



## Nearest and farthest points in spaces of curvature bounded below

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Received 20 October 2009; received in revised form 6 February 2010; accepted 17 February 2010

Available online 25 February 2010

Communicated by Günther Nürnberger

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### Abstract

Let  $A$  be a nonempty closed subset (resp. nonempty bounded closed subset) of a metric space  $(X, d)$  and  $x \in X \setminus A$ . The nearest point problem (resp. the farthest point problem) w.r.t.  $x$  considered here is to find a point  $a_0 \in A$  such that  $d(x, a_0) = \inf\{d(x, a) : a \in A\}$  (resp.  $d(x, a_0) = \sup\{d(x, a) : a \in A\}$ ). We study the well posedness of nearest point problems and farthest point problems in geodesic spaces. We show that if  $X$  is a space of curvature bounded below then the complement of the set of all points  $x \in X$  for which the nearest point problem (resp. the farthest point problem) w.r.t.  $x$  is well posed is  $\sigma$ -porous in  $X \setminus A$ . Our results extend and/or improve some recent results about proximality in geodesic spaces as well as the corresponding ones previously obtained in uniformly convex Banach spaces. In particular, the result regarding the nearest point problem answers affirmatively an open problem proposed by Kaewcharoen and Kirk [A. Kaewcharoen, W.A. Kirk, Proximality in geodesic spaces, *Abstr. Appl. Anal.* 2006 (2006) 1–10].

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*Keywords:* Nearest and farthest points; Geodesic spaces

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## 1. Introduction

Let  $(X, d)$  be a metric space and  $A$  a nonempty closed subset of  $X$ . The metric projection (or nearest point mapping)  $P_A$  onto  $A$  is defined by

$$P_A(x) := \{y \in A : d(x, y) = \text{dist}(x, A)\}, \quad \forall x \in X,$$

where  $\text{dist}(x, A) := \inf\{d(x, y) : y \in A\}$  for each  $x \in X$ . Similarly, if  $A$  is additionally bounded, we use  $F_A$  to denote the farthest point mapping onto  $A$  defined by

$$F_A(x) := \{y \in A : d(x, y) = \sup\{d(x, z) : z \in A\}\}, \quad \forall x \in X.$$

Usually, for a point  $x \in X$ , if  $P_A(x) \neq \emptyset$ , any element of  $P_A(x)$  (resp.  $F_A(x)$ ) is called a best approximation or nearest point (resp. a farthest point) to  $x$ . The basic and interesting problems are the problem of existence and the problem of uniqueness of the nearest point (resp. the farthest point). Recall that the set  $A$  is a Chebyshev set if  $P_A(x)$  is a singleton for each  $x \in X$ . In the case when  $X$  is a normed space, it is well-known that any nonempty closed convex subset  $A$  of  $X$  is a Chebyshev set if and only if  $X$  is reflexive and strictly convex. However, this result is no longer true for nonconvex subsets, in general. Thus “how many” points  $x \in X$  there are such that  $P_A(x)$  is a singleton becomes an interesting problem. In the 60’s of last century, Stečkin established in [34] some fundamental generic uniqueness and existence results for nearest points for arbitrary subsets of normed spaces. In particular, he proved that each closed nonempty subset of a uniformly convex Banach space is an almost Chebyshev set (by which we mean that the set of all points  $x \in X$  such that  $P_A(x)$  fails to be a singleton is a first category set in  $X$ ). This result was extended to locally uniformly convex reflexive Banach spaces in [21] and to the more general Kadec and strictly convex Banach spaces in [19]. Moreover, Stečkin’s result was also sharpened by Edelstein in [14], Konjagin in [18], Zamfirescu in [35] and De Blasi, Myjak and Papini in [4,5]. The key tool used by De Blasi, Myjak and Papini is the so-called *Stečkin’s lens Lemma*, which gives a quantitative version of a result previously proved in [34] by Stečkin, and ultimately leads to a result stating that the set of all points  $x \in X$  such that the nearest point problem w.r.t.  $x$  fails to be well posed is, at most, a  $\sigma$ -porous set in  $X \setminus A$ .

Issues regarding the farthest point problem are somewhat similar and the readers are referred to [1,5,6,13,20]. For more developments and extensions in this direction, the readers are referred to [2,3,7,8,11,22–29,31–33] and the surveys [12,30].

Recent interests are focused on the extension of Stečkin’s idea to geodesic spaces regarding the existence and uniqueness of the nearest point. This line of research was initiated by Zamfirescu in [17,36]. Then the same author proved in [37, Theorem 1] that if  $X$  is a complete geodesic space without bifurcating geodesics and  $A$  is compact, then  $P_A$  is single-valued at most points of  $X$  in the category sense; while Kaewcharoen and Kirk proved in [16, Theorems 3.1 and 3.2] that if  $X$  is a geodesic space of curvature bounded above by 0 and below by  $\kappa > -\infty$  and with the property of the geodesic extension, then  $P_A$  is single-valued on a set of second Baire category, and  $F_A$  is single-valued on a dense subset of  $X$ . The approach used there depends closely upon the uniform convexity possessed by the geodesic space with non-positive curvature.

The purpose of the present paper is to extend the porosity results on nearest point problems and farthest point problems of De Blasi, Myjak and Papini to the geodesic space setting. For this purpose, we first establish in Section 3 an analogous to the Stečkin’s Lemma in geodesic spaces of curvature bounded below. We also provide an example in Section 6 to show that the condition on the bound of the curvature cannot be removed from the Stečkin’s Lemma. Suppose that  $X$  is a geodesic space of curvature bounded below (but not necessarily bounded above) and

$A$  is a nonempty closed subset of  $X$ . Then we show in Section 4 that the set of all points  $x \in X$  such that the nearest point problem w.r.t.  $x$  fails to be well posed is a  $\sigma$ -porous set in  $X \setminus A$ . Furthermore, if additionally  $X$  enjoys the geodesic extension property, we show in Section 5 that the set of all points  $x \in X$  such that the farthest point problem w.r.t.  $x$  fails to be well posed is a  $\sigma$ -porous set in  $X$ . Clearly, our results extend and/or improve the corresponding ones in [16,36,37]. In particular, our results regarding the nearest point problem give an affirmative answer to an open problem in [16] (cf. [16, Problem 3.11]); while the ones regarding the farthest point problem improve [16, Theorem 3.2].

**2. Preliminaries**

Let  $(X, d)$  be a metric space and let  $x, y \in X$ . A *geodesic* in  $X$  is an isometry from  $\mathbb{R}$  into  $X$  (we may also refer to the image of this isometry as a geodesic). A *geodesic path or geodesic segment* joining  $x$  to  $y$  is a map  $c : [0, l] \subseteq \mathbb{R} \rightarrow X$  such that  $c(0) = x, c(l) = y$ , and  $d(c(t), c(t')) = |t - t'|$  for all  $t, t' \in [0, l]$ . Clearly,  $c$  is an isometry between  $[0, l]$  and  $c([0, l])$ , and  $d(x, y) = l$ . Usually, the image  $c([0, l])$  of  $c$  is called a *geodesic segment* joining  $x$  and  $y$ . A geodesic segment joining  $x$  and  $y$  is not necessarily unique in general. In particular, if no confusion arises, we use  $[x, y]$  (resp.  $(x, y)$ ) to denote a geodesic segment joining  $x$  and  $y$  (resp. the interior of  $[x, y]$ ). Thus  $z \in [x, y]$  (resp.  $z \in (x, y)$ ) if and only if there exists  $t \in [0, 1]$  (resp.  $t \in (0, 1)$ ) such that  $d(z, x) = (1 - t)d(x, y)$  and  $d(z, y) = td(x, y)$ , and we will write  $z = tx + (1 - t)y$  for simplicity. The space  $(X, d)$  is called a *geodesic space* if each pair of two points of  $X$  are joined by a geodesic segment.  $(X, d)$  is said to have *the geodesic extension property* if each geodesic segment is contained in a geodesic.

One classic and important example of geodesic spaces with the geodesic extension property is the *hyperbolic  $m$ -space*  $\mathbb{H}^m$ , where  $m$  is a positive integer, which is defined as follows. Consider the quadratic form given by

$$\langle u, v \rangle := \sum_{i=1}^m u_i v_i - u_{m+1} v_{m+1}, \quad \forall u, v \in \mathbb{R}^{m+1}.$$

Following [9] (see also [10]), the hyperbolic  $m$ -space  $\mathbb{H}^m$  is the set defined by

$$\mathbb{H}^m := \{u = (u_1, u_2, \dots, u_{m+1}) \in \mathbb{R}^{m+1} : \langle u, u \rangle = -1, u_{m+1} > 0\}.$$

Then  $\mathbb{H}^m$  is a geodesic metric space (in fact it is a Hadamard manifold) of constant curvature  $-1$  and the hyperbolic distance  $d : \mathbb{H}^m \times \mathbb{H}^m \rightarrow \mathbb{R}^+$  is given by

$$\cosh d(\cdot, \cdot) = -\langle \cdot, \cdot \rangle.$$

Furthermore, a geodesic  $c : \mathbb{R} \rightarrow \mathbb{H}^m$  starting at  $x \in \mathbb{H}^m$  can be given by

$$c(t) = (\cosh t)x + (\sinh t)v, \quad \forall t \in \mathbb{R}, \tag{2.1}$$

where  $v \in T_x \mathbb{H}^m$  is a unit vector. In this case we say that the geodesic (or *hyperbolic segment*)  $c$  has the initial unit vector  $v$ . By definition, the *hyperbolic angle*  $\alpha \in [0, \pi]$  between two hyperbolic segments starting from a point in  $\mathbb{H}$ , with initial vectors  $u$  and  $v$ , is defined by

$$\cos \alpha = \langle u | v \rangle.$$

Let  $\kappa \leq 0$  and let  $M_\kappa^m$  denote the metric space obtained from  $(\mathbb{H}^m, d)$  by multiplying the distance function by  $1/\sqrt{-\kappa}$  if  $\kappa < 0$  and the  $m$ -dimensional Euclidean space  $\mathbb{R}^m$  if  $\kappa = 0$ . Then  $M_\kappa^m$  is

a metric space of constant curvature  $\kappa$ . We use the same symbol  $d$  to denote the distance in  $M_\kappa^m$  if no confusion arises.

A *geodesic triangle*  $\Delta(x_1, x_2, x_3)$  in a metric space  $(X, d)$  consists of three points in  $X$  (the *vertices* of  $\Delta$ ) and three geodesic segments joining each pair of vertices (the *edges* of  $\Delta$ ). A *comparison triangle* in  $M_\kappa^2$  for a geodesic triangle  $\Delta(x_1, x_2, x_3)$  in  $(X, d)$  is a triangle  $\Delta_\kappa(\bar{x}_1, \bar{x}_2, \bar{x}_3)$  in  $M_\kappa^2$  such that  $d(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$  for any  $i, j \in \{1, 2, 3\}$ . Note that comparison triangles always exist and are unique up to isometry, see [9, Paragraph I.2.14]. Let  $x, y$  and  $z$  be three points in  $X$  and let  $\Delta_\kappa(\bar{x}, \bar{y}, \bar{z})$  be a comparison triangle in  $M_\kappa^2$ . We define the  $\kappa$ -*comparison angle* between  $y$  and  $z$  at  $x$ , which is denoted by  $\angle_x^{(\kappa)}(y, z)$ , to be the interior angle of  $\Delta_\kappa(\bar{x}, \bar{y}, \bar{z})$  in  $M_\kappa^2$  at  $\bar{x}$ .

The law of cosines, which establishes a relationship between the interior angles and the edges of a triangle in  $M_\kappa^2$ , plays an important role for our study.

**Theorem 2.1** (*Hyperbolic Law of Cosines*). *Let  $\Delta$  be a triangle in  $M_\kappa^2$  with vertices  $A, B, C$ . Let  $a = d(B, C)$ ,  $b = d(A, C)$  and  $c = d(A, B)$ . Let  $\gamma$  denote the vertex angle at  $C$ . Then the following equality holds:*

$$\cosh(\sqrt{-\kappa}c) = \cosh(\sqrt{-\kappa}a) \cosh(\sqrt{-\kappa}b) - \sinh(\sqrt{-\kappa}a) \sinh(\sqrt{-\kappa}b) \cos(\gamma).$$

Following [10], we give in the following definition the notion of the Alexandrov angles for two geodesic paths in the metric space  $X$ .

**Definition 2.2.** Let  $X$  be a metric space and let  $x \in X$ . Let  $c : [0, a] \rightarrow X$  and  $c' : [0, a'] \rightarrow X$  be two geodesic paths with  $c(0) = c'(0) = x$ . For each  $t \in (0, a]$  and  $t' \in (0, a']$ , let  $\angle_x^{(0)}(c(t), c'(t'))$  be the 0-comparison angle between  $c(t)$  and  $c'(t')$  at  $x$ . The (Alexandrov) angle or the upper angle  $\angle(c, c')$  between the geodesic paths  $c$  and  $c'$  is defined by:

$$\angle(c, c') := \limsup_{t, t' \rightarrow 0^+} \angle_x^{(0)}(c(t), c'(t')).$$

In particular, the angle between the geodesic segments  $[p, x]$  and  $[p, y]$  is denoted by  $\angle_p(x, y)$ .

Now we are ready to define the notion of a metric space of curvature bounded below. Among all the possible equivalent definitions we have chosen the one fitting the best our purposes.

**Definition 2.3.** Let  $\kappa \in (-\infty, 0)$ . The metric space  $X$  is said to be a metric space of curvature bounded below by  $\kappa$  if for each triangle  $\Delta(x_1, x_2, x_3)$  in  $X$  and its comparison triangle  $\Delta_\kappa(\bar{x}_1, \bar{x}_2, \bar{x}_3)$  in  $M_\kappa^2$ , one has that

$$\angle_{x_i}(x_j, x_k) \geq \angle_{\bar{x}_i}^{(\kappa)}(\bar{x}_j, \bar{x}_k)$$

for any different  $i, j, k \in \{1, 2, 3\}$  and for any two shortest paths  $[x, y]$  and  $[p, q]$  in  $X$  with  $p$  an inner point of  $[x, y]$ , on has that  $\angle_p(x, q) + \angle_p(q, y) = \pi$ . We say that the metric space  $X$  is of curvature bounded below if there exists  $\kappa \in (-\infty, 0)$  such that  $X$  is of curvature bounded below by  $\kappa$ .

**Remark 2.4.** The definition of a metric space of curvature bounded below given here is taken from [10], which is certainly equivalent to the one in [16]. For other equivalent definitions, the reader is referred to [10, Chapter 4 and Section 10.1].

**Remark 2.5.** A metric space is said to be without bifurcating geodesics, see [37], if for any two segments with same initial point and having another common point (different to the initial one), this second point is a common endpoint of both or one segment contains the other. Notice that if this is not the case then we can find a triangle  $\Delta(p, x, y)$  such that  $p, x$  and  $y$  do not lay on a same geodesic but  $\angle_p(x, y) = 0$ . Therefore a metric space with bifurcating geodesics cannot be a space of curvature bounded below, see Definition 2.3.

Let  $x \in X$  and  $r > 0$ . We use  $\mathbf{B}(x, r)$  to denote the open ball with center at  $x$  and radius  $r$ ; while the corresponding closed ball is denoted by  $\overline{\mathbf{B}}(x, r)$ . We end this section with the notion of porosity.

**Definition 2.6.** A subset  $C$  of  $X$  is said to be porous in  $X$  if there exist  $0 < \alpha \leq 1$  and  $t_0 > 0$  such that for every  $x \in X$  and  $0 < t \leq t_0$  there is  $y \in X$  such that  $\mathbf{B}(y, \alpha t) \subseteq \mathbf{B}(x, t) \setminus C$ .  $C$  is said to be  $\sigma$ -porous in  $X$  if it is a countable union of sets which are porous.

Note that in this definition, the statement “for every  $x \in X$ ” can be replaced by “for every  $x \in \overline{C}$ ”. Moreover, it is clear that a porous set in  $X$  is also a set of first Baire category in  $X$ , but the converse is not true in general; see for example [4,5].

### 3. Stečkin’s Lemma in spaces of curvature bounded below

Let  $(X, d)$  be a geodesic space. Let

$$x \in X, r > 0, \quad y \in \mathbf{B}(x, r/2) \setminus \{x\} \quad \text{and} \quad 0 \leq \sigma \leq 2d(x, y). \tag{3.2}$$

Define

$$D(x, y; r, \sigma) := \overline{\mathbf{B}}(y, r - d(x, y) + \sigma) \setminus \mathbf{B}(x, r).$$

Throughout the remainder we always let  $\kappa \in (-\infty, 0)$ . The key fact to achieve our results is the estimation on the diameter of the sets  $D(x, y; r, \sigma)$ . For this purpose, we need to introduce the real function  $F_\kappa$  on  $\mathbb{R}_+^3$ , which is defined by

$$F_\kappa(d, r, \sigma) := \frac{2}{\sqrt{-\kappa}} \operatorname{arccosh} \left( \cosh^2(\sqrt{-\kappa}(r - d + \sigma)) - \frac{\sinh(\sqrt{-\kappa}(r - d + \sigma))}{\sinh(\sqrt{-\kappa}d)} \right) \\ \times \left[ \cosh(\sqrt{-\kappa}r) - \cosh(\sqrt{-\kappa}d) \cosh(\sqrt{-\kappa}(r - d + \sigma)) \right]$$

for each  $(d, r, \sigma) \in \mathbb{R}_+^3$ . For the sake of simplicity we will write  $F_\kappa$  as  $F$  for  $\kappa = -1$ . Furthermore, for any  $z \in X$  and  $A \subset X$ , we set  $r_z(A) := \sup\{d(z, y) : y \in A\}$ .

**Lemma 3.1.** *Let  $X = M_\kappa^2$ . Let  $x, y \in X$  and  $r, \sigma$  satisfy (3.2). Let  $z$  be the point in the geodesic determined by  $x$  and  $y$  such that*

$$d(z, x) = r + \sigma \quad \text{and} \quad d(z, y) = r - d(x, y) + \sigma. \tag{3.3}$$

Then the following equality holds:

$$F_\kappa(d(x, y), r, \sigma) = 2r_z(D(x, y; r, \sigma)). \tag{3.4}$$

**Proof.** By the definition of the metric in the space  $M_\kappa^2$ , we may assume, without loss of generality, that  $\kappa = -1$ .

By definition, it is easy to see that  $z \in D(x, y; r, \sigma)$ . By assumptions,  $0 \leq \sigma \leq 2d(x, y)$ ; hence there exists  $u \in X$  such that  $d(x, u) = r$  and  $d(u, y) = r - d(x, y) + \sigma$ . We first prove that

$$r_z(D(x, y; r, \sigma)) = d(z, u). \tag{3.5}$$

To do this, let  $v \in D(x, y; r, \sigma)$ . Then

$$d(v, x) \geq r = d(u, x) \quad \text{and} \quad d(y, v) \leq r - d(x, y) + \sigma = d(y, u). \tag{3.6}$$

It suffices to verify that  $d(v, z) \leq d(u, z)$ . For this purpose, consider the triangles  $\Delta(y, z, v)$  and  $\Delta(y, z, u)$ . Let  $\gamma := \angle_y(z, v)$  and  $\gamma' := \angle_y(z, u)$ . Then, from the law of cosines, we have that

$$\cosh d(z, v) = \cosh d(v, y) \cosh d(z, y) - \sinh d(v, y) \sinh d(z, y) \cos \gamma$$

and

$$\cosh d(z, u) = \cosh d(u, y) \cosh d(z, y) - \sinh d(u, y) \sinh d(z, y) \cos \gamma'.$$

Therefore, by subtracting,

$$\begin{aligned} \cosh d(z, v) - \cosh d(z, u) &= \cosh d(z, y)(\cosh d(v, y) - \cosh d(u, y)) \\ &\quad - \sinh d(z, y)(\sinh d(v, y) \cos \gamma - \sinh d(u, y) \cos \gamma'). \end{aligned} \tag{3.7}$$

Similarly, we consider the triangles  $\Delta(x, y, u)$  and  $\Delta(x, y, v)$ . Note that  $\angle_y(x, u) = \pi - \gamma$  and  $\angle_y(x, v) = \pi - \gamma'$ . Applying again the cosine law as above and the fact that  $d(v, x) \geq d(u, x)$  (cf. (3.6)), we obtain that

$$\begin{aligned} \cosh d(x, y)[\cosh d(y, v) - \cosh d(y, u)] \\ \geq \sinh d(x, y)[\sinh d(y, u) \cos \gamma' - \sinh d(y, v) \cos \gamma]. \end{aligned}$$

Since  $d(y, v) \leq d(y, u)$  by (3.6), it follows that

$$\sinh d(y, u) \cos \gamma' - \sinh d(y, v) \cos \gamma \leq 0.$$

Combining this with (3.7), we have that

$$\cosh d(z, v) - \cosh d(z, u) \leq \cosh d(z, y)(\cosh d(v, y) - \cosh d(u, y)) \leq 0;$$

hence  $d(v, z) \leq d(u, z)$  and the assertion (3.5) holds.

Thus to complete the proof, it suffices to prove that

$$\begin{aligned} \cosh d(u, z) &= \cosh^2(r - d(x, y) + \sigma) - \frac{\sinh(r - d(x, y) + \sigma)}{\sinh d(x, y)} \\ &\quad \times [\cosh r - \cosh d(x, y) \cosh(r - d(x, y) + \sigma)]. \end{aligned} \tag{3.8}$$

To do this, consider again the triangles  $\Delta(y, z, u)$  and  $\Delta(x, y, u)$  as well as the angles  $\gamma$  and  $\pi - \gamma$  defined above. Then, we apply the law of cosines to conclude that

$$\cosh d(u, z) = \cosh^2(r - d(x, y) + \sigma) - \sinh^2(r - d(x, y) + \sigma) \cos \gamma$$

and

$$\cosh r = \cosh d(x, y) \cosh(r - d(x, y) + \sigma) + \sinh d(x, y) \sinh(r - d(x, y) + \sigma) \cos \gamma.$$

Eliminating the term  $\cos \gamma$  from the above equations, (3.8) holds and the proof is completed.  $\square$

Some useful properties of the functions  $F_\kappa$  are described in the following proposition.

**Proposition 3.2.** *The function  $F_\kappa$  is continuous on  $\mathbb{R}_+^3$  and the following statements hold:*

- (i) *For any  $d \geq 0$  and  $r \geq 0$ , we have that  $F_\kappa(d, r, 0) = 0$ .*
- (ii) *For any  $d > 0$ ,  $r > 0$  and  $\sigma \geq 0$  with  $r/2 \geq d$  and  $\sigma \leq 2d$ , we have that  $F_\kappa(d, r, \sigma) \geq 2\sigma$ .*
- (iii) *For any  $r > 0$  and  $\sigma \geq 0$ , the function  $F_\kappa(\cdot, r, \sigma)$  is nonincreasing on  $(\sigma/2, r)$ .*

**Proof.** The continuity of the function  $F_\kappa$  is clear. By an elementary calculation, one can check that  $\cosh(F_\kappa(d, r, 0)) = 1$  for any  $d \geq 0$  and  $r \geq 0$ . Consequently, the conclusion (i) is proved.

To show (ii) and (iii), observe first that  $F_\kappa(d(x, y), r, \sigma) = 2r_z(D(x, y; r, \sigma))$  holds for any  $x, y, r$  and  $\sigma$  satisfying (3.2) and  $z$  as in the proof of Lemma 3.1. Then, choosing  $x, y$  such that  $d(x, y) = d$  and noting that  $r_z(D(x, y; r, \sigma)) \geq d(z, x) = \sigma$ , one sees that (ii) holds. As to (iii), we fix  $r > 0, \sigma \geq 0$  and  $x$ . Let  $z_0$  be the point in the geodesic determined by  $x$  and  $y$  such that  $d(z_0, x) = r$  and  $d(z_0, y) = r - d(x, y)$ . Then the corresponding  $z$  is fixed when  $y$  varies in  $(x, z_0)$ . Note the sets  $D(x, y; r, \sigma)$  get bigger as  $y$  varies in  $(x, z_0)$  such that  $d(x, y)$  decreases. This implies that  $r_z(D(x, y; r, \sigma))$  does not get smaller as  $d(x, y)$  decreases and the conclusion (iii) is easily seen to hold.  $\square$

The following lemma is an extension of [4, Lemma 6] from Hilbert spaces to geodesic spaces with nonpositive constant curvature. Let  $\text{diam}(Z)$  denote the diameter of the subset  $Z \subset X$ .

**Lemma 3.3.** *Let  $X = M_\kappa^2$ . Let  $x, y \in X$  and  $r, \sigma$  satisfy (3.2). Then the following inequality holds:*

$$\text{diam}(D(x, y; r, \sigma)) \leq F_\kappa(d(x, y), r, \sigma). \tag{3.9}$$

**Proof.** Let  $z$  be the point in the geodesic determined by  $x$  and  $y$  satisfying (3.3). Then, by Lemma 3.1, equality (3.4) holds. Note that

$$\begin{aligned} d(v, w) &\leq d(v, z) + d(z, w) \leq 2d(u, z) \leq 2r_z(D(x, y; r, \sigma)) \\ &\text{for any } u, w \in D(x, y; r, \sigma). \end{aligned}$$

This together with (3.4) implies that (3.9) holds.  $\square$

Next we show that Lemma 3.3 admits an extension to general spaces of curvature bounded below.

**Proposition 3.4.** *Let  $X$  be a geodesic space of curvature bounded below by  $\kappa$ . Let  $x, y, r$  and  $\sigma$  satisfy (3.2). Suppose that there exists  $u \in X$  in a geodesic passing through  $x$  and  $y$  such that  $d(x, u) = r$  and  $d(y, u) = r - d(x, y)$ . Then the following estimate holds:*

$$\text{diam}(D(x, y; r, \sigma)) \leq F_\kappa(d(x, y), r, \sigma) + 2\sigma. \tag{3.10}$$

**Proof.** Let  $v$  be any point in  $D(x, y; r, \sigma)$  and consider a comparison triangle  $\Delta_\kappa(\bar{y}, \bar{u}, \bar{v})$  of the triangle  $\Delta(y, u, v)$  in  $M_\kappa^2$ . Let  $\tilde{x}$  be the point in the geodesic line determined by  $\bar{y}$  and  $\bar{u}$  and such that

$$d(x, y) = d(\tilde{x}, \bar{y}) \quad \text{and} \quad d(x, u) = d(\tilde{x}, \bar{u}). \tag{3.11}$$

We claim that

$$\bar{v} \in D(\tilde{x}, \bar{y}; r, \sigma). \tag{3.12}$$

In fact, since  $\bar{v} \in \bar{\mathbf{B}}(\bar{y}, r - d(x, y) + \sigma)$  from identities of comparison triangles, it suffices to prove that  $d(\bar{x}, \bar{v}) \geq d(x, v)$ . Since  $X$  is of curvature bounded below, we have that  $\angle_y(x, v) + \angle_y(u, v) = \pi$  and  $\angle_y(u, v) \geq \angle_{\bar{y}}^{(\kappa)}(\bar{u}, \bar{v})$ . Therefore

$$\angle_y(x, v) \leq \angle_{\bar{y}}^{(\kappa)}(\bar{x}, \bar{v}) \tag{3.13}$$

because  $\angle_{\bar{y}}^{(\kappa)}(\bar{u}, \bar{v}) + \angle_{\bar{y}}^{(\kappa)}(\bar{x}, \bar{v}) = \pi$ .

Consider now a comparison triangle  $\Delta_\kappa(\bar{y}, \bar{x}, \bar{v})$  of  $\Delta(y, x, v)$  in  $M_\kappa^2$  with the corresponding vertices  $\bar{y}, \bar{x}$  and  $\bar{v}$ . Then  $\angle_y(x, v) \geq \angle_{\bar{y}}^{(\kappa)}(\bar{x}, \bar{v})$ . This together with (3.13) implies that  $\angle_{\bar{y}}^{(\kappa)}(\bar{x}, \bar{v}) \geq \angle_{\bar{y}}^{(\kappa)}(\bar{x}, \bar{v})$ . Since the adjacent sides in  $\Delta_\kappa(\bar{y}, \bar{u}, \bar{v})$  to the angle  $\angle_{\bar{y}}^{(\kappa)}(\bar{x}, \bar{v})$  are equal to the corresponding ones in  $\Delta_\kappa(\bar{y}, \bar{x}, \bar{v})$  to the angle  $\angle_{\bar{y}}^{(\kappa)}(\bar{x}, \bar{v})$ , it follows from the Hyperbolic law of cosines that  $d(\bar{x}, \bar{v}) \geq d(\bar{x}, \bar{v})$ . Noting that  $\Delta_\kappa(\bar{y}, \bar{x}, \bar{v})$  is a comparison triangle of  $\Delta(y, x, v)$ , we have that  $d(\bar{x}, \bar{v}) = d(x, v)$  and so  $d(\bar{x}, \bar{v}) \geq d(x, v)$ , which completes the proof of the claim.

To complete the proof of the proposition, let  $\bar{z}$  be the point in the geodesic determined by  $\bar{x}$  and  $\bar{y}$  such that (3.3) holds with  $\bar{x}, \bar{y}, \bar{z}$  in place of  $x, y, z$  respectively. Then  $d(\bar{z}, \bar{u}) = \sigma$ . Furthermore, by (3.4) and (3.12), one has that

$$d(\bar{v}, \bar{z}) \leq r_{\bar{z}}(D(\bar{x}, \bar{y}; r, \sigma)) = \frac{F_\kappa(d(\bar{x}, \bar{y}), r, \sigma)}{2} = \frac{F_\kappa(d(x, y), r, \sigma)}{2},$$

where the last equality follows from (3.11) by noting that  $d(\bar{x}, \bar{y}) = d(x, y)$ . Since  $\Delta_\kappa(\bar{y}, \bar{u}, \bar{v})$  is a comparison triangle of  $\Delta(y, u, v)$ , it follows that

$$d(v, u) = d(\bar{v}, \bar{u}) \leq d(\bar{v}, \bar{z}) + d(\bar{z}, \bar{u}) \leq \frac{F_\kappa(d(x, y), r, \sigma)}{2} + \sigma.$$

Thus the estimate (3.10) is easily seen to hold.  $\square$

The following corollary is immediate.

**Corollary 3.5.** *Let  $X$  be a geodesic space of curvature bounded below by  $\kappa$  with the property of the geodesic extension. Let  $x, y, r$  and  $\sigma$  satisfy (3.2). Then the estimate (3.10) holds.*

**Remark 3.6.** Notice that Corollary 3.5 here gives a quantitative version of Proposition 3.3 in [16]; moreover, a number of assumptions of this proposition are dropped. As a matter of fact, Proposition 3.3 in [16] assumes not only the space  $X$  of curvature bounded below by  $\kappa$  but also of curvature bounded above by 0 (i.e., a CAT(0) space). Then assumption that  $X$  is a CAT(0) space turns out to be much stronger than what they actually need. In fact, by the proof of Proposition 3.3 in [16] one realizes that what is actually required from the CAT(0) assumption is the uniform convexity of the metric (cf. [15] for more about uniform convexity of metric spaces). Notice that our Corollary 3.5 does not require any assumption about the uniform convexity of the metric.

#### 4. Nearest point problems

For the remainder of this paper, we will always assume that  $X$  is a complete geodesic space and  $A$  is a nonempty closed subset of  $X$ . Following [5], we define

$$\lambda(x) := \text{dist}(x, A) = \inf\{d(a, x) : a \in A\}, \quad \forall x \in X.$$

Recall that a sequence  $\{a_n\} \subseteq A$  is a minimizing sequence of the nearest point problem w.r.t.  $x$  if  $\lim d(a_n, x) = \lambda(x)$ , and that the nearest point problem w.r.t.  $x$  is well posed if  $x$  has a unique nearest point in  $A$  and each minimizing sequence of the nearest point problem w.r.t.  $x$  converges to the nearest point.

Let  $K(A)$  denote the set of all points  $x \in X \setminus A$  such that the nearest point problem w.r.t.  $x$  is well posed. Furthermore, define for each  $x \in X \setminus A$  and  $\sigma > 0$

$$Y_\sigma(x) := \overline{\mathbf{B}}(x, \lambda(x) + \sigma) \cap A$$

and

$$d_0(x) := \lim_{\sigma \rightarrow 0^+} \text{diam}(Y_\sigma(x)).$$

Then “by definition” one can verify directly that the nearest point problem w.r.t.  $x$  is well posed if and only if  $d_0(x) = 0$ ; hence,

$$K(A) = \{x \in X \setminus A : d_0(x) = 0\}. \tag{4.14}$$

In [16, Theorem 3.1] it is proved that if  $X$  is a CAT(0) space of curvature bounded below and the geodesic extension property, then the set  $K(A)$  is of the second Baire category in  $X$  (note that there the uniform convexity property of CAT(0) plays a crucial role), and presented the following open problem (cf. [16, Problem 3.11]):

$$\text{Does the conclusion remain true if the geodesic extension property is dropped?} \tag{4.15}$$

The following theorem improves [16, Theorem 3.1] and, in particular, gives an affirmative answer to question (4.15).

**Theorem 4.1.** *Suppose that  $X$  is a geodesic metric space of curvature bounded below. Then the set  $K(A)$  is a dense  $G_\delta$ -set in  $X \setminus A$ .*

**Proof.** Let  $\alpha > 0$  and let  $H_\alpha := H_\alpha(A)$  denote the set of all points  $x \in X$  such that  $d_0(x) \geq 1/\alpha$ . It is not hard to prove (cf., [12, Proposition 3.13]) that the set  $H_\alpha$  is closed. Moreover, we have that

$$K(A) = X \setminus \bigcup_{n=1}^\infty H_n = \bigcap_{n=1}^\infty (X \setminus H_n).$$

Since  $H_n$  is closed for each  $n$ , it is enough to prove that  $X \setminus H_n$  is dense in  $X \setminus A$  for each  $n$ .

To do this, let  $n = 1, 2, \dots$  and let  $x \in H_n$ . Let  $0 < d < \lambda(x)/2$  and  $0 < \varepsilon < 1/n$ . By assumption, let  $\kappa \in (-\infty, 0)$  be such that  $X$  is of curvature bounded below by  $\kappa$ . Then, by Proposition 3.2(i), we can fix  $\sigma_0 > 0$  such that  $F_\kappa(r, d, 2\sigma) + 4\sigma < \varepsilon$  for any  $0 \leq \sigma \leq \sigma_0$ . Let  $0 < \sigma \leq \sigma_0$ . Let  $a \in A$  such that  $d(x, a) < r + \sigma$  and choose  $y \in [x, a]$  such that  $d(x, y) = d$ . Then

$$Y_\sigma(y) \subseteq \overline{\mathbf{B}}(y, r - d + 2\sigma) \setminus \mathbf{B}(x, r) = \mathbf{D}(x, y; r, 2\sigma).$$

Note that  $d(x, a) \geq r$  and  $y \in [x, a]$ . There exists  $u \in [y, a]$  such that  $d(x, u) = r$  and  $d(y, u) = r - d(x, y)$ . Therefore we can apply Proposition 3.4 to conclude that

$$\text{diam}(Y_\sigma(y)) \leq \text{diam}(\mathbf{D}(x, y; r, 2\sigma)) \leq F_\kappa(d, r, 2\sigma) + 4\sigma \leq F_\kappa(d, r, 2\sigma_0) + 4\sigma_0 < \varepsilon,$$

where the second inequality holds because of Proposition 3.2(iii). Consequently we have that  $d_0(y) = \lim_{\sigma \rightarrow 0^+} \text{diam}(Y_\sigma(y)) \leq \varepsilon < 1/n$ . This means that  $y \in X \setminus H_n$ ; hence  $X \setminus H_n$  is dense in  $X \setminus A$  as  $d(x, y) = d$  and  $d > 0$  is arbitrary.  $\square$

We will show that Theorem 4.1 can be improved in the sense that the set of points where the nearest point problem  $(A, x)$  is well posed has  $\sigma$ -porous complement.

Let  $G_\kappa$  be the function defined by

$$G_\kappa(d, r, \sigma) := F_\kappa(d, r, \sigma) + 2\sigma \quad \text{for each } (d, r, \sigma) \in \mathbb{R}_+^3.$$

Then  $G_\kappa$  is continuous on  $\mathbb{R}_+^3$ . For the sake of simplicity we will write  $G$  for  $G_{-1}$ . Fixed  $x, y \in X$  and  $r > 0$ , we also define the function  $\tau_{xyr}^\kappa$  on  $\mathbb{R}_+$  by

$$\tau_{xyr}^\kappa(\varepsilon) := \sup\{\sigma \geq 0 : G_\kappa(d(x, y), r, \sigma) < \varepsilon\} \quad \text{for each } \varepsilon \in \mathbb{R}_+.$$

Then it is clear that

$$\tau_{xyr}^\kappa(\varepsilon) > 0 \quad \text{for each } \varepsilon > 0. \tag{4.16}$$

The proof of our next lemma is similar to that of [4, Lemma 7], but we prefer to write the proof here for completeness. Recall that  $P_A$  stands for the metric projection on  $A$ , that is

$$P_A(x) = \{a_x \in A : d(x, a_x) = \lambda(x)\} \quad \text{for each } x \in X.$$

**Lemma 4.2.** *Suppose that  $X$  is a geodesic metric space of curvature bounded below by  $\kappa$ . Let  $x \in X \setminus A$  and  $a_x \in P_A(x)$ . Let  $\varepsilon > 0$  and  $y \in (x, a_x)$ . Define*

$$\tau_{xy}^\kappa(\varepsilon) = \min \left\{ \frac{\tau_{xy\lambda(x)}^\kappa(\varepsilon)}{3}, \frac{2}{3}d(x, y) \right\}.$$

Then

$$\text{diam}(Y_{\tau_{xy}^\kappa(\varepsilon)}(z)) \leq \varepsilon \quad \text{for each } z \in \mathbf{B}(y, \tau_{xy}^\kappa(\varepsilon)).$$

**Proof.** Let  $z \in \mathbf{B}(y, \tau_{xy}^\kappa(\varepsilon))$ . Then, for each  $a \in Y_{\tau_{xy}^\kappa(\varepsilon)}(z)$ ,

$$d(a, y) \leq d(a, z) + d(z, y) \leq \lambda(z) + \tau_{xy}^\kappa(\varepsilon) + \tau_{xy}^\kappa(\varepsilon) \leq \lambda(y) + 3\tau_{xy}^\kappa(\varepsilon),$$

and so,  $Y_{\tau_{xy}^\kappa(\varepsilon)}(z) \subseteq Y_{3\tau_{xy}^\kappa(\varepsilon)}(y)$ . Recalling that  $y \in (x, a_x)$  and  $a_x \in P_A(x)$ , one sees that  $\lambda(y) = \lambda(x) - d(x, y)$  and so

$$\begin{aligned} Y_{3\tau_{xy}^\kappa(\varepsilon)}(y) &= \{a \in A : d(a, y) \leq \lambda(x) - d(x, y) + 3\tau_{xy}^\kappa(\varepsilon)\} \\ &\subseteq \overline{\mathbf{B}}(y, \lambda(x) - d(x, y) + 3\tau_{xy}^\kappa(\varepsilon)) \setminus \mathbf{B}(x, \lambda(x)) \\ &= \mathbf{D}(x, y; \lambda(x), 3\tau_{xy}^\kappa(\varepsilon)). \end{aligned}$$

Moreover, the existence of  $a_x$  guarantees that Proposition 3.4 is applicable, and so

$$\text{diam}(\mathbf{D}(x, y; \lambda(x), 3\tau_{xy}^\kappa(\varepsilon))) \leq G(d(x, y), \lambda(x), \tau_{xy}^\kappa(\varepsilon)) \leq \varepsilon,$$

which completes the proof.  $\square$

Next we prove the main result of this section, whose proof closely follows that of [4, Theorem 8].

**Theorem 4.3.** *Suppose that  $X$  is a geodesic metric space of curvature bounded below. Then the set  $X \setminus K(A)$  is  $\sigma$ -porous in  $X \setminus A$ .*

**Proof.** Recall that  $K(A)$  denote the set of all points  $x \in X \setminus A$  such that the nearest point problem w.r.t.  $x$  is well posed. Then  $K(A)$  is dense in  $X \setminus A$  by [Theorem 4.1](#). For each  $x \in K(A)$ , let  $a_x \in P_A(x)$  and set  $I_x := (x, \frac{x+a_x}{2}]$ . Let  $\{\varepsilon_k\} \subseteq (0, 1]$  be a decreasing null convergent sequence. Without loss of generality, we assume that  $X$  is of curvature bounded below by  $\kappa = -1$ . Write  $\tau_{xy}(\varepsilon_k) := \tau_{xy}^{-1}(\varepsilon_k)$ , where  $\tau_{xy}^{-1}(\varepsilon_k)$  is defined as in [Lemma 4.2](#) with  $\kappa = -1$ . Then we define

$$X^* := \bigcap_{k \in \mathbb{N}} \bigcup_{x \in K(A)} \bigcup_{y \in I_x} \mathbf{B}(y, \tau_{xy}(\varepsilon_k)).$$

We claim that  $X^* \subseteq K(A)$ . Indeed, let  $z \in X^*$  and  $k \in \mathbb{N}$ . Then there exist  $x_k \in K(A)$  and  $y_k \in I_{x_k}$  such that  $z \in \mathbf{B}(y_k, \tau_{x_k y_k}(\varepsilon_k))$ . Without loss of generality, assume that  $y_k \neq x_k$ . Then,  $\tau_{x_k y_k}(\varepsilon_k) > 0$  by [\(4.16\)](#) and  $\text{diam}(Y_{\tau_{x_k y_k}(\varepsilon_k)}(z)) \leq \varepsilon_k$  by [Lemma 4.2](#). This implies that

$$d_0(z) = \lim_{\sigma \rightarrow 0^+} \text{diam}(Y_\sigma(z)) = 0$$

and so  $z \in K(A)$  by [\(4.14\)](#). Therefore the claim holds. Thus it suffices to show that  $X \setminus X^*$  is  $\sigma$ -porous in  $X \setminus A$ . For this purpose, let  $k, h \in \mathbb{N}$  and define

$$X_k = (X \setminus A) \setminus \bigcup_{x \in K(A)} \bigcup_{y \in I_x} \mathbf{B}(y, \tau_{xy}(\varepsilon_k)), \quad X_{kh} = \{z \in X_k : 1/h < \lambda(z) < h\}.$$

Then

$$(X \setminus A) \setminus X^* = \bigcup_{k \in \mathbb{N}} X_k = \bigcup_{k \in \mathbb{N}} \bigcup_{h \in \mathbb{N}} X_{kh}.$$

To complete the proof, we only need to show that  $X_{kh}$  is porous in  $X \setminus A$ . Let  $0 < \rho \leq 1/h$  and set

$$\alpha := \inf \left\{ \frac{\tau_{xy}(\varepsilon_k)}{4h} : x \in K(A), y \in I_x, 1/h \leq \lambda(x) \leq h \text{ and } d(x, y) \geq \rho/4 \right\}.$$

Then  $\alpha \in (0, 1/6)$ . In fact, since  $x \in X \setminus A$ , it follows that  $\tau_{xy}(\varepsilon_k) \leq \frac{2}{3} d(x, y) \leq \frac{2}{3} h$  and so  $\alpha \leq \frac{1}{6}$ . To see that  $\alpha > 0$ , we suppose on the contrary that  $\alpha = 0$ . Then there exist  $x_i \in K(A)$ ,  $y_i \in I_{x_i}$  with  $1/h \leq \lambda(x_i) \leq h$  and  $d(x_i, y_i) \geq \rho/4$  such that

$$\lim_{i \rightarrow \infty} \tau_{x_i y_i}(\varepsilon_k) = 0. \tag{4.17}$$

Without loss of generality, we may assume that  $\lambda(x_i) \rightarrow r_0$  and  $\rho_i := d(x_i, y_i) \rightarrow \rho_0$  with  $r_0 \in [1/h, h]$  and  $\rho_0 > 0$ . Thus, by the definition of  $\tau_{x_i y_i}(\varepsilon_k)$ , [\(4.17\)](#) entails that

$$\lim_{i \rightarrow \infty} \sup \{ \sigma : G(\rho_i, r_i, \sigma) < \varepsilon_k \} = 0.$$

By the continuity of the function  $G$ , we have that  $G(\rho_0, r_0, 0) = \varepsilon_k$ , which contradicts [Proposition 3.2\(i\)](#) and shows that  $\alpha > 0$ .

Let  $t_0 = 1/(2h)$ . To complete the proof it suffices to prove that for each  $z \in X_{kh}$  and  $\rho \in (0, t_0]$  there exists  $y \in X \setminus A$  such that

$$\mathbf{B}(y, \alpha\rho) \cap (X \setminus A) \subseteq \mathbf{B}(z, \rho) \cap [(X \setminus A) \setminus X_k]. \tag{4.18}$$

Granting this, one sees that  $X_{kh}$  is porous in  $X \setminus A$  which completes the proof.

Let  $z \in X_{kh}$  and  $\rho \in (0, t_0]$ . Choose  $x \in K(A)$  such that  $d(x, z) < \rho/2$  and  $1/h < \lambda(x) < h$ . Let  $a_x \in P_A(x)$  and  $x_1 := \frac{1}{2}x + \frac{1}{2}a_x$ . Note that

$$|\lambda(x) - \lambda(z)| \leq d(x, z) \quad \text{for any } x, z \in X.$$

Then

$$d(x_1, z) \geq d(a_x, z) - d(a_x, x_1) \geq \lambda(z) - \frac{1}{2}\lambda(x) > \frac{1}{2h} - \frac{d(x, z)}{2} \geq t_0 - \frac{\rho}{4} \geq \frac{3\rho}{4},$$

and so there exists  $y \in I_x$  such that  $d(y, z) = \frac{3}{4}\rho$ . Consequently,  $d(y, x) \geq d(y, z) - d(z, x) \geq \rho/4$ . This together with the definition of  $\alpha$  implies that

$$\alpha\rho \leq \alpha 4d(x, y) \leq \alpha 4h \leq \tau_{xy}(\varepsilon_k)$$

(noting that  $1/h \leq \lambda(x) \leq h$  and  $\alpha \leq 1/6$ ). Hence we have that  $\mathbf{B}(y, \alpha\rho) \subseteq \mathbf{B}(y, \tau_{xy}(\varepsilon_k))$ . Moreover, because, for each  $w \in \mathbf{B}(y, \alpha\rho)$ ,

$$d(w, z) \leq d(w, y) + d(y, z) \leq \alpha\rho + \frac{3}{4}\rho \leq \rho,$$

we have that  $\mathbf{B}(y, \alpha\rho) \subseteq \mathbf{B}(z, \rho)$ . Therefore, (4.18) is proved.  $\square$

**Remark 4.4.** The interested reader may check the close relation relation between Theorem 4.3 and Theorem 2 in [36], both leading to a  $\sigma$ -porosity result in connection to related problems.

### 5. Farthest point problems

This section is devoted to the study of the well posedness of the farthest point problem. The approach used in this section is similar to the one used in the previous section.

Throughout the whole section, we assume that  $X$  is a complete geodesic space and  $A$  is a nonempty bounded and closed subset of  $X$ . Define

$$\mu(x) := \sup\{d(a, x) : a \in A\} \quad \text{for each } x \in X.$$

Similarly to the case for nearest point problems, a sequence  $\{a_n\} \subseteq A$  such that  $\lim d(a_n, x) = \mu(x)$  is called a maximizing sequence of the farthest point problem w.r.t.  $x$ , and the farthest point problem w.r.t.  $x$  is said to be well posed if  $x$  has a unique farthest point in  $A$  and each maximizing sequence of the farthest point problem w.r.t.  $x$  converges to the farthest point.

Let  $L(A)$  denote the set of all points in  $X$  such that the farthest point problem w.r.t.  $x$  is well posed. Given  $\sigma > 0$ , we set

$$M_\sigma(x) := \{a \in A : d(a, x) \geq \mu(x) - \sigma\}$$

and define

$$e_0(x) := \lim_{\sigma \rightarrow 0^+} \text{diam}(M_\sigma(x)) \quad \text{for each } x \in X.$$

Then the farthest point problem w.r.t.  $x$  is well posed if and only if  $e_0(x) = 0$ ; hence,

$$L(A) = \{x \in X : e_0(x) = 0\}. \tag{5.19}$$

The following theorem is an improvement of [16, Theorem 3.2] and plays a key role in the proof of the main result of this section.

**Theorem 5.1.** *Suppose that  $X$  is a geodesic space of curvature bounded below and with the geodesic extension property. Then the set  $L(A)$  is a dense  $G_\delta$ -set in  $X$ .*

**Proof.** Let  $\alpha > 0$  and let  $G_\alpha := G_\alpha(A)$  denote the set of all points  $x \in X$  such that  $e_0(x) \geq 1/\alpha$ . Then, in a similar way to the case for nearest point problems, we have that the set  $G_\alpha$  is closed and

$$L(A) = X \setminus \bigcup_{n=1}^\infty G_n = \bigcap_{n=1}^\infty (X \setminus G_n).$$

Thus, to complete the proof, it suffices to prove that  $X \setminus G_n$  is dense in  $X$  for each  $n$ .

Let  $n = 1, 2, \dots$  and let  $x \in G_n$ . Without loss of generality, assume that  $A$  is not a singleton. Then  $\mu(x) > 0$ . Let  $0 < d < r/2$  and  $0 < \varepsilon < 1/n$ . Assume that  $X$  is of curvature bounded below by  $\kappa$ . Then by Proposition 3.2(i), we can fix  $\sigma_0 > 0$  such that  $F_\kappa(r, d, 2\sigma) + 4\sigma < \varepsilon$  for any  $0 \leq \sigma \leq \sigma_0$ . Let  $0 < \sigma \leq \sigma_0$ . Let  $a \in A$  such that  $d(x, a) > \mu(x) - \sigma$ . Then by the geodesic extension property, there exists  $y \in X$  such that  $x \in [y, a]$  and  $d(x, y) = d$ . Clearly, we have that  $M_\sigma(y) \subseteq \overline{\mathbf{B}}(x, \mu(x))$  as  $A \subseteq \overline{\mathbf{B}}(x, \mu(x))$ . Furthermore, since

$$\mu(y) \geq d(y, a) = d(x, a) + d(x, y) \geq \mu(x) + d(x, y) - \sigma = \mu(x) + d - \sigma,$$

it follows that

$$M_\sigma(y) \subseteq \overline{\mathbf{B}}(x, \mu(x)) \setminus \mathbf{B}(y, \mu(x) + d - 2\sigma).$$

Write  $r := \mu(x) + d - 2\sigma$ . Then we have that

$$M_\sigma(y) \subseteq \overline{\mathbf{B}}(x, r - d + 2\sigma) \setminus \mathbf{B}(y, r) = D(y, x; r, 2\sigma).$$

Consequently, Proposition 3.4 can be applied to conclude that

$$\text{diam}(M_\sigma(y)) \leq \text{diam}(D(y, x; r, 2\sigma)) \leq F_\kappa(d, r, 2\sigma) + 4\sigma \leq F_\kappa(d, r, 2\sigma_0) + 4\sigma_0 < \varepsilon,$$

where the second inequality holds because of Proposition 3.2(iii). This implies that  $d_0(y) = \lim_{\sigma \rightarrow 0^+} \text{diam}(M_\sigma(y)) \leq \varepsilon < 1/n$ ; hence  $y \in X \setminus H_n$ . As  $d(x, y) = d$  and  $d > 0$  is arbitrary, one sees that  $X \setminus H_n$  is dense in  $X \setminus A$ .  $\square$

Recall that  $F_A$  is the farthest point mapping defined by

$$F_A(x) = \{y \in A : d(x, y) = \sup\{d(x, z) : z \in A\}\}.$$

**Lemma 5.2.** *Suppose that  $X$  is a geodesic space of curvature bounded below by  $\kappa$  with the geodesic extension property. Let  $x \in X$  and  $b_x \in F_A(x)$ . Set  $I_x := \{y : y = tx + (1-t)b_x, 1 < t \leq 3/2\}$ . Let  $\varepsilon > 0$  and  $y \in I_x$ . Define*

$$\delta_{xy}^\kappa(\varepsilon) := \min \left\{ \frac{\tau_{xy\mu(y)}^\kappa(\varepsilon)}{3}, \frac{2}{3}d(x, y), \frac{\mu(x)}{6} \right\}.$$

Then

$$\text{diam}(M_{\delta_{xy}^\kappa(\varepsilon)}(z)) \leq \varepsilon \quad \text{for each } z \in \mathbf{B}(y, \delta_{xy}^\kappa(\varepsilon)).$$

**Proof.** For simplicity, we assume that  $\kappa = -1$  and denote  $\delta_{xy}^\kappa(\varepsilon)$  as  $\delta_{xy}(\varepsilon)$ . Let  $z \in \mathbf{B}(y, \delta_{xy}(\varepsilon))$ . As in the proof of Lemma 4.2, we can obtain that

$$M_{\delta_{xy}(\varepsilon)}(z) \subseteq M_{3\delta_{xy}(\varepsilon)}(y).$$

By Remark 2.5, there are no bifurcating geodesics and it is easy to see that  $\mu(y) = \mu(x) + d(y, x)$ . Then we have

$$M_{3\delta_{xy}(\varepsilon)}(y) = \{a \in A : d(a, y) \geq \mu(x) + d(x, y) - 3\delta_{xy}(\varepsilon)\} \\ \subseteq \overline{\mathbf{B}}(x, \mu(x)) \setminus \mathbf{B}(y, \mu(x) + d(x, y) - 3\delta_{xy}(\varepsilon)).$$

Now, letting  $r = \mu(x) + d(x, y) - 3\delta_{xy}(\varepsilon)$ , we have

$$M_{3\delta_{xy}(\varepsilon)}(y) \subseteq \overline{\mathbf{B}}(x, r - d(x, y) + 3\delta_{xy}(\varepsilon)) \setminus \mathbf{B}(y, r) \\ = \mathbf{D}(y, x; r, 3\delta_{xy}(\varepsilon)).$$

Now, observing that  $0 < d(x, y) \leq r/2$  and that  $0 < 3\delta_{xy}(\varepsilon) \leq 2d(x, y)$ , we can apply Corollary 3.5 to deduce that

$$\text{diam}(\mathbf{D}(y, x; r, 3\delta_{xy}(\varepsilon))) \leq G(d(x, y), r, 3\delta_{xy}(\varepsilon)) \leq \varepsilon.$$

Since  $z$  is arbitrary in  $\mathbf{B}(y, \delta_{xy}(\varepsilon))$ , the conclusion follows.  $\square$

Next we show the main result about farthest point problems.

**Theorem 5.3.** *Suppose that  $X$  is a geodesic space of curvature bounded below and with the geodesic extension property. Then the set  $X \setminus L(A)$  is  $\sigma$ -porous in  $X$ .*

**Proof.** Again we assume that  $\kappa = -1$  and that  $A$  is not a singleton. The proof follows the same patterns than that of Theorem 4.3. From Theorem 5.1 we know that the set  $L(A)$  is dense in  $X$ . Let  $\{\varepsilon_k\} \subseteq (0, 1]$  be a decreasing null convergent sequence and define

$$X^* = \bigcap_{k \in \mathbb{N}} \bigcup_{x \in L(A)} \bigcup_{y \in I_x} \mathbf{B}(y, \delta_{xy}(\varepsilon_k)),$$

where  $I_x$  and  $\delta_{xy}^{\kappa}(\varepsilon_k)$  are defined as in Lemma 5.2.

As in the proof of Theorem 4.3, we have from (5.19) and Lemma 5.2 that  $X^* \subseteq L(A)$ . Therefore it suffices to prove that  $X \setminus X^*$  is  $\sigma$ -porous in  $X$ . For this purpose, we define, for  $k, h \in \mathbb{N}$ ,

$$X_k := X \setminus \bigcup_{x \in L(A)} \bigcup_{y \in I_x} \mathbf{B}(y, \delta_{xy}(\varepsilon_k)) \quad \text{and} \quad X_{kh} = \left\{ z \in X_k : \frac{1}{h} < \mu(z) < h \right\}.$$

Then

$$X \setminus X^* = \bigcup_{k \in \mathbb{N}} X_k = \bigcup_{k \in \mathbb{N}} \bigcup_{h \in \mathbb{N}} X_{kh}.$$

We claim that  $X_{kh}$  is porous in  $X$  for each pair  $(k, h)$ . Let  $k, h \in \mathbb{N}$  and  $0 < \rho \leq t_0 := \frac{1}{h}$ . Set

$$\alpha := \inf \left\{ \frac{\delta_{xy}(\varepsilon_k)}{4h} : x \in L(A), y \in I_x, 1/h \leq \mu(x) \leq h \text{ and } d(x, y) \geq \rho/4 \right\}.$$

Then we have that  $\alpha \in (0, 1/6)$ . Notice here that  $\alpha$  is estimated for  $\mu(y) \in (\frac{1}{h}, \frac{3}{2}h)$ .

Let  $z \in X_{kh}$  and  $x \in L(A)$  be such that  $d(x, z) < \rho/4$  and  $1/h < \mu(x) < h$ . Let  $b_x \in F_A(x)$  and  $x_1 \in X$  such that  $x = \frac{2}{3}x_1 + \frac{1}{3}b_x$ . Then

$$d(x_1, z) \geq d(x_1, b_x) - d(b_x, z) \geq \frac{3}{2}\mu(x) - \mu(z) \geq \mu(x) - d(x, z) \geq t_0 - \frac{\rho}{4} \geq \frac{3\rho}{4},$$

where the third inequality holds because of the following Lipschitz continuity:

$$|\mu(x) - \mu(z)| \leq d(x, z) \quad \text{for any } x, z \in X.$$

Consequently, there exists  $y \in I_x$  such that  $d(y, z) = \frac{3}{4}\rho$  and so  $d(y, x) \geq \rho/4$ . Therefore, noting that  $\alpha \leq 1/4$ , we have that  $\mathbf{B}(y, \alpha\rho) \subseteq \mathbf{B}(z, \rho)$ . Also, since  $\rho \leq 4d(x, y)$  and  $1/h \leq \mu(x) \leq h$ , it follows from the definition of  $\alpha$  that

$$\alpha\rho \leq \alpha 4d(x, y) \leq \alpha 4h \leq \delta_{xy}(\varepsilon_k)$$

and so  $\mathbf{B}(y, \alpha\rho) \subseteq \mathbf{B}(y, \delta_{xy}(\varepsilon_k))$ . Consequently,

$$\mathbf{B}(y, \alpha\rho) \subseteq X \setminus X_k \subseteq X \setminus X_{kh}.$$

This shows that  $X_{kh}$  is porous in  $X$  and completes the proof.  $\square$

### 6. Concluding remarks and an example

In view of [16, Example 3.9], the condition that geodesics do not bifurcate is a necessary condition for Theorems 4.3 and 5.3 to hold. A question raised in [16] asks whether this condition is also sufficient, that is, whether the condition of curvature bounded below in Theorems 4.3 and 5.3 can be replaced by the weaker one of not having bifurcating geodesics. We do not have an answer to this question. However, we give in the following an example showing that the Stečkin’s Lemma (Proposition 3.4), which is the main tool in our proof of Theorems 4.3 and 5.3, does not hold under this weaker condition.

**Example 6.1.** Consider the collection of model spaces given by  $M_{-n}^2$  for  $n \in \mathbb{N}$ . Consider a line (isometric to the real line) in each of these spaces and glue all them through this line as it is shown in [9, Chapter II.11]. The resulting space, which we denote by  $\bigsqcup_{n=1, I}^{\infty} M_{-n}^2$  where  $I$  stands for the gluing line, is a geodesic space with no bifurcating geodesics and which curvature is not bounded below, notice also that this is a CAT(0) space. Let  $x, y \in I$  such that  $d(x, y) = 1/2$  and make  $r = 1$  and  $\sigma > 0$ . Then, from Lemma 3.1, we know that  $\text{diam}(D(x, y; 1, \sigma)) \geq \frac{1}{2}F_{-n}(1/2, 1, \sigma)$  for each  $n \in \mathbb{N}$ .

We are going to show that

$$\lim_{n \rightarrow \infty} \frac{1}{2}F_{-n}(1/2, 1, \sigma) = 1 + 2\sigma \quad \text{for each } \sigma > 0, \tag{6.20}$$

granting this, the Stečkin’s Lemma fails for this space.

Define a real function  $y(\cdot)$  on  $(0, +\infty)$  by

$$y(t) := \cosh^2((1 + 2\sigma) \ln t) - \frac{\sinh((1 + 2\sigma) \ln t)}{\sinh(\ln t)} \\ \times [\cosh(2(\ln t)) - \cosh(\ln t) \cosh((1 + 2\sigma) \ln t)]$$

for each  $t \in (0, +\infty)$ . Then

$$\frac{1}{2}F_{-n}(1/2, 1, \sigma) = \frac{\text{arccosh}[y(e^{\sqrt{n}}/2)]}{\sqrt{n}} \quad \text{for any } n \in \mathbb{N} \text{ and } \sigma > 0.$$

Thus it suffices to verify that

$$\lim_{t \rightarrow +\infty} \frac{\text{arccosh}[y(t)]}{2 \ln t} = 1 + 2\sigma \quad \text{for each } \sigma > 0. \tag{6.21}$$

For this purpose, let  $\sigma > 0$ . By definitions of the functions  $\sinh$  and  $\cosh$ ,  $y$  can be expressed as

$$y(t) = \frac{t^{2+4\sigma} + y_0(t)}{2(1 - t^{-2})} \quad \text{for each } t \in (0, +\infty), \tag{6.22}$$

where  $y_0$  is of the form

$$y_0(t) = \sum_{i=1}^9 a_i t^{\beta_i} \quad \text{for each } t \in (0, +\infty)$$

with each  $\beta_i \in [-4 - 2\sigma, 2 + 2\sigma]$  (only depend on  $\sigma$ ) and each  $a_i$  being constant. Therefore we have that

$$\lim_{t \rightarrow +\infty} y(t) = +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} \frac{t \cdot y'_0(t)}{t^{2+4\sigma} + y_0(t)} = 0. \tag{6.23}$$

By L'Hôpital's Rule in element calculus we can obtain that

$$\lim_{t \rightarrow +\infty} \frac{\operatorname{arccosh}[y(t)]}{\ln[y(t)]} = \lim_{u \rightarrow +\infty} \frac{\operatorname{arccosh}(u)}{\ln u} = \lim_{v \rightarrow +\infty} \frac{v}{\ln \cosh(v)} = 1. \tag{6.24}$$

Furthermore, by (6.22) and (6.23), we can use L'Hôpital's Rule to conclude that

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{\ln[y(t)]}{\ln t} &= \lim_{t \rightarrow +\infty} \frac{\ln[t^{2+4\sigma} + y_0(t)] - \ln[2(1 - t^{-2})]}{\ln t} \\ &= \lim_{t \rightarrow +\infty} \frac{\ln[t^{2+4\sigma} + y_0(t)]}{\ln t} = 2 + 4\sigma. \end{aligned} \tag{6.25}$$

Thus combining (6.24) and (6.25), one sees that (6.21) holds and the proof is complete.

**Acknowledgments**

The research of the first author was partially supported by DGES, Grant MTM2009-1110696-C02-01 and Junta de Andalucía, Grant FQM-127. The research of the second author was partially supported by the National Natural Science Foundation of China (grant 10731060). The research of the third author was partially supported by DGES, Grant MTM2009-1110696-C02-01 and Junta de Andalucía, Grant FQM-127.

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