



# The Bregman distance, approximate compactness and convexity of Chebyshev sets in Banach spaces

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## Abstract

We present some sufficient conditions ensuring the upper semicontinuity and the continuity of the Bregman projection operator  $\Pi_C^g$  and the relative projection operator  $P_C^g$  in terms of the  $D$ -approximate (weak) compactness for a nonempty closed set  $C$  in a Banach space  $X$ . We next present certain sufficient conditions as well as equivalent conditions for the convexity of a Chebyshev subset of a Banach space  $X$ . Our results extend the corresponding results of [H.H. Bauschke, X.F. Wang, J. Ye and X. M. Yuan, Bregman distances and Chebyshev sets, J. Approx. Theory 159 (2009) 3–25] to infinite dimensional spaces.

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## 1. Introduction

Let  $X$  be a real normed space with the dual space  $X^*$ . Let  $C \subset X$  be a nonempty subset of  $X$ . As usual, the metric projection operator on  $C$  is denoted by  $P_C: X \rightrightarrows C$  and defined by

$$P_C(x) := \left\{ z \in C : \|x - z\| = \inf_{y \in C} \|x - y\| \right\} \quad \text{for each } x \in X.$$

We recall (cf. [37]) that  $C$  is said to be Chebyshev if  $P_C(x)$  is a singleton for each  $x \in X$ .

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It is known (cf. [37]) that each nonempty closed convex subset of  $X$  is Chebyshev if and only if  $X$  is reflexive and strictly convex. In particular, each nonempty closed convex subset of a Hilbert space is Chebyshev. As to the converse, the famous convexity problem of Chebyshev sets inquires:

“Is a Chebyshev set in a Hilbert space necessarily convex?” (1.1)

The study of this problem has a long history. An affirmative answer to this problem for the Euclidean space  $\mathbb{R}^n$  was given independently by Bunt in 1934, Motzkin in 1935 and others; however, the problem is still open in the case of infinite dimensional Hilbert spaces (e.g., see [5,10,21,25,37]). Johnson has constructed a nonconvex Chebyshev set in an infinite dimensional pre-Hilbert space in [27], where there is a minor gap and a corrected version was provided in [26] by Jiang. Very recently, a conjecture aiming for the construction of a nonconvex Chebyshev set in a Hilbert space was proposed in [23] by Faraci and Iannizzotto. On the other hand, the answer becomes affirmative only if some very mild condition (e.g., weak closedness, weak approximate compactness, continuity or maximal monotonicity of  $P_C$ ) is imposed on  $C$ ; see for example [2,4,22,29,37,36] and the survey [20].

The convexity problem of Chebyshev sets in general Banach spaces was also studied extensively, and many sufficient conditions for a Chebyshev set to be convex have been obtained. In particular, Busemann [12] pointed out that each Chebyshev set in a smooth, strictly convex finite dimensional space is convex. Klee [29] showed that any weakly closed Chebyshev set in a uniformly convex and uniformly smooth Banach space or, more generally, any Chebyshev set in a smooth reflexive space with the weakly continuous projection is convex. This result was extended in [34] by Vlasov to the setting of Banach spaces with round dual spaces. For the details and other related results, the readers are referred to [19,22,33,37,30] and the surveys [5,35]. Note that the continuity of the projection  $P_C$  plays a key role in the study mentioned above.

Recent interests are focused on some similar problems but with the Bregman distance instead of the norm on  $X$ . The setting is as follows. Let  $g: X \rightarrow \overline{\mathbb{R}} := (-\infty, +\infty]$  be a proper convex function with its domain  $\text{dom } g$ . The right-hand side derivative of  $g$  at  $x \in \text{dom } g$  in the direction  $h$  is given by

$$g'_+(x, h) := \lim_{t \rightarrow 0^+} \frac{g(x + th) - g(x)}{t}. \quad (1.2)$$

The Bregman distance with respect to  $g$  between the points  $x, y \in \text{dom } g$  is defined by

$$D_g(y, x) := g(y) - g(x) - g'_+(x, y - x). \quad (1.3)$$

In 1976, Bregman discovered an elegant and effective technique for the use of the function  $D_g$  in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique is applied in various ways in order to design and analyze iterative algorithms not only for solving feasibility and optimization problems, but also for solving variational inequalities and computing fixed points of nonlinear mappings and more; see [7,13,15,14,17,18,31] and the references therein.

Let  $C \subset \text{dom } g$  be a nonempty subset. The Bregman projection on  $C$  with respect to  $g$ , denoted by  $\Pi_C^g$ , is defined as the set of the solutions of the optimization problem  $\min_{y \in C} D_g(y, x)$ , i.e.,

$$\Pi_C^g(x) := \arg \min_{y \in C} D_g(y, x) \quad \text{for each } x \in \text{dom } g. \quad (1.4)$$

Bauschke et al. started in [8] to consider the convexity problem of Chebyshev sets in the sense of Bregman distance in the Euclidean space  $\mathbb{R}^n$ . Under the assumption that  $g$  is a convex function of Legendre type and 1-coercive, they proved that each Chebyshev subset of  $\mathbb{R}^n$  (in the sense of Bregman distance) is convex. The techniques used there are closely dependent upon the properties possessed by the Euclidean space.

The purpose of the present paper is to explore the convexity problem of Chebyshev sets (in the sense of Bregman distance) in general Banach spaces. Our approach is based on the study of the Bregman projection  $\Pi_C^g$  as well as the relative projection  $P_C^g : X^* \rightarrow C$ , which is defined by

$$P_C^g(x^*) := \arg \min_{y \in C} W^g(y, x^*) \quad \text{for each } x^* \in X^*,$$

where  $W^g$  is the function defined by

$$W^g(x, x^*) := g(x) - \langle x^*, x \rangle + g^*(x^*) \quad \text{for each pair } (x, x^*) \in X \times X^*.$$

Remembering that the continuity of the projection on  $C$  is a powerful tool, some useful conditions ensuring the upper semicontinuity and/or the continuity of the Bregman projections and the relative projections in terms of the  $D$ -approximate (weak) compactness of  $C$  are present in Section 3. The main results obtained in this section, which themselves have independent interest, are improvements and extensions of some known ones due to [33,30,19,24]. We next present in the last section several equivalent conditions (such as  $D$ -approximate compactness of the set  $C$ , continuity or maximal monotonicity of the Bregman projection  $\Pi_C^g$ , and the differentiability of the Bregman distance function  $D_C^g$ , etc.) for a Chebyshev subset  $C$  in Banach spaces to be convex. In particular, our results extend and/or improve both the corresponding ones for the Euclidean spaces in [8] and the well known results on convexity of Chebyshev sets in Hilbert spaces (in norm distance) to general infinite dimensional Banach spaces.

## 2. Preliminaries

Let  $X$  be a Banach space and  $g : X \rightarrow \overline{\mathbb{R}}$  be a proper convex function. As usual, the closed unit ball and unit sphere of  $X$  are denoted by  $\mathbf{B}$  and  $\mathbf{S}$ , respectively. We also denote by  $\mathbf{B}(x, \delta)$  the closed ball centered at  $x$  with radius  $\delta$ . Moreover, we use  $\text{dom } g$  to denote the domain of  $g$ . Let  $x \in \text{dom } g$ . The *subdifferential* of  $g$  at  $x$  is the convex set defined by

$$\partial g(x) := \{x^* \in X^* : g(x) + \langle x^*, y - x \rangle \leq g(y) \text{ for each } y \in X\};$$

while the *conjugate function* of  $g$  is the function  $g^* : X^* \rightarrow \overline{\mathbb{R}}$  defined by

$$g^*(x^*) := \sup\{\langle x^*, x \rangle - g(x) : x \in X\}.$$

Then, by [39, Theorem 2.4.2(iii)], the Young–Fenchel inequality holds:

$$\langle x^*, x \rangle \leq g(x) + g^*(x^*) \quad \text{for each pair } (x^*, x) \in X^* \times X, \tag{2.1}$$

and the equality holds if and only if  $x^* \in \partial g(x)$ , i.e.,

$$\langle x^*, x \rangle = g(x) + g^*(x^*) \iff x^* \in \partial g(x) \quad \text{for each pair } (x^*, x) \in X^* \times X. \tag{2.2}$$

The domain and the image of  $\partial g$  are denoted by  $\text{dom}(\partial g)$  and  $\text{Im}(\partial g)$ , respectively, which are defined by

$$\text{dom}(\partial g) := \{x \in \text{dom } g : \partial g(x) \neq \emptyset\}$$

and

$$\text{Im}(\partial g) := \{x^* \in X^* : x^* \in \partial g(x), x \in \text{dom}(\partial g)\}.$$

Recall that the Bregman distance with respect to  $g$  is defined by

$$D_g(y, x) := g(y) - g(x) - g'_+(x, y - x) \quad \text{for any } x, y \in \text{dom } g. \tag{2.3}$$

**Remark 2.1.** There is another way (cf. [6,28]) to define the Bregman distance with respect to  $g$ :

$$D_g(y, x) := g(y) - g(x) + g'_+(x, x - y) \quad \text{for any } x, y \in \text{dom } g.$$

Clearly, the two definitions coincide for any  $x$  where  $g$  is Gâteaux differentiable.

According to [14], we define the modulus of total convexity at  $x$  by

$$v_g(x, t) := \inf\{D_g(y, x) : y \in \text{dom } g, \|y - x\| = t\} \quad \text{for each } t \geq 0. \tag{2.4}$$

By [18, Proposition 2.1], the modulus of total convexity at  $x$  has the following properties:

- (P1)  $v_g(x, ct) \geq cv_g(x, t)$  for all  $c \geq 1$  and  $t \geq 0$ ;
- (P2)  $v_g(x, t) = \inf\{D_g(y, x) : y \in \text{dom } g, \|y - x\| \geq t\}$  for each  $t \geq 0$ ;
- (P3)  $v_g(x, \cdot)$  is nondecreasing;  $v_g(x, \cdot)$  is strictly increasing on its domain if and only if  $v_g(x, t) > 0$  for each  $t > 0$ .

For our study, we need to introduce the locally uniform modulus of total convexity at  $x$ , which is defined by

$$v_g^{loc}(x, t) := \lim_{\delta \rightarrow 0^+} \inf\{v_g(u, t) : u \in \mathbf{B}(x, \delta) \cap \text{dom } g\} \quad \text{for each } t \geq 0.$$

**Definition 2.1.** Let  $x \in \text{dom } g$ . The function  $g: X \rightarrow \mathbb{R}$  is said to be:

- (a) totally convex at  $x$  if its modulus is positive on  $(0, \infty)$ , i.e.  $v_g(x, t) > 0$  for each  $t > 0$ ;
- (b) locally uniformly totally convex at  $x$  if its locally uniform modulus is positive on  $(0, \infty)$ , i.e.,  $v_g^{loc}(x, \cdot) > 0$  for each  $t > 0$ ;
- (c) essentially strictly convex if  $(\partial g)^{-1}$  is locally bounded on its domain and  $g$  is strictly convex on every convex subset of  $\text{dom}(\partial g)$ .

**Remark 2.2.** (a) The notion of total convexity at a point was first introduced in [13] but using the terminology “very convex”; while the notion of the essentially strict convexity was introduced in [6].

(b) Clearly, the locally uniform total convexity at a point implies the total convexity at the same point. It was proved in [14] (see also [18, Proposition 2.2]) that if  $g$  is totally convex at any point of  $\text{dom } g$ , then it is strictly convex on  $\text{dom } g$ , and in [18, Proposition 2.13] that if  $X$  is reflexive and  $g$  is totally convex at any point of  $\text{dom}(\partial g)$ , then it is essentially strictly convex.

(c) By [31, Proposition 2.2], the function  $g$  is totally convex at  $x \in \text{dom } g$  if and only if for any sequence  $\{y_n\} \subset \text{dom } g$ ,

$$\lim_{n \rightarrow \infty} D_g(y_n, x) = 0 \implies \lim_{n \rightarrow \infty} \|y_n - x\| = 0. \tag{2.5}$$

Similarly, we can prove that  $g$  is locally uniformly totally convex at  $x$  if and only if for any sequence  $\{y_n\} \subset \text{dom } g$  and any sequence  $\{x_n\} \subset \text{dom } g$  convergent to  $x$ ,

$$\lim_{n \rightarrow \infty} D_g(y_n, x_n) = 0 \implies \lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \tag{2.6}$$

Recall from [38] that  $g$  is uniformly convex at  $x$  if the uniform convexity modulus  $\mu_g(x, t)$  of  $g$  at  $x$  is positive for each  $t > 0$ , where  $\mu_g(x, t)$  is defined by

$$\mu_g(x, t) = \inf \left\{ \frac{\lambda g(x) + (1 - \lambda)g(y) - g(\lambda x + (1 - \lambda)y)}{\lambda(1 - \lambda)} : \begin{array}{l} \lambda \in (0, 1), y \in \text{dom } g, \\ \|y - x\| = t \end{array} \right\}. \tag{2.7}$$

The following proposition provides the relationships among the total convexity, the locally uniform total convexity and the locally uniform convexity.

**Proposition 2.1.** *Suppose that  $g: X \rightarrow \overline{\mathbb{R}}$  is a lower semicontinuous proper convex function. Let  $\bar{x} \in \text{int}(\text{dom } g)$ . Consider the following assertions.*

- (i) *The function  $g$  is uniformly convex at  $\bar{x}$ .*
- (ii) *The function  $g$  is locally uniformly totally convex at  $\bar{x}$ .*
- (iii) *The function  $g$  is totally convex at  $\bar{x}$ .*

*Then (i)  $\implies$  (ii)  $\implies$  (iii). Furthermore, if  $g$  is Fréchet differentiable at  $\bar{x}$ , then they are equivalent to each other.*

**Proof.** The equivalence between (i) and (iii) under the assumption that  $g$  is Fréchet differentiable at  $\bar{x}$  was proved in [17, Proposition 2.3]. The implication (ii)  $\implies$  (iii) is obvious. Hence we need only prove the implication (i)  $\implies$  (ii). For this purpose, define for any  $x \in X$  and  $t \geq 0$

$$\begin{aligned} \bar{\mu}_g(x, t) &:= \inf \left\{ g(x) + g(y) - 2g\left(\frac{x+y}{2}\right) : y \in \text{dom } g, \|y - x\| = t \right\} \\ &= \inf \left\{ g(x) + g(x + tu) - 2g\left(x + \frac{t}{2}u\right) : u \in \mathbf{S} \right\} \end{aligned} \tag{2.8}$$

(with the convention  $\infty - \infty = \infty$ ). Then we easily get that

$$v_g(x, t) \geq \bar{\mu}_g(x, t) \geq \frac{1}{2}\mu_g(x, t) \quad \text{for each } x \in X \text{ and each } t \geq 0. \tag{2.9}$$

Note that  $\bar{x} \in \text{int}(\text{dom } g)$  and that  $g$  is lower semicontinuous. It follows that  $g$  is locally Lipschitz around  $\bar{x}$ , that is, there exist  $\delta > 0$  and  $L > 0$  such that

$$|g(x) - g(y)| \leq L\|x - y\| \quad \text{for any } x, y \in \mathbf{B}(\bar{x}, 2\delta).$$

This together with (2.8) implies that

$$\bar{\mu}_g(x, t) \geq \bar{\mu}_g(\bar{x}, t) - 4L\|x - \bar{x}\| \quad \text{for each } x \in \mathbf{B}(\bar{x}, \delta) \text{ and } 0 < t < \delta. \tag{2.10}$$

Now suppose on the contrary that assertion (ii) does not hold. Then there exist  $t_0 > 0$  and a sequence  $\{x_n\} \subset \text{int}(\text{dom } g)$  such that  $\|x_n - \bar{x}\| \rightarrow 0$  and  $v_g(x_n, t_0) \rightarrow 0$ . By the property (P1) and Eq. (2.9), we deduce that

$$\bar{\mu}_g(x_n, t) \rightarrow 0 \quad \text{for each } 0 < t \leq t_0. \tag{2.11}$$

Applying (2.10) (to  $\bar{t} := \min\{t_0, \delta\}$  and  $x_n$  in place of  $t$  and  $x$ ) and taking limits, we have

$$\bar{\mu}_g(\bar{x}, \bar{t}) \leq \lim_n \bar{\mu}_g(x_n, \bar{t}) + \lim_n 4L\|x_n - \bar{x}\| = 0.$$

This means that  $\mu_g(\bar{x}, \bar{t}) = 0$  due to (2.9) and so  $g$  is not uniformly convex at  $\bar{x}$ . Thus we complete the proof.  $\square$

Combining Proposition 2.1 and [18, Theorem 2.14], we have the following proposition, which shows that all convexities are equivalent for a real-valued convex function  $g$  on the Euclidean space  $\mathbb{R}^n$ .

**Proposition 2.2.** *Let  $g: \mathbb{R}^n \rightarrow \mathbb{R}$  be a lower semicontinuous proper convex function. Then the following conditions are equivalent.*

- (i) *The function  $g$  is strictly convex.*
- (ii) *The function  $g$  is essentially strictly convex.*
- (iii) *The function  $g$  is totally convex at any  $x \in \mathbb{R}^n$ .*
- (iv) *The function  $g$  is uniformly convex at any  $x \in \mathbb{R}^n$ .*
- (v) *The function  $g$  is locally uniformly totally convex at any  $x \in \mathbb{R}^n$ .*

One important and interesting family of continuous convex functions on  $X$  is the family consisting of convex functions  $g_p$  with  $p > 1$  defined by

$$g_p(x) := \frac{1}{p} \|x\|^p \quad \text{for each } x \in X, \quad (2.12)$$

which has been extensively studied and applied in the building up of Bregman type algorithms; see for example [38,16,17,31,39]. It is known from [38, Theorem 4.1] that  $g_p$  with  $p > 1$  is uniformly convex at any point  $x \in X$  if and only if  $X$  is locally uniformly convex. In particular, following Resmerita in [31], we say that a Banach space  $X$  is locally totally convex if the function  $g_2$  defined by (2.12) is totally convex at each point  $x \in \mathbf{S}$ . The following proposition on characterizing the locally total convexity of  $X$  was proved in [18].

**Proposition 2.3.** *The space  $X$  is locally totally convex if and only if for any  $x \in \mathbf{S}$  and any real number  $\varepsilon > 0$ , there exists  $\delta = \delta(x, \varepsilon) > 0$  such that, for any  $y \in \mathbf{S}$  with  $\|y - x\| \geq \varepsilon$ , there exists  $\lambda_0 \in (0, 1)$  satisfying*

$$\|(1 - \lambda_0)x + \lambda_0 y\| < 1 - \lambda_0 \delta. \quad (2.13)$$

We end this section with two propositions on some properties of convex functions, which will be frequently used in subsequent sections; see [9, Proposition 2.11] and [3, Corollary 3.1, Corollary 3.2] for the first one, and [17, Proposition 3.4] for the second one.

**Proposition 2.4.** *Suppose that  $g: X \rightarrow \overline{\mathbb{R}}$  is a lower semicontinuous proper convex function which is Gâteaux differentiable (resp. Fréchet differentiable) on  $\text{int}(\text{dom } g)$ . Then  $g$  is continuous and its Gâteaux derivative  $\nabla g$  is norm-weak\* continuous (resp. continuous) on  $\text{int}(\text{dom } g)$ .*

**Proposition 2.5.** *Suppose that  $g: X \rightarrow \overline{\mathbb{R}}$  is a lower semicontinuous proper, convex function. Let  $x \in \text{dom } g$  and suppose that  $g$  is totally convex at  $x$ . Then  $\partial g(x) \subseteq \text{int}(\text{dom } g^*)$  and  $g^*$  is Fréchet differentiable at each point  $x^* \in \partial g(x)$ . Furthermore, there exists a nondecreasing function  $\theta: [0, +\infty) \rightarrow [0, +\infty)$  with  $\lim_{t \rightarrow 0^+} \theta(t) = 0$  such that, for any pair  $(y, y^*) \in X \times X^*$  with  $y^* \in \partial g(y)$ , one has*

$$\|y - x\| \leq \theta(\|y^* - x^*\|).$$

### 3. Approximate compactness and continuity of projection operators

Let  $X, Y$  be Banach spaces, and let  $g: X \rightarrow \overline{\mathbb{R}}$  be a lower semicontinuous proper convex function. Throughout the paper, let  $C \subseteq \text{int}(\text{dom } g)$  be a nonempty set and assume that  $g$  is

Gâteaux differentiable on  $\text{int}(\text{dom } g)$  with its Gâteaux derivative denoted by  $\nabla g$ . Recall that the Bregman distance  $D_g$  with respect to  $g$  is defined by (2.3). In particular, we have

$$D_g(y, x) = g(y) - g(x) - \langle \nabla g(x), y - x \rangle \quad \text{for each pair } (y, x) \in X \times \text{int}(\text{dom } g). \tag{3.1}$$

Clearly,  $D_g(\cdot, x)$  is convex for each  $x \in \text{int}(\text{dom } g)$  and the following equality holds for any  $\hat{y}, x \in \text{int}(\text{dom } g)$  and  $y \in X$ :

$$D_g(y, \hat{y}) = D_g(y, x) - D_g(\hat{y}, x) + \langle \nabla g(\hat{y}) - \nabla g(x), \hat{y} - y \rangle. \tag{3.2}$$

We define the Bregman distance function of  $C$  by

$$D_C^g(x) := \inf_{y \in C} D_g(y, x) \quad \text{for each } x \in \text{int}(\text{dom } g) \tag{3.3}$$

and the Bregman projection onto  $C$  by

$$\Pi_C^g(x) := \{y \in C : D_g(y, x) = D_C^g(x)\} \quad \text{for each } x \in \text{int}(\text{dom } g). \tag{3.4}$$

One key tool for our study is the function  $W^g : X \times X^* \rightarrow \overline{\mathbb{R}}$  associated with  $g$ , which is defined by

$$W^g(y, x^*) := g(y) - \langle x^*, y \rangle + g^*(x^*) \quad \text{for each pair } (y, x^*) \in X \times X^*.$$

Clearly,  $W^g$  is nonnegative, and the function  $W^g(\cdot, x^*)$  is convex for any  $x^* \in \text{dom } g^*$ . Moreover the following equality holds:

$$W^g(y, y^*) = W^g(y, x^*) + g^*(y^*) - g^*(x^*) - \langle y^* - x^*, y \rangle \tag{3.5}$$

for any  $x^*, y^* \in X^*$  and  $y \in X$ .

Like for the case of Bregman distances, we define the relative distance function of  $C$  by

$$W_C^g(x^*) := \inf_{y \in C} W^g(y, x^*) \quad \text{for each } x^* \in \text{dom } g^*. \tag{3.6}$$

Then the relative projection operator onto  $C$  (relative to  $g$ ) is defined by

$$P_C^g(x^*) := \{y \in C : W^g(y, x^*) = W_C^g(x^*)\} \quad \text{for each } x^* \in \text{dom } g^*. \tag{3.7}$$

Such projection operators were introduced in [18] by Butnariu and Resmerita to generalize the Bregman projection and the generalized projection defined and studied by Alber in [1]. In the case where  $X$  is a Hilbert space and  $g(\cdot) = \frac{1}{2} \|\cdot\|^2$ , the operators  $\Pi_C^g$  and  $P_C^g$  coincide and are equal to the metric projection operator onto the set  $C$ . For the general case, the relationships between the two operators are described in the following proposition, which is a direct consequence of the Young–Fenchel inequality and the definition of subdifferential of convex functions (cf. (2.1) and (2.2)).

**Proposition 3.1.** *The following assertions hold:*

$$D_g(y, x) = W^g(y, \nabla g(x)) \quad \text{for each pair } (y, x) \in \text{int}(\text{dom } g) \times \text{int}(\text{dom } g); \tag{3.8}$$

$$\Pi_C^g(x) = P_C^g(\nabla g(x)) \quad \text{for each } x \in \text{int}(\text{dom } g). \tag{3.9}$$

Let  $x \in \text{int}(\text{dom } g)$  and  $\{y_n\} \subset C$ . The sequence  $\{y_n\} \subseteq C$  is called a  $D$ -minimizing sequence of  $x$  if

$$\lim_{n \rightarrow \infty} D_g(y_n, x) = D_C^g(x). \tag{3.10}$$

The following notions of  $D$ -approximate compactness and  $D$ -approximate weak compactness are taken from [7].

**Definition 3.1.** The set  $C$  is said to be  $D$ -approximately compact (resp.  $D$ -approximately weakly compact) if, for any  $x \in \text{int}(\text{dom } g)$ , each  $D$ -minimizing sequence of  $x$  has a subsequence converging (resp. weakly converging) to an element of  $C$ .

Replacing the Bregman distance by the norm distance,  $D$ -approximate compactness reduces to the original approximate compactness introduced by Efimov and Stechkin in [22].

Clearly, if  $C$  is  $D$ -approximately compact, then it is  $D$ -approximately weakly compact. The converse is also true under some additional conditions as shown in the following proposition.

**Proposition 3.2.** *Suppose that  $g$  is totally convex at any point of  $\text{int}(\text{dom } g)$ . Then  $C$  is  $D$ -approximately weakly compact if and only if it is  $D$ -approximately compact.*

**Proof.** Assume that  $C$  is  $D$ -approximately weakly compact. For every  $x \in \text{int}(\text{dom } g)$ , if  $\{y_n\} \subset C$  satisfies  $\lim_{n \rightarrow \infty} D_g(y_n, x) = D_C^g(x)$ , then  $\{y_n\}$  contains a subsequence  $\{y_{n_k}\}$  which converges weakly to some element  $\hat{y} \in C$ . It follows from the weak lower semicontinuity of  $g$  that

$$D_g(\hat{y}, x) \leq \lim_{k \rightarrow \infty} D_g(y_{n_k}, x) = D_C^g(x).$$

This implies that

$$\lim_{k \rightarrow \infty} D_g(y_{n_k}, x) = D_g(\hat{y}, x).$$

By (3.2), we have

$$D_g(y_{n_k}, \hat{y}) = D_g(y_{n_k}, x) - D_g(\hat{y}, x) + \langle \nabla g(\hat{y}) - \nabla g(x), \hat{y} - y_{n_k} \rangle \quad \text{for each } k \in \mathbb{N}.$$

Taking the limit, one gets that  $\lim_{k \rightarrow \infty} D_g(y_{n_k}, \hat{y}) = 0$ . It follows by (2.5) that  $\lim_{k \rightarrow \infty} y_{n_k} = \hat{y}$ . Therefore  $C$  is  $D$ -approximately compact.  $\square$

Recall that  $C$  is boundedly compact (resp. boundedly weakly compact) if for any  $\delta > 0$  the intersection  $C \cap \mathbf{B}(0, \delta)$  is empty or compact (resp. weakly compact). Clearly, the bounded compactness of  $C$  implies its bounded weak compactness.

For the remainder, we need the notion of the 1-coercivity, or super-coercivity (cf. [6]). We say that  $g$  is 1-coercive if

$$\lim_{\|y\| \rightarrow \infty} \frac{g(y)}{\|y\|} = \infty.$$

It is easy to see (cf. [6]) that  $g$  is 1-coercive if and only if

$$\text{int}(\text{dom } g^*) = \text{dom } g^* = X^*. \tag{3.11}$$

The following proposition shows that if  $g$  is 1-coercive or totally convex at any point of  $\text{int}(\text{dom } g)$  then the bounded compactness implies the  $D$ -approximate compactness.

**Proposition 3.3.** *Suppose that  $C$  is boundedly compact (resp. boundedly weakly compact). Then the following assertions hold.*

- (i) *If  $g$  is 1-coercive, then  $C$  is  $D$ -approximately compact (resp.  $D$ -approximately weakly compact).*
- (ii) *If  $g$  is totally convex at any point of  $\text{int}(\text{dom } g)$ , then  $C$  is  $D$ -approximately compact.*

**Proof.** (i) Assume that  $g$  is 1-coercive. Let  $x \in \text{int}(\text{dom } g)$  and let  $\{y_n\} \subset C$  be a  $D$ -minimizing sequence of  $x$ , that is, (3.10) holds. This means that  $\{D_g(y_n, x)\}$  is bounded. By the definition of  $D_g$ , we have

$$D_g(y_n, x) \geq \|y_n\| \left( \frac{g(y_n)}{\|y_n\|} - \|\nabla g(x)\| \right) + \langle \nabla g(x), x \rangle - g(x). \tag{3.12}$$

It follows from the 1-coercivity that  $\{y_n\}$  is bounded. Since  $C$  is boundedly (resp. boundedly weakly) compact,  $\{y_n\}$  contains a subsequence which converges (resp. weakly converges) to some element of  $C$ . This completes the proof of assertion (i).

(ii) Assume that  $g$  is totally convex at any point of  $\text{int}(\text{dom } g)$ . We need only consider the case where  $C$  is boundedly weakly compact. Let  $x \in \text{int}(\text{dom } g)$  and let  $\{y_n\} \subset C$  satisfy (3.10). We claim that  $\{y_n\}$  is bounded. Indeed, otherwise, there exists a subsequence  $\{y_{n_k}\}$  of  $\{y_n\}$  such that  $\lim_{k \rightarrow \infty} \|y_{n_k} - x\| = \infty$ . Without loss of generality, we may assume that  $\|y_{n_k} - x\| \geq 1$  for each  $k$ . By (P1) and the definition of the total convexity modulus, we have

$$D_g(y_{n_k}, x) \geq v_g(x, \|y_{n_k} - x\|) \geq \|y_{n_k} - x\| v_g(x, 1). \tag{3.13}$$

Since  $v_g(x, 1) > 0$  by the total convexity assumption and since  $D_g(y_{n_k}, x) \rightarrow D_C^g(x)$  by (3.10), we get a contradiction by letting  $k \rightarrow \infty$  in (3.13). Therefore, the claim holds and it follows that  $\{y_n\}$  contains a subsequence  $\{y_{n_k}\}$  which converges weakly to some element  $\hat{y} \in C$ . This shows that  $C$  is  $D$ -approximately weakly compact and thus the conclusion follows from Proposition 3.2.  $\square$

The following corollary is a direct consequence of Proposition 3.3.

**Corollary 3.1.** *Suppose that  $X$  is reflexive and that  $C$  is weakly closed. Then the following assertions hold.*

- (i) *If  $g$  is 1-coercive, then  $C$  is  $D$ -approximately weakly compact.*
- (ii) *If  $g$  is totally convex at any point of  $\text{int}(\text{dom } g)$ , then  $C$  is  $D$ -approximately compact.*

The following definition is taken from [7].

**Definition 3.2.** The set  $C$  is said to be:

- (a)  $D$ -proximal if  $\Pi_C^g(x) \neq \emptyset$  for any  $x \in \text{int}(\text{dom } g)$ ;
- (b)  $D$ -semi-Chebyshev if  $\Pi_C^g(x)$  contains at most a point for any  $x \in \text{int}(\text{dom } g)$ ;
- (c)  $D$ -Chebyshev if it is  $D$ -proximal and  $D$ -semi-Chebyshev.

**Remark 3.1.** Suppose that  $C$  is closed and convex. Then the following assertions hold by [18, Proposition 4.2], [6, Theorem 7.8] and [14, Proposition 2.15].

- (i) If  $X$  is reflexive, and if  $g$  is totally convex at any point of  $\text{int}(\text{dom } g)$  or 1-coercive, then  $C$  is  $D$ -proximal.
- (ii) If  $g$  is strictly convex, then  $C$  is  $D$ -semi-Chebyshev.
- (iii) If  $X$  is reflexive, and if  $g$  is totally convex at any point of  $\text{int}(\text{dom } g)$  or  $g$  is essentially strictly convex, then  $C$  is  $D$ -Chebyshev.

**Remark 3.2.** By the definition of the  $D$ -Chebyshev set and the formula (3.9), one sees that if  $C$  is  $D$ -Chebyshev, then  $P_C^g(x^*)$  is single-valued for each  $x^* \in \nabla g(\text{int}(\text{dom } g))$ .

**Proposition 3.4.** *Suppose that  $g$  is strictly convex and that  $\text{cl } C \subseteq \text{int}(\text{dom } g)$ . If  $C$  is  $D$ -proximal, then  $C$  is closed.*

**Proof.** Let  $x \in \text{int}(\text{dom } g)$  and  $\{y_n\} \subset C$  be such that  $\lim_n y_n = x$ . Then  $g(y_n) \rightarrow g(x)$  by the continuity of  $g$  on  $\text{int}(\text{dom } g)$ ; hence  $D_g(y_n, x) \rightarrow D_g(x, x) = 0$ . Since  $C$  is  $D$ -proximal,  $\Pi_C^g(x)$  is nonempty. Taking  $y \in \Pi_C^g(x)$ , we have

$$D_g(y, x) \leq D_g(y_n, x) \rightarrow D_g(x, x) = 0,$$

which together with the strict convexity assumption implies that  $x = y \in C$  and so  $C$  is closed.  $\square$

Throughout the remainder, we always assume that  $C$  is closed. The result described in Proposition 3.5 was proved in [7] but here we provide a direct proof. For this purpose, we first prove a simple lemma.

**Lemma 3.1.** *Let  $x \in \text{int}(\text{dom } g)$  and let  $\{y_n\} \subseteq C$  be a  $D$ -minimizing sequence of  $x$ . If  $\bar{y} \in C$  is a weakly cluster point of  $\{y_n\}$ , then  $\bar{y} \in \Pi_C^g(x)$ .*

**Proof.** We may assume, without loss of generality, that  $y_n \xrightarrow{w} \bar{y}$  as  $n \rightarrow \infty$ . Since  $g$  is weakly lower semicontinuous, we have  $g(\bar{y}) \leq \liminf_{n \rightarrow \infty} g(y_n)$ ; consequently

$$\begin{aligned} D_g(\bar{y}, x) &= g(\bar{y}) - g(x) - \langle \nabla g(x), \bar{y} - x \rangle \\ &\leq \liminf_{n \rightarrow \infty} (g(y_n) - g(x) - \langle \nabla g(x), y_n - x \rangle) \\ &= \lim_{n \rightarrow \infty} D_g(y_n, x) \\ &= D_C^g(x). \end{aligned} \tag{3.14}$$

Hence  $\bar{y} \in \Pi_C^g(x)$ .  $\square$

**Proposition 3.5.** *Suppose that  $C$  is  $D$ -approximately weakly compact. Then  $C$  is  $D$ -proximal.*

**Proof.** Fix  $x \in \text{int}(\text{dom } g)$  and take a sequence  $\{y_n\} \subseteq C$  such that (3.10) holds. Since  $C$  is  $D$ -approximately weakly compact,  $\{y_n\}$  has a subsequence which is weakly convergent to an element of  $C$ . Without loss of generality, we may assume that  $y_n \xrightarrow{w} \bar{y} \in C$ ; hence  $\bar{y}$  is a weakly cluster point of  $\{y_n\}$ . Thus, by Lemma 3.1,  $\bar{y} \in \Pi_C^g(x)$  and  $C$  is  $D$ -proximal because  $x \in \text{int}(\text{dom } g)$  is arbitrary.  $\square$

In the sequel we will make use of the notion of level boundedness in the following definition, which is an extension to infinite dimensional space setting of the corresponding one for finite dimensional spaces; see for example [8].

**Definition 3.3.** Let  $\phi: Y \times X \rightarrow \overline{\mathbb{R}}$  and let  $\bar{x} \in X$ . We say that  $\phi$  is level bounded in the first variable locally uniformly at  $\bar{x}$ , if for every  $\alpha \in \mathbb{R}$ , there exists  $\delta > 0$  such that

$$\bigcup_{x \in \mathbf{B}(\bar{x}, \delta)} \{y \in Y: \phi(y, x) \leq \alpha\}$$

is bounded.

Clearly,  $\phi$  is level bounded in the first variable locally uniformly at  $\bar{x}$  if and only if, for any sequences  $\{x_n\} \subset X$  and  $\{y_n\} \subset Y$ , the following implication holds:

$$\phi(y_n, x_n) \text{ is bounded and } x_n \rightarrow \bar{x} \implies \{y_n\} \text{ is bounded.}$$

**Lemma 3.2.** *Let  $\bar{x} \in \text{int}(\text{dom } g)$ . Suppose that  $g$  is 1-coercive or locally uniformly totally convex at  $\bar{x}$ . Then the following assertions hold.*

- (i) *The function  $D_g$  is level bounded in the first variable locally uniformly at  $\bar{x}$ .*
- (ii) *The function  $W^g$  is level bounded in the first variable locally uniformly at  $\nabla g(\bar{x})$ .*

**Proof.** (i) Suppose on the contrary that  $D_g$  is not level bounded in the first variable locally uniformly at some point  $\bar{x}$ . Then there exist sequences  $\{x_n\} \subset \text{dom } g$ ,  $\{y_n\} \subset \text{dom } g$  and  $\alpha \in \mathbb{R}$  such that

$$D_g(y_n, x_n) \leq \alpha, \quad x_n \rightarrow \bar{x} \text{ and } \|y_n\| \rightarrow \infty. \tag{3.15}$$

We first assume that  $g$  is 1-coercive. By the definition of  $D_g$ , we have

$$\begin{aligned} D_g(y_n, x_n) &\geq \|y_n\| \left( \frac{g(y_n)}{\|y_n\|} - \|\nabla g(x_n)\| \right) + \langle \nabla g(x_n), x_n \rangle - g(x_n) \\ &\geq \|y_n\| \left( \frac{g(y_n)}{\|y_n\|} - \|\nabla g(x_n)\| \right) - \|\nabla g(x_n)\| \cdot \|x_n\| - |g(x_n)|. \end{aligned} \tag{3.16}$$

By Proposition 2.4,  $g$  is continuous and  $\nabla g$  is norm-weak\* continuous at  $\bar{x}$ . It follows that  $\{\|\nabla g(x_n)\|\}$  and  $\{|g(x_n)|\}$  are bounded as  $x_n \rightarrow \bar{x}$ . Thus (3.16) together with the 1-coercivity assumption implies that  $\{D_g(y_n, x_n)\}$  is unbounded, which is a contradiction to (3.15).

Now, we assume that  $g$  is locally uniformly totally convex at  $\bar{x}$ . Then there exists  $\delta > 0$  such that  $\tau := \inf\{v_g(u, 1) : u \in \mathbf{B}(\bar{x}, \delta) \cap \text{dom } g\} > 0$ . By (3.15), we may assume that  $\{x_n\} \subset \mathbf{B}(\bar{x}, \delta)$ , and  $\|y_n - x_n\| \geq 1$  for all  $n \geq 1$ . By property (P1) and the definition of the total convexity modulus, we have that for all  $n$

$$\tau \|y_n - x_n\| \leq \|x_n - y_n\| v_g(x_n, 1) \leq v_g(x_n, \|x_n - y_n\|) \leq D_g(y_n, x_n) \leq \alpha. \tag{3.17}$$

Since  $\{x_n\}$  is bounded, it follows that  $\{y_n\}$  is bounded, which contradicts (3.15). Thus we complete the proof of assertion (i).

(ii) Suppose on the contrary that  $W^g$  is not level bounded in the first variable locally uniformly at  $x^* := \nabla g(\bar{x})$ . Then there exist sequences  $x_n^* \in \text{dom } g^*$ ,  $\{y_n\} \subset \text{dom } g$  and  $\alpha \in \mathbb{R}$  such that

$$W^g(y_n, x_n^*) \leq \alpha, \quad x_n^* \rightarrow x^* \text{ and } \|y_n\| \rightarrow \infty. \tag{3.18}$$

Assume first that  $g$  is 1-coercive. Then  $\text{int}(\text{dom } g^*) = \text{dom } g^* = X^*$ , and so  $x^*, x_n^* \in \text{int}(\text{dom } g^*)$ . Consequently,  $g^*$  is continuous at  $x^*$ ; hence  $g^*(x_n^*) \rightarrow g^*(x^*)$ . From the definition of  $W^g$ , we get that

$$W^g(y_n, x_n^*) \geq g^*(x_n^*) + \|y_n\| \left( \frac{g(y_n)}{\|y_n\|} - \|x_n^*\| \right) \quad \text{for each } n \in \mathbb{N}.$$

This contradicts (3.18).

We now assume that  $g$  is locally uniformly totally convex at  $\bar{x}$ . Since  $x^* \in \nabla g(\text{int}(\text{dom } g))$ , it follows from Proposition 2.5 that  $x^* \in \text{int}(\text{dom } g^*)$ . Without loss of generality, we may assume that  $\{x_n^*\} \subset \text{int}(\text{dom } g^*)$  and so  $\partial g^*(x_n^*) \neq \emptyset$  for each  $n \in \mathbb{N}$ . Let  $n \in \mathbb{N}$ , and take  $x_n \in \partial g^*(x_n^*)$ . Then  $x_n^* \in \partial g(x_n)$  and  $x^* \in \partial g(\bar{x})$  by (2.2). Since  $g$  is locally uniformly totally convex at  $\bar{x}$ , it follows from Proposition 2.5 that  $x_n \rightarrow \bar{x}$  because  $x_n^* \rightarrow x^*$ . Therefore, we may assume that  $x_n \in \text{int}(\text{dom } g)$  for each  $n \in \mathbb{N}$ . Thus, by Proposition 3.1, we have  $W^g(y_n, x_n^*) = D_g(y_n, x_n)$ . This, together with (i), implies that  $\{y_n\}$  is bounded, which is a contradiction. The proof is complete.  $\square$

**Theorem 3.1.** *Suppose that  $g$  is 1-coercive or locally uniformly totally convex at any point of  $\text{int}(\text{dom } g)$ . Then the following assertions hold.*

- (i) *The function  $W_C^g(\cdot)$  is continuous on  $\nabla g(\text{int}(\text{dom } g))$ .*
- (ii) *If  $g$  is Fréchet differentiable on  $\text{int}(\text{dom } g)$ , then  $D_C^g(\cdot)$  is continuous on  $\text{int}(\text{dom } g)$ .*

**Proof.** (i) Let  $x^* \in \nabla g(\text{int}(\text{dom } g))$  and  $\{x_n^*\} \subset \text{dom } g^*$  be such that  $x_n^* \rightarrow x^*$ . Then we have the assertion  $x^* \in \text{int}(\text{dom } g^*)$ , which is true by (3.11) in the case where  $g$  is 1-coercive and by Proposition 2.5 in the case where  $g$  is locally uniformly totally convex at any point of  $\text{int}(\text{dom } g)$ . Consequently,

$$g^*(x_n^*) \rightarrow g^*(x^*). \tag{3.19}$$

Fix  $n \in \mathbb{N}$ . By (3.5), we have

$$W^g(y, x_n^*) = W^g(y, x^*) + g^*(x_n^*) - g^*(x^*) - \langle x_n^* - x^*, y \rangle \quad \text{for each } y \in C. \tag{3.20}$$

It follows that

$$W^g(y, x_n^*) \rightarrow W^g(y, x^*) \quad \text{for each } y \in C. \tag{3.21}$$

Thus,

$$W_C^g(x_n^*) \leq W^g(y, x_n^*) \longrightarrow W^g(y, x^*) \quad \text{for each } y \in C;$$

hence

$$\limsup_{n \rightarrow \infty} W_C^g(x_n^*) \leq W_C^g(x^*). \tag{3.22}$$

Below we verify that

$$W_C^g(x^*) \leq \liminf_{n \rightarrow \infty} W_C^g(x_n^*). \tag{3.23}$$

Guaranteeing this together with (3.22), we complete the proof of assertion (i).

To show (3.23), let  $\epsilon > 0$  be arbitrary and let  $\{y_n\} \subseteq C$  be such that

$$W^g(y_n, x_n^*) \leq W_C^g(x_n^*) + \epsilon \quad \text{for each } n = 1, 2, \dots$$

Taking  $y_0 \in C$ , we have that  $W^g(y_0, x_n^*) \rightarrow W^g(y_0, x^*)$  by (3.21). Since for each  $n$

$$W^g(y_n, x_n^*) \leq W_C^g(x_n^*) + \epsilon \leq W^g(y_0, x_n^*) + \epsilon \rightarrow W^g(y_0, x^*) + \epsilon,$$

it follows Lemma 3.2 that  $\{y_n\}$  is bounded. By (3.5),

$$W^g(y_n, x^*) = W^g(y_n, x_n^*) + g^*(x^*) - g^*(x_n^*) + \langle x_n^* - x^*, y_n \rangle \quad \text{for each } n \in \mathbb{N}. \tag{3.24}$$

Letting  $n \rightarrow +\infty$  in (3.24) and using (3.19), we get that

$$W_C^g(x^*) \leq \liminf_{n \rightarrow \infty} W^g(y_n, x_n^*) \leq \liminf_{n \rightarrow \infty} W_C^g(x_n^*) + \epsilon.$$

This completes the proof of (3.23).

(ii) Since  $g$  is Fréchet differentiable on  $\text{int}(\text{dom } g)$ , by Proposition 2.4, one has that  $\nabla g$  is continuous on  $\text{int}(\text{dom } g)$ . By Proposition 3.1, we have  $D_C^g(x) = W_C^g(\nabla g(x))$  for every  $x \in \text{int}(\text{dom } g)$ . Hence the assertion follows directly from assertion (i).  $\square$

In the case where  $C$  is a closed and convex subset of  $\text{int}(\text{dom } g)$ , **Theorem 3.1(i)** can be deduced from [18, Theorem 4.5]; while **Theorem 3.1(ii)** was proved by Resmerita (see [31, Proposition 4.1]) under the condition that  $g$  is totally convex at any point of  $\text{int}(\text{dom } g)$  and  $R_g^\alpha(y, C) := \{x \in C : D_g(y, x) \leq \alpha\}$  is bounded whenever  $\alpha \in (0, \infty)$  and  $y \in C$ .

Let  $Z$  be a Banach space and let  $T : Z \rightrightarrows X$  be a set-valued mapping. The domain of  $T$  is denoted by  $\mathcal{D}(T)$  and defined by

$$\mathcal{D}(T) := \{z \in Z : T(z) \neq \emptyset\}.$$

**Definition 3.4.** The set-valued mapping  $T$  is said to be:

- (a) upper semicontinuous (resp. norm-weak upper semicontinuous) at  $z_0 \in \mathcal{D}(T)$  if for every open (resp. weakly open) set  $U \supset T(z_0)$ , there exists  $\delta > 0$  such that  $T(z) \subset U$  for every  $z \in \mathbf{B}(z_0, \delta) \cap \mathcal{D}(T)$ ;
- (b) upper semicontinuous (resp. norm-weak upper semicontinuous) on  $Z_0 \subset \mathcal{D}(T)$  if it is upper semicontinuous (resp. norm-weak upper semicontinuous) at each  $z \in Z_0$ .

**Proposition 3.6.** Let  $x \in \text{int}(\text{dom } g)$ . Suppose that  $g$  is totally convex at  $x$  and that  $\Pi_C^g(x)$  is nonempty weakly compact. Then the following statements hold.

- (i) The operator  $P_C^g(\cdot)$  is upper semicontinuous at  $\nabla g(x)$  if and only if it is norm-weak upper semicontinuous at  $\nabla g(x)$ .
- (ii) If  $g$  is Fréchet differentiable at  $x$ , then  $\Pi_C^g(\cdot)$  is upper semicontinuous at  $x$  if and only if it is norm-weak upper semicontinuous at  $x$ .

**Proof.** (i) It is sufficient to verify the sufficient part. To this end, we suppose on the contrary that  $P_C^g(\cdot)$  is norm-weak upper semicontinuous, but not upper semicontinuous at  $x^* := \nabla g(x)$ . Then there exist an open subset  $U$  of  $X$  with  $P_C^g(x^*) \subseteq U$  and sequences  $\{x_n^*\} \subseteq \text{dom } g^*$ ,  $\{y_n\} \subseteq X$  with each  $y_n \in P_C^g(x_n^*)$  such that  $x_n^* \rightarrow x^*$  and  $\{y_n\} \subset X \setminus U$ . Since  $g$  is totally convex at  $x$ , one has  $x^* \in \text{int}(\text{dom } g^*)$  by **Proposition 2.5**. Without loss of generality, we may assume that  $\{x_n^*\} \subset \text{int}(\text{dom } g^*)$ .

Since  $\Pi_C^g(x)$  is weakly compact, it is easy to prove by the definition and the assumed norm-weak upper semicontinuity of  $P_C^g(\cdot)$  that  $P_C^g(\{x_n^*\} \cup \{x^*\})$  is weakly compact. This means that  $\{y_n\}$  converges weakly to some point  $y \in P_C^g(x^*) \subseteq C$  (using a subsequence if necessary) since  $P_C^g(\cdot)$  is norm-weak upper semicontinuous at  $x^*$ .

By the definition of  $P_C^g$  and the continuity of  $g^*$  at  $x^*$ , we have that

$$\limsup_n W^g(y_n, x_n^*) \leq \limsup_n W^g(y, x_n^*) = \lim_n W^g(y, x_n^*) = W^g(y, x^*).$$

Consequently, by the weak lower semicontinuity of  $g$  and (3.5), we have

$$\begin{aligned} W^g(y, x^*) &\leq \liminf_n W^g(y_n, x^*) \\ &\leq \limsup_n W^g(y_n, x^*) \\ &= \limsup_n [W^g(y_n, x_n^*) + g^*(x^*) - g^*(x_n^*) + \langle x_n^* - x^*, y_n \rangle] \\ &\leq W^g(y, x^*). \end{aligned}$$

This implies that

$$\lim_n D_g(y_n, x) = \lim_n W^g(y_n, x^*) = W^g(y, x^*) = D_g(y, x).$$

Thus using (3.2), we have that

$$\lim_n D_g(y_n, y) = \lim_n [D_g(y_n, x) - D_g(y, x) + \langle \nabla g(y) - \nabla g(x), y - y_n \rangle] = 0. \tag{3.25}$$

This together with the assumed totally convexity implies that  $y_n \rightarrow y$ . Since  $X \setminus U$  is closed and  $\{y_n\} \subset X \setminus U$ , it follows that  $y \in X \setminus U$ , which contradicts  $y \in P_C^g(x^*)$ . This completes the proof of the first assertion.

(ii) Suppose that  $g$  is Fréchet differentiable at  $x$ . Then, by Proposition 2.4,  $\nabla g$  is continuous at  $x$ . Hence the result follows from assertion (i) and the fact  $\Pi_C^g(x) = P_C^g(\nabla g(x))$  (cf. (3.9)). The proof is complete.  $\square$

In particular, in the case where  $C$  is  $D$ -Chebyshev, we get the following corollary.

**Corollary 3.2.** *Suppose that  $C$  is  $D$ -Chebyshev and that  $g$  is totally convex at any point of  $\text{int}(\text{dom } g)$ . Then the following statements hold.*

- (i) *The operator  $P_C^g(\cdot)$  is continuous on  $\nabla g(\text{int}(\text{dom } g))$  if and only if it is norm-weak continuous on  $\nabla g(\text{int}(\text{dom } g))$ .*
- (ii) *If  $g$  is Fréchet differentiable on  $\text{int}(\text{dom } g)$ , then  $\Pi_C^g(\cdot)$  is continuous on  $\text{int}(\text{dom } g)$  if and only if it is norm-weak continuous on  $\text{int}(\text{dom } g)$ .*

**Theorem 3.2.** *Suppose that  $g$  is 1-coercive or locally uniformly totally convex at any point of  $\text{int}(\text{dom } g)$ . If  $C$  is  $D$ -approximately compact (resp.  $D$ -approximately weakly compact), then the following statements hold.*

- (i) *The operator  $P_C^g(\cdot)$  is upper semicontinuous (resp. norm-weak upper semicontinuous) on  $\nabla g(\text{int}(\text{dom } g))$ .*
- (ii) *If  $g$  is Fréchet differentiable on  $\text{int}(\text{dom } g)$ , then the operator  $\Pi_C^g(\cdot)$  is upper semicontinuous (resp. norm-weak upper semicontinuous) on  $\text{int}(\text{dom } g)$ .*

**Proof.** (i) By Proposition 3.5,  $C$  is  $D$ -proximal; hence  $P_C^g(\nabla g(x)) = \Pi_C^g(x) \neq \emptyset$  for each  $x \in \text{int}(\text{dom } g)$ .

We only prove the conclusion for the case where  $C$  is  $D$ -approximately compact (as the case of  $D$ -approximate weak compactness is similar). Suppose on the contrary that  $P_C^g$  is not upper semicontinuous at  $x^* := \nabla g(x)$  for some  $x \in \text{int}(\text{dom } g)$ . Then there exist an open set  $U \supset P_C^g(x^*)$ , a sequence  $\{x_n^*\} \subset \text{dom } g^*$  with  $x_n^* \rightarrow x^*$ , and  $y_n \in P_C^g(x_n^*)$  such that  $y_n \in X \setminus U$ . Then, by (3.5),

$$W^g(y_n, x^*) = W^g(y_n, x_n^*) + g^*(x^*) - g^*(x_n^*) + \langle x_n^* - x^*, y_n \rangle \quad \text{for each } n \in \mathbb{N}. \tag{3.26}$$

Since  $x^* \in \text{int}(\text{dom } g^*)$  by (3.11) and Proposition 2.5, it follows that  $g^*(x_n^*) \rightarrow g^*(x^*)$ . Furthermore, by Theorem 3.1, we have that  $W^g(y_n, x_n^*) = W_C^g(x_n^*) \rightarrow W_C^g(x^*)$ . Taking limits in (3.26), we get that

$$D_g(y_n, x) = W^g(y_n, x^*) \rightarrow W_C^g(x^*) = D_C^g(x).$$

Since  $C$  is  $D$ -approximately compact, we have that  $\{y_n\}$  has a subsequence which converges to some  $\bar{y} \in C$ . Hence  $\bar{y} \in \Pi_C^g(x) = P_C^g(x^*)$  by Lemma 3.1. Noting that each  $y_n \in X \setminus U$ , we have that  $\bar{y} \in X \setminus U$ , which contradicts  $\bar{y} \in P_C^g(x^*)$ . Therefore  $P_C^g$  is upper semicontinuous at  $x^*$ .

(ii) Suppose that  $g$  is Fréchet differentiable on  $\text{int}(\text{dom } g)$ . Then, by Proposition 2.4,  $\nabla g$  is continuous on  $\text{int}(\text{dom } g)$ . Hence the assertion follows from (i) because  $\Pi_C^g(x) = P_C^g(\nabla g(x))$  for each  $x \in \text{int}(\text{dom } g)$ .  $\square$

**Corollary 3.3.** *Suppose that  $g$  is locally uniformly totally convex at any point of  $\text{int}(\text{dom } g)$ . Let  $C$  be a  $D$ -approximately weakly compact subset of  $\text{int}(\text{dom } g)$ . Then the following assertions hold.*

- (i) *The operator  $P_C^g(\cdot)$  is upper semicontinuous on  $\nabla g(\text{int}(\text{dom } g))$ .*
- (ii) *If  $g$  is Fréchet differentiable on  $\text{int}(\text{dom } g)$ , then the operator  $\Pi_C^g(\cdot)$  is upper semicontinuous on  $\text{int}(\text{dom } g)$ .*

**Proof.** This results from Proposition 3.2 and Theorem 3.2.  $\square$

Combining Theorem 3.2 and Corollary 3.2, we easily get the following results.

**Corollary 3.4.** *Let  $C$  be a  $D$ -approximately compact (resp.  $D$ -approximately weakly compact) and  $D$ -Chebyshev subset of  $\text{int}(\text{dom } g)$ . Suppose that  $g$  is 1-coercive. Then the following statements hold.*

- (i) *The operator  $P_C^g(\cdot)$  is continuous (resp. norm-weak continuous) on  $\nabla g(\text{int}(\text{dom } g))$ .*
- (ii) *If  $g$  is Fréchet differentiable on  $\text{int}(\text{dom } g)$ , then the operator  $\Pi_C^g(\cdot)$  is continuous (resp. norm-weak continuous) on  $\text{int}(\text{dom } g)$ .*

**Corollary 3.5.** *Let  $C$  be a  $D$ -approximately weakly compact and  $D$ -Chebyshev subset of  $\text{int}(\text{dom } g)$ . Suppose that  $g$  is locally uniformly totally convex at any point of  $\text{int}(\text{dom } g)$ . Then the following statements hold.*

- (i) *The operator  $P_C^g(\cdot)$  is continuous on  $\nabla g(\text{int}(\text{dom } g))$ .*
- (ii) *If  $g$  is Fréchet differentiable on  $\text{int}(\text{dom } g)$ , then the operator  $\Pi_C^g(\cdot)$  is continuous on  $\text{int}(\text{dom } g)$ .*

**Corollary 3.6.** *Let  $C$  be a weakly closed  $D$ -Chebyshev subset of  $\text{int}(\text{dom } g)$ . Suppose that  $X$  is reflexive, and that  $g$  is locally uniformly totally convex at any point of  $\text{int}(\text{dom } g)$  (resp.  $g$  is 1-coercive). Then the following assertions hold.*

- (i) *The operator  $P_C^g(\cdot)$  is continuous (resp. norm-weak continuous) on  $\nabla g(\text{int}(\text{dom } g))$ .*
- (ii) *If  $g$  is Fréchet differentiable on  $\text{int}(\text{dom } g)$ , then the operator  $\Pi_C^g(\cdot)$  is continuous (resp. norm-weak continuous) on  $\text{int}(\text{dom } g)$ .*

In the case where  $C$  is a closed and convex subset of  $\text{int}(\text{dom } g)$ , Corollary 3.6(i) (for the norm-weak continuity) can be deduced from [18, Theorem 4.5]; while Corollary 3.6(ii) (for the continuity) was proved by Resmerita (see [31, Proposition 4.3]) under the condition that  $g$  is totally convex at any point of  $\text{int}(\text{dom } g)$  and  $R_\alpha^g(y, C) := \{x \in C : D_g(y, x) \leq \alpha\}$  is bounded whenever  $\alpha \in (0, \infty)$  and  $y \in C$ .

Applying Theorem 3.2 and Corollary 3.4 to the special convex function  $g = g_2$  given by (2.12) for  $p = 2$  (noting that  $g$  is clearly 1-coercive), we obtain the following corollary. In particular, the assertion (ii) was proved in [24, Proposition 2.7, 2.8] for the case where  $C$  is  $D$ -approximately compact. For simplicity, we write  $D_C = D_C^{g_2}$ ,  $P_C = P_C^{g_2}$ , and  $\Pi_C = \Pi_C^{g_2}$ .

**Corollary 3.7.** *Let  $C$  be a  $D$ -approximately compact (resp.  $D$ -approximately weakly compact) subset of  $X$ . Then the following statements hold.*

- (i) *If  $X$  is a smooth Banach space, then the operator  $P_C(\cdot)$  is upper semicontinuous (resp. norm-weak upper semicontinuous) on  $X^*$ . If, in addition,  $C$  is  $D$ -Chebyshev, then the operator  $P_C(\cdot)$  is continuous (resp. norm-weak upper semicontinuous) on  $X^*$ .*

- (ii) If the norm of  $X$  is Fréchet differentiable, then the operator  $\Pi_C$  is upper semicontinuous (resp. norm-weak upper semicontinuous) on  $X$ . If, in addition,  $C$  is  $D$ -Chebyshev, then the operator  $\Pi_C(\cdot)$  is continuous (resp. norm-weak continuous) on  $X$ .

#### 4. Convexity of $D$ -Chebyshev sets

As assumed in the previous section, let  $g: X \rightarrow \overline{\mathbb{R}}$  be a lower semicontinuous proper convex function and let  $C \subseteq \text{int}(\text{dom } g)$  be a nonempty closed subset. This section is devoted to providing some characterizations of the convexity of  $D$ -Chebyshev sets in reflexive Banach spaces. For this purpose, we need to introduce the notions of essentially smooth convex functions and Legendre convex functions, which have been studied extensively in [6].

**Definition 4.1.** The function  $g$  is said to be:

- (a) essentially smooth if  $\partial g$  is both locally bounded and single-valued on its domain;
- (b) Legendre if  $g$  is both essentially strictly convex and essentially smooth.

The following proposition is useful and known from [6, Theorems 5.4 and 5.6].

**Proposition 4.1.** *The following assertions hold.*

- (i) *The function  $g$  is essentially smooth if and only if  $\text{dom}(\partial g) = \text{int}(\text{dom } g) \neq \emptyset$  and  $\partial g$  is single-valued on its domain.*
- (ii) *If  $X$  is reflexive, then  $g$  is essentially smooth if and only if  $g^*$  is essentially strictly convex.*

**Remark 4.1.** (a) By (2.2), the following equivalence holds:

$$x \in \partial g^*(x^*) \iff x^* \in \partial g(x) \quad \text{for each pair } (x, x^*) \in X \times X^*; \tag{4.1}$$

hence

$$x \in (\partial g^* \circ \nabla g)(x) \quad \text{for each } x \in \text{int}(\text{dom } g). \tag{4.2}$$

(b) By (a) and Proposition 4.1(i), if  $g$  is essentially smooth, then

$$\text{Im}(\partial g^*) = \text{dom}(\partial g) = \text{int}(\text{dom } g). \tag{4.3}$$

(c) If  $g$  is 1-coercive and  $g$  is essentially smooth, then

$$\nabla g(\text{int}(\text{dom } g)) = \text{dom}(\partial g^*) = X^* \tag{4.4}$$

and

$$D_C^g \circ \partial g^* = W_C^g \quad \text{and} \quad \Pi_C^g \circ \partial g^* = P_C^g. \tag{4.5}$$

In fact, by (4.3) and the 1-coercivity assumption, one has that  $\text{dom}(\partial g^*) = \text{dom } g^* = X^*$  and  $\text{Im}(\partial g^*) = \text{int}(\text{dom } g)$ . Thus (4.4) follows from (4.1); while (4.5) holds because of Proposition 3.1 and (4.2).

(d) If both  $g$  and  $g^*$  are essentially smooth (e.g., if  $X$  is reflexive and  $g$  is Legendre), then  $\nabla g : \text{int}(\text{dom } g) \rightarrow \text{int}(\text{dom } g^*)$  is a bijection satisfying

$$(\nabla g)^{-1} = \nabla g^*. \tag{4.6}$$

Let  $T : X^* \rightrightarrows X$  be a set-valued mapping. Recall that  $T$  is monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0 \quad \text{for any } x^*, y^* \in \mathcal{D}(T) \text{ and } x \in T(x^*), y \in T(y^*). \tag{4.7}$$

A monotone set-valued mapping  $T$  is maximal monotone if, for any monotone mapping  $T' : X^* \rightrightarrows X$ , the condition that  $T(x^*) \subset T'(x^*)$  for each  $x^* \in \mathcal{D}(T)$  implies that  $T = T'$ .

**Proposition 4.2.** *The projection operator  $P_C^g$  is monotone.*

**Proof.** Let  $x^*, y^* \in \text{dom } P_C^g$  and  $x \in P_C^g(x^*), y \in P_C^g(y^*)$  be arbitrary elements. Then, by the definition of  $P_C^g$ , one has that

$$W^g(x, x^*) \leq W^g(y, x^*) \quad \text{and} \quad W^g(y, y^*) \leq W^g(x, y^*).$$

Adding these inequalities, one obtains

$$\langle x - y, x^* - y^* \rangle \geq 0,$$

which shows that  $P_C^g$  is monotone.  $\square$

The maximal monotonicity of  $P_C^g$  for the case where  $C$  is a closed and convex subset of  $\text{int}(\text{dom } g)$  has been proved by Butnariu and Resmerita (see Proposition 4.7 in [18]).

**Proposition 4.3.** *Suppose that  $g$  is 1-coercive and essentially smooth. Then the following assertions hold.*

- (i) *If  $C$  is a  $D$ -Chebyshev set, then  $P_C^g$  is single-valued on  $X^*$ .*
- (ii) *If  $C$  is a  $D$ -Chebyshev set and if  $P_C^g$  is norm-weak continuous, then  $P_C^g$  is maximal monotone.*
- (iii) *If  $X$  is reflexive and  $P_C^g$  is maximal monotone, then  $C$  is convex.*

**Proof.** (i) By Remark 4.1(c),  $\nabla g(\text{int}(\text{dom } g)) = \text{dom}(\partial g^*) = X^*$ . Thus, if  $C$  is a  $D$ -Chebyshev set, then  $\text{dom } P_C^g = X^*$ , and  $P_C^g(x^*)$  is a singleton for each  $x^* \in X^*$  by Remark 3.2.

(ii) By (i),  $P_C^g$  is single-valued on  $X^*$ . Thus, if  $P_C^g$  is norm-weak continuous, then  $P_C^g$  is maximal monotone by a well known fact about maximal monotonicity (cf. [11]).

(iii) Note that

$$C \supset \text{Im}(P_C^g) \supset P_C^g(\nabla g(\text{int}(\text{dom } g))) \supset P_C^g(\nabla g(C)) = C.$$

This implies that  $\text{cl}[\text{Im}(P_C^g)] = \text{cl } C = C$ . Since  $P_C^g$  is maximal monotone and  $X$  is reflexive, it follows from [32, Theorem 1] that  $\text{cl}[\text{Im}(P_C^g)]$  is convex; hence  $C$  is convex. This completes the proof.  $\square$

Let  $I_C$  denote the indicate function of the set  $C$ , that is,

$$I_C(x) := \begin{cases} 0 & \text{for each } x \in C, \\ +\infty & \text{for each } x \in X \setminus C. \end{cases}$$

**Lemma 4.1.** *Let  $x^* \in (\text{dom } g^*)$ . Then the following assertions hold:*

$$(g + I_C)^*(x^*) = g^*(x^*) - W_C^g(x^*); \tag{4.8}$$

$$P_C^g(x^*) \subseteq \partial(g + I_C)^*(x^*). \tag{4.9}$$

Consequently, if  $g$  is 1-coercive, then (4.8) holds for each  $x^* \in X^*$ .

**Proof.** We observe that

$$\begin{aligned} W_C^g(x^*) &= \inf_{x \in C} \{g(x) + g^*(x^*) - \langle x^*, x \rangle\} \\ &= g^*(x^*) - \sup_{x \in X} \{\langle x^*, x \rangle - (g + I_C)(x)\} \\ &= g^*(x^*) - (g + I_C)^*(x^*). \end{aligned} \tag{4.10}$$

Hence (4.8) is proved.

To show (4.9), we first note the following equivalences for  $x \in C$ :

$$\begin{aligned} x \in P_C^g(x^*) &\iff W^g(x, x^*) = W_C^g(x^*) \\ &\iff (g + I_C)(x) + g^*(x^*) - \langle x^*, x \rangle = W_C^g(x^*) \\ &\iff (g + I_C)(x) + (g + I_C)^*(x^*) = \langle x^*, x \rangle. \end{aligned}$$

Now let  $x \in P_C^g(x^*)$ . Then  $(g + I_C)(x) + (g + I_C)^*(x^*) = \langle x^*, x \rangle$ . Since  $(g + I_C)^{**}(x) \leq (g + I_C)(x)$ , we have

$$(g + I_C)^{**}(x) + (g + I_C)^*(x^*) \leq \langle x^*, x \rangle, \tag{4.11}$$

and hence

$$(g + I_C)^{**}(x) + (g + I_C)^*(x^*) = \langle x^*, x \rangle$$

because the inverse of (4.11) holds automatically. This implies that  $x \in \partial(g + I_C)^*(x^*)$  and completes the proof.  $\square$

Now we are ready to prove the following theorem.

**Theorem 4.1.** *Suppose that  $X$  is a reflexive Banach space and that  $g$  is a 1-coercive, essentially smooth function. Let  $C \subset \text{int}(\text{dom } g)$  be a  $D$ -Chebyshev set. Then the following assertions are equivalent.*

- (i) *The set  $C$  is convex.*
- (ii) *The set  $C$  is weakly closed.*
- (iii) *The set  $C$  is boundedly weakly compact.*
- (iv) *The set  $C$  is  $D$ -approximately weakly compact.*
- (v) *The operator  $\Pi_C^g \circ \partial g^* (= P_C^g)$  is norm-weak continuous on  $X^*$ .*
- (vi) *The operator  $\Pi_C^g \circ \partial g^* (= P_C^g)$  is maximal monotone.*  
*Furthermore, if  $g$  is additionally Legendre, then each of assertions (i)–(vi) is equivalent to the following one.*
- (vii) *The function  $D_C^g \circ \nabla g^* (= W_C^g)$  is Gâteaux differentiable on  $X^*$ .*

**Proof.** The implications (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) are trivial. To show implications (iv) $\implies$ (v) $\implies$ (vi), we first note that  $\Pi_C^g \circ \partial g^* = P_C^g$  by (4.5). Thus the implication (iv) $\implies$ (v) follows from Theorem 3.2(i) and (4.4); while the implications (v) $\implies$ (vi) $\implies$ (i) hold by Proposition 4.3.

Furthermore, suppose that  $g$  is additionally Legendre. Then  $D_C^g \circ \nabla g^* = W_C^g$  by (4.5). Moreover, (4.8) holds for each  $x^* \in X^*$  by Lemma 4.1. Since  $g^*$  is Gâteaux differentiable on  $X^*$ , it follows that  $W_C^g$  is Gâteaux differentiable on  $X^*$  if and only if so is  $(g + I_C)^*$ . This together with (4.9) implies that (vii) is equivalent to  $P_C^g = \partial(g + I_C)^*$ , which is in turn equivalent to  $P_C^g$  being maximal monotone because  $\partial(g + I_C)^*$  is a monotone extension of  $P_C^g$  by (4.9) and Proposition 4.2. Hence the implication (vii) $\iff$ (vi) is proved.  $\square$

**Theorem 4.2.** *Suppose that  $X$  is reflexive. Suppose that  $g$  is essentially smooth, 1-coercive, and totally convex at any point of  $\text{int}(\text{dom } g)$ . Let  $C \subset \text{int}(\text{dom } g)$  be a  $D$ -proximal set.*

- (1) *The following conditions are equivalent.*
  - (i) *The set  $C$  is convex.*
  - (ii) *The set  $C$  is  $D$ -approximately compact and  $D$ -Chebyshev.*
  - (iii) *The operator  $\Pi_C^g \circ \nabla g^*$  is single-valued and continuous on  $X^*$ .*
  - (iv) *The operator  $\Pi_C^g \circ \nabla g^*$  is maximal monotone.*
  - (v) *The function  $D_C^g \circ \nabla g^*$  is Fréchet differentiable on  $X^*$ .*
- (2) *If  $g$  is Fréchet differentiable at each point of  $\text{int}(\text{dom } g)$ , (i)–(v) are equivalent to the following assertion:*
  - (vi) *The operator  $\Pi_C^g$  is single-valued and continuous on  $\text{int}(\text{dom } g)$ .*
- (3) *If  $\nabla g$  and  $\nabla g^*$  are Fréchet differentiable at each point of  $\text{int}(\text{dom } g)$  and  $\text{int}(\text{dom } g^*)$ , then (i)–(vi) are equivalent to the following assertion:*
  - (vii) *The function  $D_C^g$  is Fréchet differentiable on  $\text{int}(\text{dom } g)$ .*

**Proof.** (1) The implication (i) $\implies$ (ii) follows from Remark 3.1 and Corollary 3.1.

Since  $g$  is totally convex at any point of  $\text{int}(\text{dom } g)$ , by formula (4.4) and Proposition 2.5, one sees that  $g^*$  is Fréchet differentiable on  $X^*$ . Hence  $\Pi_C^g \circ \nabla g^* = \Pi_C^g \circ \partial g^* = P_C^g$ , and thus the implication (ii) $\implies$ (iii) follows from Corollary 3.4 and Proposition 4.3.

By Proposition 4.3, (iii) $\implies$ (iv)  $\implies$ (i). Hence (i)–(iv) are equivalent.

Below we show the equivalence (iii) $\iff$ (v). Note that  $g^*$  is Fréchet differentiable on  $X^*$  by Proposition 2.5. Thus, by (4.5) and (4.8), the assertion (v) is equivalent to  $(g + I_C)^*$  being Fréchet differentiable on  $X^*$  and  $\nabla(g + I_C)^*$  is continuous on  $X^*$ , which is in turn equivalent to (iii) because  $\emptyset \neq P_C^g(x^*) \subseteq \partial(g + I_C)^*(x^*)$  for each  $x^* \in X^*$  and  $\partial(g + I_C)^*$  is the monotone extension of  $P_C^g$  (noting that  $P_C^g = \Pi_C^g \circ \nabla g^*$ ). Hence (1) is proved.

(2) Suppose that  $g$  is Fréchet differentiable at each point of  $\text{int}(\text{dom } g)$ . Then, by Proposition 2.4,  $\nabla g$  is continuous on  $\text{int}(\text{dom } g)$ . Since  $\Pi_C^g = (\Pi_C^g \circ \nabla g^*) \circ \nabla g$ , it follows that (iii) $\iff$ (vi) and the proof of (2) is complete.

(3) Finally, suppose that  $\nabla g$  and  $\nabla g^*$  are Fréchet differentiable respectively on  $\text{int}(\text{dom } g)$  and  $\text{int}(\text{dom } g^*)$ . Since  $D_C^g = (D_C^g \circ \nabla g^*) \circ \nabla g$ , we have that (v) $\iff$ (vii) and complete the proof.  $\square$

Applying the above Theorem 4.2 to the Euclidean space  $\mathbb{R}^n$ , we immediately have the following corollary, which improves the corresponding one from [8]

**Corollary 4.1.** *Let  $X = \mathbb{R}^n$  and suppose that  $g : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  is Legendre and 1-coercive. Let  $C \subset \text{int}(\text{dom } g)$  be a closed set. Then the following assertions are equivalent.*

- (i) *The set  $C$  is convex.*
- (ii) *The set  $C$  is  $D$ -Chebyshev.*
- (iii) *The operator  $\Pi_C^g$  is continuous on  $\text{int}(\text{dom } g)$ .*
- (iv) *The operator  $\Pi_C^g \circ \nabla g^*$  is maximal monotone.*
- (v) *The function  $D_C^g \circ \nabla g^*$  is differentiable on  $X^*$ .*

*If, in addition,  $g$  is second-order continuously differentiable on  $\text{int}(\text{dom } g)$  and  $\nabla^2 g$  is positive definite, then (i)–(v) are equivalent to the following assertion.*

- (vi) *The function  $D_C^g$  is differentiable on  $\text{int}(\text{dom } g)$ .*

**Proof.** We need only prove that if  $g$  is second-order continuously differentiable on  $\text{int}(\text{dom } g)$ , and for every  $x \in \text{int}(\text{dom } g)$ ,  $\nabla^2 g(x)$  is positive definite, then  $\nabla g^*$  is differentiable on  $\text{int}(\text{dom } g^*)$ . In fact, since  $g$  is Legendre, by Remark 4.1(d),  $\nabla g: \text{int}(\text{dom } g) \rightarrow \text{int}(\text{dom } g^*)$  is bijective, and  $\nabla g^* = (\nabla g)^{-1}$ . Since  $g$  is second-order continuously differentiable on  $\text{int}(\text{dom } g)$ , and for every  $x \in \text{int}(\text{dom } g)$ ,  $\nabla^2 g(x)$  is positive definite, by the well known inverse theorem,  $\nabla g^* = (\nabla g)^{-1}$  is continuously differentiable on  $\text{int}(\text{dom } g^*)$ .  $\square$

Consider the significant particular case of  $g_2$  defined by (2.12) for  $p = 2$ . Let  $J: X \rightrightarrows X^*$  and  $J^*: X^* \rightrightarrows X$  be the duality mappings, i.e.,

$$J(x) := \{x^* \in X^*: \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2\},$$

$$J^*(x^*) := \{x \in X: \langle x^*, x \rangle = \|x^*\|^2 = \|x\|^2\}.$$

It is well known that when  $X$  is a reflexive smooth and strictly convex Banach space,  $J$  is bijective and  $J^{-1} = J^*$ .

**Corollary 4.2.** *Suppose that  $X$  is a reflexive, smooth and strictly convex Banach space. Suppose that  $C$  is a  $D$ -Chebyshev subset of  $X$  with respect to the function  $g_2$  defined by (2.12) for  $p = 2$ . Then the following statements are equivalent.*

- (i) *The set  $C$  is convex.*
- (ii) *The set  $C$  is weakly closed.*
- (iii) *The set  $C$  is boundedly weakly compact.*
- (iv) *The set  $C$  is  $D$ -approximately weakly compact.*
- (v) *The operator  $\Pi_C \circ J^*$  is norm-weak continuous on  $X^*$ .*
- (vi) *The operator  $\Pi_C \circ J^*$  is maximal monotone.*
- (vii) *The function  $D_C \circ J^*$  is Gâteaux differentiable on  $X^*$ .*

Moreover, if  $X$  is locally totally convex, then (i)–(vii) are equivalent to the following assertions.

- (viii) *The set  $C$  is  $D$ -approximately compact.*
- (ix) *The operator  $\Pi_C \circ J^*$  is continuous on  $X^*$ .*
- (x) *The function  $D_C \circ J^*$  is Fréchet differentiable on  $X^*$ .*

**Proof.** Since  $X$  is smooth if and only if  $\partial g_2 = J$  is single-valued, it is clear that  $\text{dom } J = \text{int}(\text{dom } g_2) = X$ . Hence  $g_2$  is essentially smooth. By Lemma 5.8 in [6],  $X$  is strictly convex if and only if  $g_2$  is essentially strictly convex.

Moreover, the local total convexity of  $X$  implies that  $g_2$  is totally convex at any point of  $X$ . Hence the result follows from Theorems 4.1 and 4.2.  $\square$

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