$/s?

5. Perhaps for some flow levels it would be advantageous to use two turbines. If the incoming flow is 1500 ft$^3$/s, which two turbines would you recommend using? Use Lagrange multipliers to determine how the flow should be distributed between the two turbines to maximize the energy produced. For this flow, is using two turbines more efficient than using all three?

6. If the incoming flow is 3400 ft$^3$/s, what would you recommend to the company?

7. How do you find a tangent plane to each of the following types of surfaces?
   (a) A graph of a function of two variables, $z = f(x, y)$
   (b) A level surface of a function of three variables, $F(x, y, z) = k$
   (c) A parametric surface given by a vector function $\mathbf{r}(u, v)$

8. Define the linearization of $f$ at $(a, b)$. What is the corresponding linear approximation? What is the geometric interpretation of the linear approximation?

9. (a) What does it mean to say that $f$ is continuous at $(a, b)$?
   (b) If $f$ is continuous on $\mathbb{R}^2$, what can you say about its graph?

10. If $z = f(x, y)$, what are the differentials $dx$, $dy$, and $dz$?

11. State the Chain Rule for the case where $z = f(x, y)$ and $x$ and $y$ are functions of one variable. What if $x$ and $y$ are functions of two variables?

12. If $z$ is defined implicitly as a function of $x$ and $y$ by an equation of the form $F(x, y, z) = 0$, how do you find $\partial z/\partial x$ and $\partial z/\partial y$?
13. (a) Write an expression as a limit for the directional derivative of \( f \) at \((x_0, y_0)\) in the direction of a unit vector \( \mathbf{u} = (a, b) \). How do you interpret it as a rate? How do you interpret it geometrically?
(b) If \( f \) is differentiable, write an expression for \( D_uf(x_0, y_0) \) in terms of \( f_x \) and \( f_y \).

14. (a) Define the gradient vector \( \nabla f \) for a function \( f \) of two or three variables.
(b) Express \( D_uf \) in terms of \( \nabla f \).
(c) Explain the geometric significance of the gradient.

15. What do the following statements mean?
(a) \( f \) has a local maximum at \((a, b)\).
(b) \( f \) has an absolute maximum at \((a, b)\).
(c) \( f \) has a local minimum at \((a, b)\).
(d) \( f \) has an absolute minimum at \((a, b)\).
(e) \( f \) has a saddle point at \((a, b)\).

16. (a) If \( f \) has a local maximum at \((a, b)\), what can you say about its partial derivatives at \((a, b)\)?
(b) What is a critical point of \( f \)?

17. State the Second Derivatives Test.

18. (a) What is a closed set in \( \mathbb{R}^2 \)? What is a bounded set?
(b) State the Extreme Value Theorem for functions of two variables.
(c) How do you find the values that the Extreme Value Theorem guarantees?

19. Explain how the method of Lagrange multipliers works in finding the extreme values of \( f(x, y, z) \) subject to the constraint \( g(x, y, z) = k \). What if there is a second constraint \( h(x, y, z) = c \)?

\[ \text{TRUE-FALSE QUIZ} \]

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. \( f_x(a, b) = \lim_{y \to b} \frac{f(a, y) - f(a, b)}{y - b} \)

2. There exists a function \( f \) with continuous second-order partial derivatives such that \( f_1(x, y) = x + y^2 \) and \( f_2(x, y) = x - y^3 \).

3. \( f_{xy} = \frac{\partial^2 f}{\partial x \partial y} \)

4. \( D_k f(x, y, z) = f(x, y, z) \)

5. If \( f(x, y) \to L \) as \( (x, y) \to (a, b) \) along every straight line through \((a, b)\), then \( \lim_{(x, y) \to (a, b)} f(x, y) = L \).

6. If \( f_x(a, b) \) and \( f_y(a, b) \) both exist, then \( f \) is differentiable at \((a, b)\).

7. If \( f \) has a local minimum at \((a, b)\) and \( f \) is differentiable at \((a, b)\), then \( \nabla f(a, b) = \mathbf{0} \).

8. \( \lim_{(x, y) \to (1, 1)} \frac{x - y}{x^2 - y^2} = \lim_{(x, y) \to (1, 1)} \frac{1}{x + y} = \frac{1}{2} \)

9. If \( f(x, y) = \ln y \), then \( \nabla f(x, y) = 1/y \).

10. If \((2, 1)\) is a critical point of \( f \) and \( f_x(2, 1)f_x(2, 1) < [f_y(2, 1)]^2 \), then \( f \) has a saddle point at \((2, 1)\).

11. If \( f(x, y) = \sin x + \sin y \), then \( -\sqrt{2} \leq D_uf(x, y) \leq \sqrt{2} \).

12. If \( f(x, y) \) has two local maxima, then \( f \) must have a local minimum.
1–2 ■ Find and sketch the domain of the function.
1. \( f(x, y) = \sin^{-1}x + \tan^{-1}y \)
2. \( f(x, y, z) = \sqrt{x^2 - y^2 - z^2} \)

3–4 ■ Sketch the graph of the function.
3. \( f(x, y) = 1 - x^2 - y^2 \)
4. \( f(x, y) = \sqrt{x^2 + y^2 - 1} \)

5–6 ■ Sketch several level curves of the function.
5. \( f(x, y) = e^{-(x^2+y^2)} \)
6. \( f(x, y) = x^2 + 4y \)

7. Make a rough sketch of a contour map for the function whose graph is shown.

8. A contour map of a function \( f \) is shown. Use it to make a rough sketch of the graph of \( f \).

9–10 ■ Evaluate the limit or show that it does not exist.
9. \( \lim_{(x, y) \to (1, 1)} \frac{2xy}{x^2 + 2y^2} \)
10. \( \lim_{(x, y) \to (0, 0)} \frac{2xy}{x^2 + 2y^2} \)

11. A metal plate is situated in the \( xy \)-plane and occupies the rectangle \( 0 \leq x \leq 10, 0 \leq y \leq 8 \), where \( x \) and \( y \) are measured in meters. The temperature at the point \( (x, y) \) in the plate is \( T(x, y) \), where \( T \) is measured in degrees Celsius. Temperatures at equally spaced points were measured and recorded in the table.
(a) Estimate the values of the partial derivatives \( T_x(6, 4) \) and \( T_y(6, 4) \). What are the units?

(b) Estimate the value of \( D_u T(6, 4) \), where \( u = (i + j)/\sqrt{2} \). Interpret your result.
(c) Estimate the value of \( T_y(6, 4) \).

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
x & y & 0 & 2 & 4 & 6 & 8 \\
\hline
0 & 30 & 38 & 45 & 51 & 55 & \\
2 & 52 & 56 & 60 & 62 & 61 & \\
4 & 78 & 74 & 72 & 68 & 66 & \\
6 & 98 & 87 & 80 & 75 & 71 & \\
8 & 96 & 90 & 86 & 80 & 75 & \\
10 & 92 & 92 & 91 & 87 & 78 & \\
\hline
\end{array}
\]

12. Find a linear approximation to the temperature function \( T(x, y) \) in Exercise 11 near the point \( (6, 4) \). Then use it to estimate the temperature at the point \( (5, 3.8) \).

13–17 ■ Find the first partial derivatives.
13. \( f(x, y) = \sqrt{2x + y^2} \)
14. \( u = e^{-x} \sin 2\theta \)
15. \( g(u, v) = u \tan^{-1}v \)
16. \( w = \frac{x}{y - z} \)
17. \( T(p, q, r) = p \ln(q + e^r) \)

18. The speed of sound traveling through ocean water is a function of temperature, salinity, and pressure. It has been modeled by the function
\[
C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 \\
\quad + (1.34 - 0.017)(S - 35) + 0.016D
\]

where \( C \) is the speed of sound (in meters per second), \( T \) is the temperature (in degrees Celsius), \( S \) is the salinity (the concentration of salts in parts per thousand, which means the number of grams of dissolved solids per 1000 g of water), and \( D \) is the depth below the ocean surface (in meters). Compute \( \partial C/\partial T \), \( \partial C/\partial S \), and \( \partial C/\partial D \) when \( T = 10^\circ \text{C}, S = 35 \text{ parts per thousand}, \) and \( D = 100 \text{ m} \).

19–22 ■ Find all second partial derivatives of \( f \).
19. \( f(x, y) = 4x^3 - xy^2 \)
20. \( z = xe^{-2y} \)
21. \( f(x, y, z) = x^3y^2z^m \)
22. \( v = r \cos(x + 2t) \)

23. If \( u = x^t \), show that \( \frac{x}{y} \frac{\partial u}{\partial x} + \frac{1}{\ln x} \frac{\partial u}{\partial y} = 2u \).
24. If \( \rho = \sqrt{x^2 + y^2 + z^2} \), show that
\[
\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2} - \frac{2}{\rho}
\]
25–29 ■ Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.
25. \( z = 3x^2 - y^2 + 2x \), \((1, -2, 1)\)
26. \( z = e^x \cos y \), \((0, 0, 1)\)
27. \( x^2 + 2y^2 - 3z^2 = 3 \), \((2, -1, 1)\)
28. \( xy + yz + zx = 3 \), \((1, 1, 1)\)
29. \( r(u, v) = (u + v) \mathbf{i} + u^2 \mathbf{j} + v^2 \mathbf{k} \), \((3, 4, 1)\)
30. Use a computer to graph the surface \( z = x^3 + 2xy \) and its tangent plane and normal line at \((1, 2, 5)\) on the same screen. Choose the domain and viewpoint so that you get a good view of all three objects.
31. Find the points on the sphere \( x^2 + y^2 + z^2 = 1 \) where the tangent plane is parallel to the plane \( 2x + y - 3z = 2 \).
32. Find \( dz \) if \( z = x^2 \tan^{-1} y \).
33. Find the linear approximation of the function \( f(x, y, z) = x^3y^{1/2} + z^2 \) at the point \((2, 3, 4)\) and use it to estimate the number (1.98013)\( \sqrt{(3.01)^2 + (3.97)^2} \).
34. The two legs of a right triangle are measured as \( 5 \) m and 12 m with a possible error in measurement of at most 0.2 cm in each. Use differentials to estimate the maximum error in the calculated value of (a) the area of the triangle and (b) the length of the hypotenuse.
35. If \( w = \sqrt{x + y^2/z} \), where \( x = e^{2t}, y = t^3 + 4t \), and \( z = t^2 - 4 \), use the Chain Rule to find \( dw/dt \).
36. If \( z = \cos xy + y \cos x \), where \( x = u^2 + v \) and \( y = u - v^3 \), use the Chain Rule to find \( \partial z/\partial u \) and \( \partial z/\partial v \).
37. Suppose \( z = f(x, y) \), where \( x = g(s, t), y = h(s, t) \), \( g(1, 2) = 3, g(1, 2) = -1, g(1, 2) = 4, h(1, 2) = 6, h(1, 2) = -5, h(1, 2) = 10, f(3, 6) = 7 \), and \( f(3, 6) = 8 \). Find \( \partial z/\partial s \) and \( \partial z/\partial t \) when \( s = 1 \) and \( t = 2 \).
38. Use a tree diagram to write out the Chain Rule for the case where \( w = f(t, u, v), t = t(p, q, r, s), u = u(p, q, r, s), \) and \( v = v(p, q, r, s) \) are all differentiable functions.
39. If \( z = y + f(x^2 - y^2) \), where \( f \) is differentiable, show that
\[
y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = x
\]
40. The length \( x \) of a side of a triangle is increasing at a rate of 3 in/s, the length \( y \) of another side is decreasing at a rate of 2 in/s, and the contained angle \( \theta \) is increasing at a rate of 0.05 radian/s. How fast is the area of the triangle changing when \( x = 40 \) in, \( y = 50 \) in, and \( \theta = \pi/6 \)?
41. If \( z = f(u, v) \), where \( u = xy, v = y/x, \) and \( f \) has continuous second partial derivatives, show that
\[
x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = -4uv \frac{\partial^2 z}{\partial u \partial v} + 2v \frac{\partial z}{\partial v}
\]
42. If \( yz^4 + x^2z^3 = e^{xy} \), find \( \partial z/\partial x \) and \( \partial z/\partial y \).
43. Find the gradient of the function \( f(x, y, z) = z^2 e^{x+y} \).
44. (a) When is the directional derivative of \( f \) a maximum? (b) When is it a minimum? (c) When is it 0? (d) When is it half of its maximum value?
45–46 ■ Find the directional derivative of \( f \) at the given point in the indicated direction.
45. \( f(x, y) = 2\sqrt{x} - y^2 \), \((1, 5)\), in the direction toward the point \((4, 1)\)
46. \( f(x, y, z) = x^2y + y\sqrt{1 + z} \), \((1, 2, 3)\), in the direction of \( v = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \)
47. Find the maximum rate of change of \( f(x, y) = x^2y + \sqrt{y} \) at the point \((2, 1)\). In which direction does it occur?
48. Find the direction in which \( f(x, y, z) = ze^{x+y} \) increases most rapidly at the point \((0, 1, 2)\). What is the maximum rate of increase?
49. The contour map shows wind speed in knots during Hurricane Andrew on August 24, 1992. Use it to estimate the value of the directional derivative of the wind speed at Homestead, Florida, in the direction of the eye of the hurricane.
50. Find parametric equations of the tangent line at the point 
\((-2, 2, 4)\) to the curve of intersection of the surface 
\(z = 2x^2 - y^2\) and the plane \(z = 4\).

51–54 ■ Find the local maximum and minimum values and saddle points of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

51. \(f(x, y) = x^2 - xy + y^2 + 9x - 6y + 10\)
52. \(f(x, y) = x^3 - 6xy + 8y^3\)
53. \(f(x, y) = 3xy - x^2y - xy^2\)
54. \(f(x, y) = (x^2 + y)e^{1/2}\)

55–56 ■ Find the absolute maximum and minimum values of \(f\) on the set \(D\).

55. \(f(x, y) = 4xy^2 - x^2y^2 - xy^4; \quad D\) is the closed triangular region in the \(xy\)-plane with vertices \((0, 0), (0, 6), \) and \((6, 0)\)
56. \(f(x, y) = e^{\frac{-x^2 - y^2}{4}}(x^2 + 2y^2); \quad D\) is the disk \(x^2 + y^2 \leq 4\)

57. Use a graph and/or level curves to estimate the local maximum and minimum values and saddle points of \(f(x, y) = x^2 - 3x + y^2 - 2y^2\). Then use calculus to find these values precisely.

58. Use a graphing calculator or computer (or Newton’s method or a computer algebra system) to find the critical points of \(f(x, y) = 12 + 10y - 2x^2 - 8xy - y^4\) correct to three decimal places. Then classify the critical points and find the highest point on the graph.

59–62 ■ Use Lagrange multipliers to find the maximum and minimum values of \(f\) subject to the given constraint(s).

59. \(f(x, y) = x^2y; \quad x^2 + y^2 = 1\)

60. \(f(x, y) = \frac{1}{x} + \frac{1}{y}; \quad \frac{1}{x^2} + \frac{1}{y^2} = 1\)
61. \(f(x, y, z) = xyz; \quad x^2 + y^2 + z^2 = 3\)
62. \(f(x, y, z) = x^2 + 2y^2 + 3z^2; \quad x + y + z = 1, \quad x - y + 2z = 2\)

63. Find the points on the surface \(xy^2z^3 = 2\) that are closest to the origin.

64. A package in the shape of a rectangular box can be mailed by U.S. Parcel Post if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length) is at most 108 in. Find the dimensions of the package with largest volume that can be mailed by Parcel Post.

65. A pentagon is formed by placing an isosceles triangle on a rectangle, as shown in the figure. If the pentagon has fixed perimeter \(P\), find the lengths of the sides of the pentagon that maximize the area of the pentagon.

66. A particle of mass \(m\) moves on the surface \(z = f(x, y)\). Let \(x = x(t), y = y(t)\) be the \(x\)- and \(y\)-coordinates of the particle at time \(t\).

(a) Find the velocity vector \(\mathbf{v}\) and the kinetic energy \(K = \frac{1}{2}m|\mathbf{v}|^2\) of the particle.
(b) Determine the acceleration vector \(\mathbf{a}\).
(c) Let \(z = x^2 + y^2\) and \(x(t) = t \cos t, y(t) = t \sin t\). Find the velocity vector, the kinetic energy, and the acceleration vector.