

Resolvents of Set-Valued Monotone Vector Fields in Hadamard Manifolds

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Abstract Firmly nonexpansive mappings are introduced in Hadamard manifolds, a particular class of Riemannian manifolds with nonpositive sectional curvature. The resolvent of a set-valued vector field is defined in this setting and by means of this concept, a strong relationship between monotone vector fields and firmly nonexpansive mappings is established. This fact is then used to prove that the resolvent of a maximal monotone vector field has full domain. The Yosida approximation of a set-valued vector field is also introduced, analyzing its properties from which the asymptotic behavior of the resolvent is studied. Regarding the singularities of a set-valued monotone vector field, existence results are proved under certain boundary condition. As a consequence, the existence of fixed points for continuous pseudo-contractive mappings is obtained.

Keywords Hadamard manifold · Firmly nonexpansive mapping · Resolvent · Yosida approximation · Maximal monotone vector field · Pseudo-contractive mapping

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1 Introduction

The resolvent of a monotone operator in the setting of Banach spaces was introduced by Brezis et al. in [1]. In this seminal paper, the fundamental properties for the resolvent are set up, with special emphasis on the strong connection between its fixed points and the zeros of the monotone operator. From this starting point, the asymptotic behavior of the resolvent has awakened the interest of many researchers. See, for instance, [2, 7, 8, 16, 17, 21, 25, 35, 37–39] and references therein.

In the framework of a Hilbert space H , given a set-valued maximal monotone operator $A : H \rightarrow 2^H$ and $\lambda > 0$, the resolvent of order λ associated to A is the single-valued mapping $J_\lambda : H \rightarrow H$ defined by

$$J_\lambda(x) = (I + \lambda A)^{-1}(x), \quad (1)$$

for any $x \in H$. Clearly, the set of singularities of A , $A^{-1}(0)$, coincides with the fixed point set of J_λ , $\text{Fix}(J_\lambda)$. Moreover, it is well-known (see, for instance, [7]) that the resolvent is firmly nonexpansive; that is,

$$\|J_\lambda(x) - J_\lambda(y)\|^2 \leq \langle x - y, J_\lambda(x) - J_\lambda(y) \rangle,$$

for all $x, y \in H$. On the other hand, any firmly nonexpansive mapping is the resolvent of a monotone operator. It is this approach, applicable to other related problems as well, which renders firmly nonexpansive mappings an important tool in monotone operator theory.

Regarding the asymptotic behavior, Reich showed in [38] that for every $x \in H$, $J_\lambda(x)$ converges strongly to $P_{A^{-1}(0)}x$ as $\lambda \rightarrow \infty$, where $P_{A^{-1}(0)}$ is the metric projection onto $A^{-1}(0)$. More precisely, Reich proved this result in the framework of uniformly smooth Banach spaces for m -accretive operators; the counterpart for maximal monotone operators was provided later by Kido in [17]. The strong convergence of $J_\lambda(x)$ as $\lambda \rightarrow 0$ was studied by Takahashi and Ueda in the case of either uniformly convex or uniformly smooth Banach spaces; see [45]. Recall that the notions of monotone operators and accretive operators coincide in the framework of Hilbert spaces.

Out of the setting of linear vector spaces, Hoyos introduced in [24] the concept of accretive type vector fields on Finsler manifolds; while the definition of monotone vector fields in the setting of Riemannian manifolds was introduced by Németh in [32]. The study of accretive and monotone type operators in metric spaces has been the focus of many authors in the last decades; see, for example, [10, 12, 13, 22, 23, 27, 28, 30, 32, 34, 40, 44]. It is worth mentioning that, as happens in the case of Hilbert spaces, in Hadamard manifolds, which are Riemannian manifolds of nonpositive sectional curvature, the class of maximal monotone vector fields coincides with the class of m -accretive vector fields, see [48].

The purpose of this paper is to introduce and study the concepts of firmly nonexpansive mappings and the resolvents of a set-valued vector field in the setting of Hadamard manifolds. The former is studied following consistently the definition given in the Hilbert ball by Goebel and Reich; see [15]. The latter coincides with the notion implicitly defined on Hilbert manifolds by Iwamiya and Okochi in [26].

The organization of the paper is as follows. Section 2 is devoted to basic notions, results and notations on Hadamard manifolds. In Section 3, the concept of firmly nonexpansive mappings in this setting is introduced and a relevant characterization

inequality for this mappings is provided. The resolvent and Yosida approximation of a set-valued vector field are defined in Section 4. Additionally, it is proved that they both enjoy as “good” properties as they do in the case of Hilbert spaces. As a consequence of these properties, in Section 5, the asymptotic behavior of the resolvent is studied and existence results for singularities of monotone vector fields and fixed points of continuous pseudo-contractive mappings are proved under certain boundary condition. Section 5 finishes with an example where it is shown how the previous results can be applied in optimization theory.

2 Preliminaries

In this section we introduce the fundamental definitions, properties and notations needed for a comprehensive reading of this paper. This can be found in any text book on Riemannian geometry, for example, [11, 42, 46].

Let M be a simply connected m -dimensional manifold. Given $x \in M$, the tangent space of M at x is denoted by T_xM and the tangent bundle of M by $TM = \bigcup_{x \in M} T_xM$, which is naturally a manifold. We always assume that M can be endowed with a Riemannian metric to become a *Riemannian* manifold. We denote by $\langle \cdot, \cdot \rangle_x$ the scalar product on T_xM with the associated norm $\| \cdot \|_x$, where the subscript x will be omitted. Given a piecewise smooth curve $\gamma : [a, b] \rightarrow M$ joining x to y (i.e. $\gamma(a) = x$ and $\gamma(b) = y$), by using the metric, we can define the length of γ as $L(\gamma) = \int_a^b \|\gamma'(t)\| dt$. Then, for any $x, y \in M$ the Riemannian distance $d(x, y)$, which induces the original topology on M , is defined by minimizing this length over the set of all such curves joining x to y .

Let ∇ be the Levi-Civita connection associated with $(M, \langle \cdot, \cdot \rangle)$. Let γ be a smooth curve in M . A vector field X is said to be parallel along γ if $\nabla_{\gamma'} X = 0$. If γ' itself is parallel along γ , we say that γ is a *geodesic*, and in this case $\|\gamma'\|$ is constant. When $\|\gamma'\| = 1$, γ is said to be normalized. A geodesic joining x to y in M is said to be minimal if its length equals $d(x, y)$.

A Riemannian manifold is complete if for any $x \in M$ all geodesics emanating from x are defined for all $-\infty < t < \infty$. By the Hopf-Rinow Theorem, we know that if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover, (M, d) is a complete metric space and bounded closed subsets are compact.

We use $P_{\gamma, \cdot}$ to denote the parallel transport on the tangent bundle TM along γ with respect to ∇ , which is defined by

$$P_{\gamma, \gamma(b), \gamma(a)}(v) = V(\gamma(b)) \quad \forall a, b \in \mathbb{R} \text{ and } v \in T_{c(a)}M,$$

where V is the unique vector field satisfying $\nabla_{\gamma'(t)} V = 0$ for all t and $V(\gamma(a)) = v$. Then, for any $a, b \in \mathbb{R}$, $P_{\gamma, \gamma(b), \gamma(a)}$ is an isometry from $T_{\gamma(a)}M$ to $T_{\gamma(b)}M$. Note that, for any $a, b, b_1, b_2 \in \mathbb{R}$,

$$P_{\gamma, \gamma(b_2), \gamma(b_1)} \circ P_{\gamma, \gamma(b_1), \gamma(a)} = P_{\gamma, \gamma(b_2), \gamma(a)} \quad \text{and} \quad P_{\gamma, \gamma(b), \gamma(a)}^{-1} = P_{\gamma, \gamma(a), \gamma(b)}.$$

We will write $P_{y,x}$ instead of $P_{\gamma, y, x}$ in the case when γ is a minimal geodesic joining x to y so no confusion arises.

Assuming that M is complete, the exponential map $\exp_x : T_xM \rightarrow M$ at x is defined by $\exp_x v = \gamma_v(1, x)$ for each $v \in T_xM$, where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic

starting at x with velocity v (i.e. $\gamma(0) = x$ and $\gamma'(0) = v$). Then $\exp_x tv = \gamma_v(t, x)$ for each real number t .

A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold. Throughout the remainder of the paper, we always assume that M is a m -dimensional Hadamard manifold. The following result is well-known and essential for our work.

Proposition 1 [42] *Let $x \in M$. Then $\exp_x : T_x M \rightarrow M$ is a diffeomorphism, and for any two points $x, y \in M$ there exists a unique normalized geodesic joining x to y , which is minimal.*

So from now on, when referring to a geodesic joining two points we mean the unique minimal one. This proposition also says that M is diffeomorphic to the Euclidean space \mathbb{R}^m . Thus M has the same topology and differential structure as \mathbb{R}^m . Moreover, Hadamard manifolds and Euclidean spaces have similar geometrical properties. Some of them are described next.

Recall that a geodesic triangle $\Delta(x_1, x_2, x_3)$ of a Riemannian manifold is a set consisting of three points x_1, x_2, x_3 and three minimal geodesics joining these points.

Proposition 2 (Comparison theorem for triangles) [42] *Let $\Delta(x_1, x_2, x_3)$ be a geodesic triangle. Denote, for each $i = 1, 2, 3 \pmod{3}$, by $\gamma_i : [0, l_i] \rightarrow M$ geodesic joining x_i to x_{i+1} , and set $l_i := L(\gamma_i)$, $\alpha_i := \angle(\gamma'_i(0), -\gamma'_{i-1}(l_{i-1}))$. Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi, \quad (2)$$

$$l_i^2 + l_{i+1}^2 - 2l_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2. \quad (3)$$

In terms of the distance and the exponential map, inequality (3) can be rewritten as

$$d^2(x_i, x_{i+1}) + d^2(x_{i+1}, x_{i+2}) - 2(\exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2}) \leq d^2(x_{i-1}, x_i), \quad (4)$$

since

$$(\exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2}) = d(x_i, x_{i+1})d(x_{i+1}, x_{i+2}) \cos \alpha_{i+1}.$$

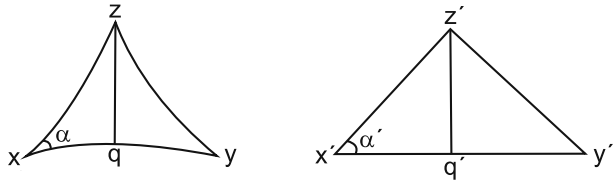
Lemma 1 [3, pag. 24] *Let $\Delta(x, y, z)$ be a geodesic triangle in a Hadamard manifold M . Then, there exists $x', y', z' \in \mathbb{R}^2$ such that*

$$d(x, y) = \|x' - y'\|, \quad d(y, z) = \|y' - z'\|, \quad d(z, x) = \|z' - x'\|$$

The triangle $\Delta(x', y', z')$ is called the comparison triangle of the geodesic triangle $\Delta(x, y, z)$, which is unique up to isometry of M . The next result shows the relation between a geodesic triangle and its comparison triangle involving angles and distances between points. This relation expresses the geometric idea of a manifold having nonpositive sectional curvature.

Lemma 2 [29] *Let $\Delta(x, y, z)$ be a geodesic triangle in a Hadamard manifold M and $\Delta(x', y', z')$ be its comparison triangle.*

Fig. 1 Comparison triangles



(1) Let α, β, γ (resp. α', β', γ') be the angles of $\Delta(x, y, z)$ (resp. $\Delta(x', y', z')$) at the vertices x, y, z (resp. x', y', z'). Then

$$\alpha' \geq \alpha, \quad \beta' \geq \beta, \quad \gamma' \geq \gamma. \tag{5}$$

(2) Given any point q belonging to the geodesic which join x to y , its comparison point is the point q' in the interval $[x', y']$ such that $d(q, x) = \|q' - x'\|$ and $d(q, y) = \|q' - y'\|$; see Fig. 1. Then

$$d(z, q) \leq \|z' - q'\|. \tag{6}$$

The following lemma is a consequence of inequality (6) and the parallelogram identity in a Euclidean space \mathbb{R}^n :

$$\|x - y\|^2 + \|x + y\|^2 = 2(\|x\|^2 + \|y\|^2), \quad \forall x, y \in \mathbb{R}^n. \tag{7}$$

Lemma 3 For all $x, y, z \in M$ and $q \in M$ with $d(x, q) = d(y, q) = d(x, y)/2$, one has

$$d^2(z, q) \leq \frac{1}{2} d^2(z, x) + \frac{1}{2} d^2(z, y) - \frac{1}{4} d^2(x, y). \tag{8}$$

Inequality (8) also appears in the context of the Hilbert ball and hyperbolic spaces (cf. page 541 of [40]). From the ‘‘law of cosines’’ in inequality (4) it readily follows the following inequality which is a general characteristic of the spaces with nonpositive curvature (see [3]).

$$\langle \exp_x^{-1} y, \exp_x^{-1} z \rangle + \langle \exp_y^{-1} x, \exp_y^{-1} z \rangle \geq d^2(x, y), \quad \forall x, y, z \in M \tag{9}$$

A subset $K \subseteq M$ is said to be convex if for any two points x and y in K , the geodesic joining x to y is contained in K ; that is, if $\gamma : [a, b] \rightarrow M$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma((1 - t)a + tb) \in K$ for all $t \in [0, 1]$. From now we will use K to denote a nonempty, closed and convex set in M .

A real-valued function f defined on M is said to be convex if for any geodesic γ of M , the composition function $f \circ \gamma : \mathbb{R} \rightarrow \mathbb{R}$ is convex; that is,

$$(f \circ \gamma)(ta + (1 - t)b) \leq t(f \circ \gamma)(a) + (1 - t)(f \circ \gamma)(b)$$

for any $a, b \in \mathbb{R}$, and $0 \leq t \leq 1$. The following proposition describes the convexity property of the distance function.

Proposition 3 [42] Let $d : M \times M \rightarrow \mathbb{R}$ be the distance function. Then d is a convex function with respect to the product Riemannian metric; that is, given any pair of

geodesics $\gamma_1 : [0, 1] \rightarrow M$ and $\gamma_2 : [0, 1] \rightarrow M$ the following inequality holds for all $t \in [0, 1]$:

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1)).$$

In particular, for each $p \in M$, the function $d(\cdot, p) : M \rightarrow \mathbb{R}$ is a convex function.

Let P_K denote the projection onto K defined by

$$P_K(x) = \{x_0 \in K : d(x, x_0) \leq d(x, y) \text{ for all } y \in K\}, \quad \forall x \in M.$$

The following proposition is well-known, see for example [27] and [47].

Proposition 4 *Let K be a closed convex subset of M . Then P_K is single-valued and, for any $x \in M$, $z = P_K(x)$ if and only if*

$$\langle \exp_z^{-1} x, \exp_z^{-1} y \rangle \leq 0, \quad \forall y \in K.$$

Using the properties of the parallel transport and the exponential map the following lemma was proved in [28].

Lemma 4 *Let $x_0 \in M$ and $\{x_n\} \subset M$ such that $x_n \rightarrow x_0$. Then the following assertions hold.*

(i) *For any $y \in M$,*

$$\exp_{x_n}^{-1} y \rightarrow \exp_{x_0}^{-1} y \quad \text{and} \quad \exp_{y_n}^{-1} x_n \rightarrow \exp_y^{-1} x_0.$$

(ii) *If $v_n \in T_{x_n} M$ and $v_n \rightarrow v_0$, then $v_0 \in T_{x_0} M$.*

(iii) *Given $u_n, v_n \in T_{x_n} M$ and $u_0, v_0 \in T_{x_0} M$, if $u_n \rightarrow u_0$ and $v_n \rightarrow v_0$, then*

$$\langle u_n, v_n \rangle \rightarrow \langle u_0, v_0 \rangle.$$

(iv) *For any $u \in T_{x_0} M$, the function $F : M \rightarrow TM$ defined by $F(x) = P_{x, x_0} u$ for each $x \in M$ is continuous on M .*

3 Firmly Nonexpansive Mappings

In this section we introduce the definition of firmly nonexpansive mappings on Hadamard manifolds. This notion was previously defined on a Banach space (cf. [4, 6, 7, 36]) and the Hilbert ball with the hyperbolic metric (cf. [15]), so-called firmly nonexpansive mapping of the first kind in the latter case. In fact, the following analysis shows that in this framework these mappings verify similar properties to those ones well-known in Hilbert spaces.

Definition 1 Given a mapping $T : K \subseteq M \rightarrow M$, we say that T is *firmly nonexpansive* if for any $x, y \in K$, the function $\Phi : [0, 1] \rightarrow [0, \infty]$ defined by

$$\Phi(t) := d(\exp_x t \exp_x^{-1} Tx, \exp_y t \exp_y^{-1} Ty), \quad \forall t \in [0, 1], \quad (10)$$

is nonincreasing.

Remark 1 By the definition, it is clear that any firmly nonexpansive mapping T is nonexpansive.

Proposition 5 *Let $T : K \subseteq M \rightarrow M$. Then the following assumptions are equivalent.*

- (i) T is firmly nonexpansive.
- (ii) For any $x, y \in K$ and $t \in [0, 1]$

$$d(T(x), T(y)) \leq d(\exp_x t \exp_x^{-1} Tx, \exp_y t \exp_y^{-1} Ty). \tag{11}$$

- (iii) For any $x, y \in K$

$$\langle \exp_{T(x)}^{-1} T(y), \exp_{T(x)}^{-1} x \rangle + \langle \exp_{T(y)}^{-1} T(x), \exp_{T(y)}^{-1} y \rangle \leq 0. \tag{12}$$

Proof Given $x, y \in K$, let $\Phi : [0, 1] \rightarrow [0, \infty]$ be the function defined by (10). Note that from Proposition 3 we deduce that Φ is a convex function.

Let us start by proving that the derivative at 1^- of the function Φ can be expressed as

$$\Phi'(1) = \langle \exp_{T(x)}^{-1} T(y), \exp_{T(x)}^{-1} x \rangle + \langle \exp_{T(y)}^{-1} T(x), \exp_{T(y)}^{-1} y \rangle. \tag{13}$$

Let $u = \exp_{T(x)}^{-1} x$ and $v = \exp_{T(y)}^{-1} y$. Then the function Φ can be written as

$$\Phi(t) = d(\exp_{T(x)}(1-t)u, \exp_{T(y)}(1-t)v), \quad \forall t \in [0, 1]. \tag{14}$$

Let γ be the geodesic joining $T(x)$ to $T(y)$ defined by

$$\gamma(t) = \exp_{T(x)} t \exp_{T(x)}^{-1} T(y), \quad \forall t \in [0, 1]. \tag{15}$$

Given $\varepsilon > 0$, we define the function $f : (-\varepsilon, \varepsilon) \times [0, 1] \rightarrow M$ by

$$f(s, t) = \exp_{\exp_{T(x)} su} t (\exp_{\exp_{T(x)} su}^{-1} \exp_{T(y)} sv), \quad \forall (s, t) \in (-\varepsilon, \varepsilon) \times [0, 1]. \tag{16}$$

Note that $f(0, \cdot) = \gamma(\cdot)$. Since the exponential map is differentiable and f is a variation of γ , it follows that $V(\cdot) = \frac{\partial f}{\partial s}(0, \cdot)$ is the variational field of f . In particular,

$$V(0) = \frac{\partial f}{\partial s}(0, 0) = u \quad \text{and} \quad V(1) = \frac{\partial f}{\partial s}(0, 1) = v. \tag{17}$$

Let $s \in (-\varepsilon, \varepsilon)$. Note that the parameterized curve $f_s : [0, 1] \rightarrow M$ given by $f_s(\cdot) = f(s, \cdot)$ is a geodesic and therefore $\|\frac{\partial f}{\partial t}(s, \cdot)\|$ is a constant. In particular, from (14),

$$\left\| \frac{\partial f}{\partial t}(s, \cdot) \right\| = \left\| \exp_{\exp_{T(x)} su}^{-1} \exp_{T(y)} sv \right\| = d(\exp_{T(x)} su, \exp_{T(y)} sv) = \Phi(1-s). \tag{18}$$

Define $L : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ by

$$L(s) = \int_0^1 \left\| \frac{\partial f}{\partial t}(s, t) \right\| dt, \quad \forall s \in (-\varepsilon, \varepsilon). \tag{19}$$

Then

$$L^2(s) = \int_0^1 \left\| \frac{\partial f}{\partial t}(s, t) \right\|^2 dt = \Phi^2(1-s)$$

(recalling that $\|\frac{\partial f}{\partial t}(s, \cdot)\|$ is a constant). Therefore, by the first variation formula stated in [42, p. 89, Proposition 2.2] (where E denotes the energy integral $E(s) = \frac{1}{2}L^2(s)$), we have that

$$\begin{aligned} \frac{1}{2} \left(\frac{d}{ds} L^2(s) \right)_{s=0} &= - \int_0^1 \left\langle V(t), \frac{D}{dt} \frac{d\gamma}{dt} \right\rangle dt - \left\langle V(0), \frac{d\gamma}{dt}(0) \right\rangle + \left\langle V(1), \frac{d\gamma}{dt}(1) \right\rangle \\ &= - \left\langle \frac{\partial f}{\partial s}(0, 0), \gamma'(0) \right\rangle + \left\langle \frac{\partial f}{\partial s}(0, 1), \gamma'(1) \right\rangle \\ &= - \left\langle \exp_{T(x)}^{-1} x, \exp_{T(x)}^{-1} T(y) \right\rangle + \left\langle \exp_{T(y)}^{-1} y, -\exp_{T(y)}^{-1} T(x) \right\rangle, \end{aligned} \tag{20}$$

where the second equality holds because $\frac{D}{dt} \frac{d\gamma}{dt} = 0$, since γ is a geodesic. Then the fact that

$$\left(\frac{d}{ds} L^2(s) \right)_{s=0} = - \left(\frac{d}{dt} \Phi^2(t) \right)_{t=1} = -(\Phi^2)'(1) = -2\Phi'(1), \tag{21}$$

together with inequality (20), implies equality (13).

By definition, the mapping T is firmly nonexpansive if and only if the convex function Φ is nonincreasing. This is equivalent to $\Phi(1) \leq \Phi(t)$ for any $t \in [0, 1]$, which is exactly inequality (11), proving the equivalence between (i) and (ii). On the other hand, since Φ is convex and differentiable at 1, the definition of firmly nonexpansive mapping is equivalent to $\Phi'(1) \leq 0$, and then to inequality (12) because of equality (13). Then (i) is proved to be equivalent to (iii). \square

The following corollary provides us with an important class of firmly nonexpansive mappings.

Corollary 1 *The metric projection onto a closed convex subset $K \subseteq M$ is a firmly nonexpansive mapping.*

Proof Let $T = P_K$ be the metric projection onto K . For any $x, y \in M$, by Proposition 4, we have that

$$\left\langle \exp_{T(x)}^{-1} T(y), \exp_{T(x)}^{-1} x \right\rangle \leq 0 \tag{22}$$

because $T(y) \in K$, and analogously, since $T(x) \in K$,

$$\left\langle \exp_{T(y)}^{-1} T(x), \exp_{T(y)}^{-1} y \right\rangle \leq 0. \tag{23}$$

By summing inequalities (22) and (23) we obtain inequality (12) which implies the firmly nonexpansivity of P_K from Proposition 5. \square

As another example of class of firmly nonexpansive mappings, we can prove that, for any nonexpansive mapping T defined on M , there exists an associated family of firmly nonexpansive mappings $\{G_t : 0 \leq t < 1\}$ whose fixed point sets, $\text{Fix}(G_t)$, coincide with the fixed point set of the mapping T , $\text{Fix}(T)$. Indeed, given $T : K \rightarrow K$ and $x \in K$, for any $t \in [0, 1)$ let T_t be the mapping defined by

$$T_t(y) := \exp_x t \exp_x^{-1} Ty, \quad \forall y \in K. \tag{24}$$

T_t is a contraction for any $t \in [0, 1)$, i.e., there exists $\alpha \in (0, 1)$ such that

$$d(T_t(y), T_t(z)) \leq \alpha d(y, z), \quad \forall y, z \in K.$$

Then the Banach contraction principle implies that there exists a unique fixed point of T_t , which is being denoted by x_t . By means of the so-called approximating curve $\{x_t\}$, for any $t \in [0, 1)$, we define the mapping $G_t : K \rightarrow K$ by

$$G_t(x) =: x_t = \exp_x t \exp_x^{-1} T(G_t(x)), \quad \forall x \in K.$$

Both the Banach space and the Hilbert ball analogs of these mappings are discussed in [15].

Proposition 6 *Let $t \in [0, 1)$. Then the following assertions hold.*

- (i) *The mapping G_t is firmly nonexpansive.*
- (ii) $\text{Fix}(G_t) = \text{Fix}(T)$.

Proof

- (i) Let $x, y \in K$. From the convexity of the distance, we have that

$$d(G_t(x), G_t(y)) \leq (1 - t)d(x, y) + td(G_t(x), G_t(y)),$$

which implies the nonexpansivity of G_t . Now consider the geodesics γ_1 and γ_2 joining x to $G_t(x)$ and y to $G_t(y)$, respectively. Let $s \in [0, 1]$ and set $r := \frac{(1-s)t}{1-st} \in [0, 1)$. Then $r \in [0, 1)$ and satisfies that

$$G_t(x) = \exp_{\gamma_1(s)} r \exp_{\gamma_1(s)}^{-1} T(G_t(x)) \tag{25}$$

and

$$G_t(y) = \exp_{\gamma_2(s)} r \exp_{\gamma_2(s)}^{-1} T(G_t(y)). \tag{26}$$

To show this fact, note that by the definition of the mapping G_t the geodesics γ_1 and γ_2 are contained in the geodesics α_1 and α_2 joining x to $T(G_t(x))$ and y to $T(G_t(y))$, respectively. In particular, the point $\gamma_1(s)$ belongs to the geodesic α_1 . Since

$$d(x, \gamma_1(s)) = s d(x, G_t(x)) = st d(x, T(G_t(x))), \tag{27}$$

it follows that

$$d(\gamma_1(s), T(G_t(x))) = (1 - st) d(x, T(G_t(x))). \tag{28}$$

Therefore, using the definitions of $\gamma_1(s)$, inequalities (27) and (28), we obtain that

$$\begin{aligned} d(\gamma_1(s), G_t(x)) &= (1 - s) d(x, G_t(x)) \\ &= (1 - s) t d(x, T(G_t(x))) \\ &= (1 - s) t \frac{1}{1 - st} d(\gamma_1(s), T(G_t(x))) \\ &= r d(\gamma_1(s), T(G_t(x))). \end{aligned}$$

This means that (25) is satisfied. Similarly, (26) is satisfied. Thus, by the convexity of the distance and the nonexpansivity of G_t , we obtain that

$$d(G_t(x), G_t(y)) \leq (1-r)d(\gamma_1(s), \gamma_2(s)) + rd(G_t(x), G_t(y)).$$

This implies that, for any $s \in [0, 1]$

$$d(G_t(x), G_t(y)) \leq d(\gamma_1(s), \gamma_2(s)). \quad (29)$$

Hence, inequality (11) holds and G_t is firmly nonexpansive by Proposition 5.

(ii) It follows from the following equivalences:

$$\begin{aligned} x = G_t(x) &\Leftrightarrow x = \exp_x(1-t)\exp_x^{-1}Tx \\ &\Leftrightarrow 0 = \exp_x^{-1}Tx \\ &\Leftrightarrow x = Tx. \end{aligned}$$

□

Remark 2 If we fix a point $x \in C$, the approximating curve $\{x_t\}$ defined by the unique fixed point of the contraction T_t in (24), that is $x_t = G_t(x)$ for all $t \in [0, 1]$, converges to $P_{\text{Fix}(T)}x$, the unique nearest point to x in $\text{Fix}(T)$. This was proved by Kirk in the more general setting of CAT(0) spaces; see [18]. The Banach space and the Hilbert ball analogs of this result can be found in [38] and [15], respectively.

4 Resolvents and Yosida Approximations of Vector Fields

Let $\mathcal{X}(M)$ denote the set of all set-valued vector fields $A : M \rightarrow 2^{TM}$ such that $A(x) \subseteq T_xM$ for each $x \in \mathcal{D}(A)$, where $\mathcal{D}(A)$ denote the domain of A defined by

$$\mathcal{D}(A) = \{x \in M \mid A(x) \neq \emptyset\}.$$

Definition 2 [9, 10, 28, 32] A vector field $A \in \mathcal{X}(M)$ is said to be

(a) *monotone* if for any $x, y \in \mathcal{D}(A)$

$$\langle u, \exp_x^{-1}y \rangle \leq \langle v, -\exp_y^{-1}x \rangle, \quad \forall u \in A(x) \text{ and } \forall v \in A(y); \quad (30)$$

(b) *strongly monotone* if there exists $\rho > 0$ such that for any $x, y \in \mathcal{D}(A)$,

$$\langle u, \exp_x^{-1}y \rangle - \langle v, -\exp_y^{-1}x \rangle \leq -\rho d^2(x, y), \quad \forall u \in A(x) \text{ and } \forall v \in A(y); \quad (31)$$

(c) *maximal monotone* if it is monotone and for any $x \in M$ and $u \in T_xM$, the condition

$$\langle u, \exp_x^{-1}y \rangle \leq \langle v, -\exp_y^{-1}x \rangle, \quad \forall y \in \mathcal{D}(A) \text{ and } v \in A(y), \quad (32)$$

implies that $u \in A(x)$.

The concept of accretivity of vector fields on Riemannian manifolds was introduced in [48] and previously studied in other nonlinear metric spaces in [23, 40, 44]. In [48] the authors proved that in the setting of Hadamard manifolds the notions of monotonicity and accretivity are equivalent.

Definition 3 A vector field $A \in \mathcal{X}(M)$ is said to be *accretive* if given $x, y \in \mathcal{D}(A)$, for any $u \in A(x)$ and $v \in A(y)$ we have that

$$d(x, y) \leq d(\exp_x(ru), \exp_y(rv)), \quad \forall r \geq 0.$$

Theorem 1 [48] *A vector field $A \in \mathcal{X}(M)$ is monotone if and only if it is accretive.*

Given a single-valued function T defined on a Hadamard manifold, Németh, in [33], introduced an associated vector field A called complementary vector field of T , which establishes a connection between nonexpansivity and monotonicity. The same concept can be defined in the set-valued case.

Definition 4 Let $T : K \subseteq M \rightarrow 2^M$. We say that $A \in \mathcal{X}(M)$ is the complementary vector field of T if

$$A(x) = -\exp_x^{-1} T(x), \quad \forall x \in K.$$

Theorem 2 [33] *Given a nonexpansive mapping $T : K \subseteq M \rightarrow M$, its complementary vector field A is monotone.*

The existence of singularities of vector fields will be a crucial fact for the resolvent to be well-defined.

Definition 5 Given a vector field $A \in \mathcal{X}(M)$, we say that $x \in \mathcal{D}(A)$ is a *singularity* of A if $0 \in A(x)$. The set of all singularities of A is denoted by

$$A^{-1}(0) := \{x \in \mathcal{D}(A) : 0 \in A(x)\}.$$

Theorem 3 [28] *Let $A \in \mathcal{X}(M)$ be a maximal strongly monotone vector field with full domain, i.e., $\mathcal{D}(A) = M$. Then $A^{-1}(0) \neq \emptyset$.*

We are now in a position to define the notions of resolvents and Yosida approximations of a vector field $A \in \mathcal{X}(M)$.

Definition 6 Given $\lambda > 0$ and $A \in \mathcal{X}(M)$, the *resolvent* and the *Yosida approximation* of A of order λ are the set-valued mappings $J_\lambda : M \rightarrow 2^M$ and $A_\lambda : M \rightarrow 2^{TM}$ defined respectively by

$$J_\lambda(x) := \{z \in M \mid x \in \exp_z \lambda Az\}, \quad \forall x \in M \tag{33}$$

and

$$A_\lambda(x) = -\frac{1}{\lambda} \exp_x^{-1} J_\lambda(x), \quad \forall x \in M. \tag{34}$$

Clearly, the *Yosida approximation* of A is the complementary vector field of the corresponding resolvent multiplied by the constant $\frac{1}{\lambda}$.

Remark 3 Let $\lambda > 0$. By the definitions of the resolvent and the Yosida approximation of a vector field, the following assertions hold.

(a) The rang of the resolvent J_λ is contained in the domain of A and

$$\text{Fix}(J_\lambda) = A^{-1}(0) = A_\lambda^{-1}(0). \tag{35}$$

(b) The domains of the resolvent J_λ and the Yosida approximation A_λ are the range of the vector field defined by $x \mapsto \exp_x \lambda Ax$. We will denote this range as $\mathcal{R}(\exp \lambda A(\cdot))$. Then we have that

$$\mathcal{D}(J_\lambda) = \mathcal{D}(A_\lambda) = \mathcal{R}(\exp \lambda A(\cdot)).$$

The resolvent and Yosida approximation were implicitly defined in the setting of differential manifolds, in particular, in Finsler manifold by J. Hoyos in [24] and in Hilbert manifolds by Iwamiya and Okochi in [26]. In fact, these two definitions can be proved to coincide with the corresponding concepts defined in Hadamard manifolds. However, it turns out that in the former settings when the resolvent and the Yosida approximation have full domain is still unknown, whereas we are proving next that under certain monotonicity conditions these operators do have full domain in a Hadamard manifold. Moreover, in the following theorem the relation between the firm nonexpansivity of the resolvent and the monotonicity of the vector field is stated as well.

Theorem 4 *Let $A \in \mathcal{X}(M)$. The following assertions hold for any $\lambda > 0$.*

- (i) *The vector field A is monotone if and only if J_λ is single-valued and firmly nonexpansive.*
- (ii) *If $\mathcal{D}(A) = M$, the vector field A is maximal monotone if and only if J_λ is single-valued, firmly nonexpansive and the domain $\mathcal{D}(J_\lambda) = M$.*
- (iii) *If A is monotone, then so is the Yosida approximation A_λ . Moreover, if A is maximal monotone with $\mathcal{D}(A) = M$ then so is A_λ .*

Proof

- (i) Suppose that A is monotone. Let $x \in \mathcal{D}(J_\lambda)$ and define the vector fields $B : M \rightarrow 2^{TM}$ by

$$B(y) = \lambda Ay - \exp_y^{-1} x, \quad \forall y \in M. \tag{36}$$

Then B is strongly monotone by inequality (9); see also [10] and [48]. Moreover,

$$z \in J_\lambda(x) \Leftrightarrow x \in \exp_z \lambda Az \Leftrightarrow 0 \in B(z). \tag{37}$$

This implies that $J_\lambda(x)$ is at most a singleton since any strongly monotone vector field has at most one singularity. Hence the resolvent is single-valued. To prove the firm nonexpansivity, by Proposition 5, it suffices to verify that for any $x, y \in M$

$$\langle \exp_{J_\lambda(x)}^{-1} J_\lambda(y), \exp_{J_\lambda(x)}^{-1} x \rangle + \langle \exp_{J_\lambda(y)}^{-1} J_\lambda(x), \exp_{J_\lambda(y)}^{-1} y \rangle \leq 0. \tag{38}$$

Considering $J_\lambda(x), J_\lambda(y) \in M$, by definition of resolvent we know that

$$\exp_{J_\lambda(x)}^{-1} x \in \lambda A(J_\lambda(x)), \quad \exp_{J_\lambda(y)}^{-1} y \in \lambda A(J_\lambda(y)).$$

Thus the monotonicity of A implies that

$$\left\langle \exp_{J_\lambda(x)}^{-1} J_\lambda(y), \frac{1}{\lambda} \exp_{J_\lambda(x)}^{-1} x \right\rangle + \left\langle \exp_{J_\lambda(y)}^{-1} J_\lambda(x), \frac{1}{\lambda} \exp_{J_\lambda(y)}^{-1} y \right\rangle \leq 0,$$

and (38) is proved.

Conversely, suppose now that J_λ is firmly nonexpansive. To show that A is monotone, let $x, y \in \mathcal{D}(A)$, $u \in A(x)$ and $v \in A(y)$. Then by definition of the resolvent J_λ , we can write $x = J_\lambda(\exp_x \lambda u)$ and $y = J_\lambda(\exp_y \lambda v)$. By using the characterization (12) of firmly nonexpansive mappings considering the points $\exp_x \lambda u$ and $\exp_y \lambda v$ we obtain that

$$\langle \exp_x^{-1} y, \lambda u \rangle + \langle \exp_y^{-1} x, \lambda v \rangle \leq 0;$$

that is,

$$\langle u, \exp_x^{-1} y \rangle \leq \langle v, -\exp_y^{-1} x \rangle.$$

Hence A is monotone as $x, y \in \mathcal{D}(A)$, $u \in A(x)$ and $v \in A(y)$ are arbitrary.

- (ii) Suppose that A is a maximal monotone vector field with full domain. Then the vector field B defined by (36) is strongly monotone and maximal monotone; see for instance [10, 28] and [48]. Then, for any $x \in M$, Theorem 3 ensures the existence and uniqueness of a singularity of B , that is, an element of $J_\lambda(x)$. Thus J_λ is single-valued and firmly nonexpansive, as we proved in (i), and moreover $\mathcal{D}(J_\lambda) = M$.

Now, suppose J_λ is firmly nonexpansive and $\mathcal{D}(J_\lambda) = M$. By (i) we know that J_λ is monotone. To prove the maximality, we take $x \in M$ and $u \in T_x M$ such that

$$\langle u, \exp_x^{-1} y \rangle \leq -\langle v, \exp_y^{-1} x \rangle, \quad \forall y \in M \text{ and } \forall v \in A(y). \tag{39}$$

Define the vector field $F : M \rightarrow 2^{TM}$ by $F(y) = A(y)$ if $x \neq y$ and $F(x) = u$. Since A is monotone, it follows from inequality (39) that F is monotone. Using the equivalence between monotonicity and accretivity established in Theorem 1, we have that F is accretive; hence

$$d(x, y) \leq d(\exp_x \lambda u, \exp_y \lambda v), \quad \forall y \in M \text{ and } \forall v \in F(y). \tag{40}$$

(noting that $u \in F(x)$). The assumption on the domain of J_λ implies that there exist $y \in M$ and $v \in A(y)$ such that $J_\lambda(\exp_x \lambda u) = y$, that is $\exp_x \lambda u = \exp_y \lambda v$. This together with inequality (40) implies that $x = y$ and therefore $u = v \in A(x)$, establishing that A is maximal monotone.

- (iii) The Yosida approximation A_λ of A is the complementary vector field of J_λ multiplied by a constant. Thus if A is monotone, since J_λ is nonexpansive by (i), A_λ is monotone by Theorem 2. Now suppose that A is maximal monotone with full domain. Since the resolvent and the exponential map are continuous, it follows that the Yosida approximation A_λ is continuous and has full domain. Thus Theorem 3.7 in [28] is applicable to concluding that A_λ is maximal monotone. □

Remark 4 Note that in the previous theorem we have proved indeed that, for each $\lambda > 0$, any firmly nonexpansive T with full domain $\mathcal{D}(T) = M$ is the resolvent J_λ^A of the maximal monotone vector field A defined by

$$A(x) := \frac{1}{\lambda} \exp_x^{-1} T^{-1}x, \quad \forall x \in M. \quad (41)$$

Indeed, the resolvent of the vector field A is the mapping T :

$$J_\lambda(x) = \{z \in M \mid x \in \exp_z \lambda Az\} = \{z \in M \mid x \in T^{-1}z\} = Tx.$$

From Theorem 4 and Remark 3, we have the following result which constitutes a counterpart in the setting of Hadamard manifolds of Minty's theorem (cf. [31]).

Corollary 2 *Let $A \in \mathcal{X}(M)$ be monotone such that $\mathcal{D}(A) = M$, and let $\lambda > 0$. Then A is maximal monotone if and only if $\mathcal{R}(\exp_\cdot \lambda A(\cdot)) = M$.*

As a byproduct of Theorem 4 (iii) we obtain the following result about the structure of the set of singularities of a maximal monotone vector field. As far as we know the only result of this type was proved in [13] under the assumption that A is smooth.

Corollary 3 *Let $A \in \mathcal{X}(M)$ be a monotone vector field. Suppose that there exists $\lambda > 0$ such that $\mathcal{D}(A) \cap \mathcal{D}(J_\lambda)$ is closed convex (e.g. $\mathcal{D}(A)$ is closed convex and $\mathcal{D}(A) \subseteq \mathcal{D}(J_\lambda)$). Then $A^{-1}(0)$ is closed and convex.*

Proof Let $\lambda > 0$ be such that $\mathcal{D}(A) \cap \mathcal{D}(J_\lambda)$ is closed convex. Then $A^{-1}(0) = \text{Fix}(J_\lambda)$ by (35). By Theorem 4, we know that J_λ is nonexpansive. Then the result can be deduced by the well-known fact that the fixed point set $\text{Fix}(T)$ of a nonexpansive mapping T on a closed convex subset of M is closed and convex. Since it seems that this result does not appear proved explicitly, we will enclose it here for the sake of completeness.

Let $T : K \rightarrow M$ be a nonexpansive mapping with the closed convex subset $K \subseteq M$. The closedness of $\text{Fix}(T)$ is obvious from the continuity of T . In order to prove the convexity, let $x, y \in \text{Fix}(T)$ and let γ be the geodesic joining x to y . Given a point z in the geodesic γ , that is $z = \gamma(t_0)$ for some $t_0 \in [0, 1]$, we just need to prove that $z \in \text{Fix}(T)$. To do this, let $\gamma_1 : [0, t_0] \rightarrow M$ and $\gamma_2 : [t_0, 1] \rightarrow M$ be the geodesics joining x to Tz and Tz to y , respectively. Consider the curve $c : [0, 1] \rightarrow M$ defined by

$$c(t) = \begin{cases} \gamma_1(t) & t \in [0, t_0], \\ \gamma_2(t) & t \in [t_0, 1]. \end{cases}$$

Since

$$d(x, Tz) \leq d(x, z) \quad \text{and} \quad d(Tz, y) \leq d(z, y), \quad (42)$$

it follows that the length $l(c)$ of c satisfies that

$$l(c) \leq l(\gamma_1) + l(\gamma_2) = d(x, Tz) + d(Tz, y) \leq d(x, z) + d(z, y) = d(x, y).$$

This means that c is the geodesic joining x to y and so $c = \gamma$. Consequently, $Tz \in \gamma([0, 1])$. This together with (42) implies that $Tz = z$, that is, $z \in \text{Fix}(T)$. \square

5 Asymptotic Behavior of the Resolvent and Singularities of Vector Fields

The asymptotic behavior of the resolvent has been studied in the setting of Banach spaces regarding the problem of approximating zeros of monotone and accretive operators; see [17, 38, 45]. In this section a parallel analysis is carried out in Hadamard manifolds.

We first prove some technical lemmas. Define

$$|Ax| := \inf\{\|v\| \mid v \in Ax\}$$

and adopt the convention that $\inf \emptyset = +\infty$.

Lemma 5 *Let $A \in \mathcal{X}(M)$ be monotone, $\lambda > 0$ and let $x \in \mathcal{D}(A_\lambda)$. Then we have that*

$$A_\lambda(x) \in P_{x, J_\lambda(x)} A J_\lambda(x) \quad \text{and} \quad \|A_\lambda(x)\| \leq |Ax|. \tag{43}$$

Proof Since A is monotone and $\mathcal{D}(A_\lambda) = \mathcal{D}(J_\lambda)$, it follows that the resolvent and the Yosida approximation of A are single-valued. By definition of resolvent, we have that $x \in \exp_{J_\lambda(x)} \lambda A J_\lambda(x)$, or equivalently, $\exp_{J_\lambda(x)}^{-1} x \in \lambda A J_\lambda(x)$. Therefore,

$$A_\lambda(x) = -\frac{1}{\lambda} \exp_x^{-1} J_\lambda(x) = \frac{1}{\lambda} P_{x, J_\lambda(x)} \exp_{J_\lambda(x)}^{-1} x \in P_{x, J_\lambda(x)} A J_\lambda(x)$$

and the first assertion of (43) is proved. To show the second assertion, we assume that $x \in \mathcal{D}(A)$ without loss of generality. Let $u \in A(x)$. Then $\exp_x \lambda u \in \exp_x \lambda A(x)$; hence $x = J_\lambda(\exp_x \lambda u)$ by definition. Since

$$\|A_\lambda(x)\| = \frac{1}{\lambda} \|\exp_x^{-1} J_\lambda(x)\| = \frac{1}{\lambda} d(x, J_\lambda(x)),$$

it follows from the nonexpansivity of the resolvent that

$$\|A_\lambda(x)\| = \frac{1}{\lambda} d(J_\lambda(\exp_x \lambda u), J_\lambda(x)) \leq \frac{1}{\lambda} d(\exp_x \lambda u, x) = \|u\|.$$

Thus the second assertion of (43) is seen to hold because $u \in A(x)$ is arbitrary. \square

For the next lemma we need the definition of upper Kuratowski semicontinuity for vector fields in the setting of Hadamard manifolds which was introduced in [28] following of the corresponding notions for operators in Banach spaces (cf. [43, pag. 55]).

Definition 7 Let $A \in \mathcal{X}(M)$ and $x_0 \in \mathcal{D}(A)$. The vector field A is said to be

- (a) *upper semicontinuous* at x_0 if for any open set V satisfying $A(x_0) \subseteq V \subseteq T_{x_0} M$, there exists an open neighborhood $U(x_0)$ of x_0 such that $P_{x_0, x} A(x) \subseteq V$ for any $x \in U(x_0)$;
- (b) *upper Kuratowski semicontinuous* at x_0 if for any sequences $\{x_k\} \subset \mathcal{D}(A)$ and $\{u_k\} \subset TM$ with each $u_k \in A(x_k)$, the relations $\lim_{k \rightarrow \infty} x_k = x_0$ and $\lim_{k \rightarrow \infty} u_k = u_0$ imply $u_0 \in A(x_0)$;

- (c) upper semicontinuous (resp. upper Kuratowski semicontinuous) on M if it is upper semicontinuous (resp. upper Kuratowski semicontinuous) at each $x_0 \in \mathcal{D}(A)$.

Remark 5 Clearly, the upper semicontinuity implies the upper Kuratowski semicontinuity. The deep relationship between upper semicontinuity and maximal monotonicity has been studied in [28].

Define

$$\vec{\mathcal{D}}(J) := \limsup_{\lambda \rightarrow \infty} \mathcal{D}(J_\lambda) = \bigcup_{\lambda > 0} \bigcap_{t \geq \lambda} \mathcal{D}(J_t).d$$

Lemma 6 *Let $A \in \mathcal{X}(M)$ be monotone and $x \in \vec{\mathcal{D}}(J)$. Suppose that A is either upper Kuratowski semicontinuous or there exists $r > 0$ such that $\mathcal{D}(A) \subseteq \mathcal{D}(J_r)$. Then the following assertions hold.*

- (i) *If there exists a sequence $\{t_n\}$ with $t_n \rightarrow \infty$, such that $\lim_{n \rightarrow \infty} J_{t_n}(x) = y$, then the limit $y \in A^{-1}(0)$ and satisfies that*

$$\langle \exp_y^{-1} x, \exp_y^{-1} z \rangle \leq 0, \quad \forall z \in A^{-1}(0). \tag{44}$$

- (ii) *If there exist two sequences $\{t_n\}$ and $\{s_n\}$ with $t_n \rightarrow \infty$ and $s_n \rightarrow \infty$, such that $\lim_{n \rightarrow \infty} J_{t_n}(x) = y$ and $\lim_{n \rightarrow \infty} J_{s_n}(x) = z$, then $y = z$.*

Proof

- (i) Let $\{t_n\}$ be a sequence with $t_n \rightarrow \infty$ such that $\lim_{n \rightarrow \infty} J_{t_n}(x) = y$. Then the sequence $\{J_{t_n}(x)\}$ is bounded and therefore there exists $\rho > 0$ such that $\|\exp_x^{-1} J_{t_n}(x)\| \leq \rho$ for all $n \geq 0$. By Lemma 5 we have that

$$|A(J_{t_n}(x))| \leq \|A_{t_n}(x)\| = \frac{1}{t_n} \|\exp_x^{-1} J_{t_n}(x)\| \leq \frac{\rho}{t_n}. \tag{45}$$

If we assume that A is upper Kuratowski semicontinuous, by letting $n \rightarrow \infty$ in (45) we obtain that $A(J_{t_n}(x)) \rightarrow 0$ and therefore $y \in A^{-1}(0)$.

Suppose now that there exists $r > 0$ such that $\mathcal{D}(A) \subseteq \mathcal{D}(J_r)$. Then, by Remark 3(a), we have that

$$\mathcal{R}(J_{t_n}) \subseteq \mathcal{D}(A) \subseteq \mathcal{D}(J_r), \quad \forall n \geq 0.$$

Let $n \geq 0$. Then, Lemma 5 is applied to get that

$$P_{x, J_{t_n}(x)} A_{t_n}(x) \in A J_{t_n}(x) \quad \text{and} \quad \|A_r(J_{t_n}(x))\| \leq |A(J_{t_n}(x))|;$$

hence, by inequality (45),

$$\|A_r(J_{t_n}(x))\| \leq |A(J_{t_n}(x))| \leq \frac{\rho}{t_n}.$$

It follows from the definition of the Yosida approximation A_r that

$$d(J_r(J_{t_n}(x)), J_{t_n}(x)) = \|\exp_{J_{t_n}(x)}^{-1} J_r(J_{t_n}(x))\| = r \|A_r(J_{t_n}(x))\| \leq \frac{r\rho}{t_n}.$$

Letting $n \rightarrow \infty$, we obtain that $y = J_r(y)$, that is, $y \in \text{Fix}(J_r) = A^{-1}(0)$.

To show (44), let $z \in A^{-1}(0)$ and $n \geq 0$. Since A is monotone and $P_{x, J_{t_n}(x)} A_{t_n}(x) \in AJ_{t_n}(x)$, we have that

$$\langle P_{J_{t_n}(x), x} A_{t_n}(x), \exp_{J_{t_n}(x)}^{-1} z \rangle \leq 0;$$

hence

$$\langle \exp_{J_{t_n}(x)}^{-1} x, \exp_{J_{t_n}(x)}^{-1} z \rangle \leq 0$$

as $A_{t_n}(x) = -\exp_x^{-1} J_{t_n}(x)$. Taking limit when $n \rightarrow \infty$ we obtain

$$\langle \exp_y^{-1} x, \exp_y^{-1} z \rangle \leq 0. \tag{46}$$

Thus (44) is seen to hold.

- (ii) Let $\{t_n\}$ and $\{s_n\}$ be sequences with $t_n \rightarrow \infty$ and $s_n \rightarrow \infty$ such that $\lim_{n \rightarrow \infty} J_{t_n}(x) = y$ and $\lim_{n \rightarrow \infty} J_{s_n}(x) = z$. Then by (i) we know that $y, z \in A^{-1}(0)$ satisfying (46) and

$$\langle \exp_z^{-1} x, \exp_z^{-1} y \rangle \leq 0. \tag{47}$$

Thus, by using inequality (9), the law of cosines for Hadamard manifolds, the previous inequalities (46) and (47) implies that

$$d^2(y, z) \leq \langle \exp_y^{-1} x, \exp_y^{-1} z \rangle + \langle \exp_z^{-1} x, \exp_z^{-1} y \rangle \leq 0.$$

Hence $y = z$. □

Theorem 5 *Let $A \in \mathcal{X}(M)$ be a monotone vector field. Suppose that A is either upper Kuratowski semicontinuous or there exists $r > 0$ such that $\mathcal{D}(A) \subseteq \mathcal{D}(J_r)$. Then the following assertions are equivalent.*

- (i) $A^{-1}(0) \neq \emptyset$;
- (ii) $\overrightarrow{\mathcal{D}}(J) \neq \emptyset$ and $\limsup_{\lambda \rightarrow \infty} \|J_\lambda(x)\| < \infty$ for each $x \in \overrightarrow{\mathcal{D}}(J)$;
- (iii) $\overrightarrow{\mathcal{D}}(J) \neq \emptyset$ and $\liminf_{\lambda \rightarrow \infty} \|J_\lambda(x)\| < \infty$ for some $x \in \overrightarrow{\mathcal{D}}(J)$;
- (iv) $\bigcap_{\lambda > 0} \mathcal{D}(J_\lambda) \neq \emptyset$ and for each $x \in \bigcap_{\lambda > 0} \mathcal{D}(J_\lambda)$ we have that

$$\lim_{\lambda \rightarrow \infty} J_\lambda(x) = y \in A^{-1}(0). \tag{48}$$

Furthermore, if $A^{-1}(0) \neq \emptyset$ is convex, then we have that

$$\lim_{\lambda \rightarrow \infty} J_\lambda(x) = P_{A^{-1}(0)}(x), \quad \forall x \in \bigcap_{\lambda > 0} \mathcal{D}(J_\lambda). \tag{49}$$

Proof

- (i) \Rightarrow (ii) Assume that $A^{-1}(0) \neq \emptyset$. Since $A^{-1}(0) = \text{Fix}(J_\lambda)$ for any $\lambda > 0$,

$$A^{-1}(0) \subseteq \bigcap_{\lambda > 0} \mathcal{D}(J_\lambda). \tag{50}$$

Hence $\overrightarrow{\mathcal{D}}(J) \neq \emptyset$. Let $x \in \overrightarrow{\mathcal{D}}(J)$ and let $\lambda > 0$ be such that $x \in \mathcal{D}(J_\lambda)$. Since A is monotone, it follows from Theorem 4 that J_λ is nonexpansive. Take $p \in \text{Fix}(J_\lambda)$. Then

$$d(J_\lambda(x), p) \leq d(x, p).$$

Therefore, the set $\{J_\lambda(x) : x \in \mathcal{D}(J_\lambda)\}$ is bounded; consequently (ii) holds.

(ii)⇒(iii) It is trivial.

(iii)⇒(iv) Assume that (iii) holds. Then, there exists a sequence $\{t_n\} \subset (0, +\infty)$ with $t_n \rightarrow \infty$ such that $\lim_{n \rightarrow \infty} J_{t_n}(x) = y$. By Lemma 6(i), $y \in A^{-1}(0)$. Hence $A^{-1}(0) \neq \emptyset$ and by (50) it follows that $\bigcap_{\lambda>0} \mathcal{D}(J_\lambda) \neq \emptyset$. Then, if $x \in \bigcap_{\lambda>0} \mathcal{D}(J_\lambda)$ Lemma 6 implies (48).

(iv)⇒(i) It is clear.

Finally, we assume that $A^{-1}(0) \neq \emptyset$ is convex and let $x \in \bigcap_{\lambda>0} \mathcal{D}(J_\lambda)$. Then (iv) holds and so $\{J_\lambda(x)\}$ is bounded as $\lambda \rightarrow +\infty$. Using Lemma 6(i) we have that the limit $y = \lim_{\lambda \rightarrow \infty} J_\lambda(x)$ satisfies (44). Since $A^{-1}(0)$ is closed and convex, it follows from Proposition 4 that $y = P_{A^{-1}(0)}(x)$. □

The Banach space analog of the previous theorem can be found in [38]. The following result is direct from Corollary 3 and Theorem 5.

Corollary 4 *Let $A \in \mathcal{X}(M)$ be monotone with closed convex domain $\mathcal{D}(A)$ and nonempty $A^{-1}(0)$. Suppose that $\mathcal{D}(A) \subseteq \mathcal{D}(J_\lambda)$ for some $\lambda > 0$. Then (49) holds, i.e.,*

$$\lim_{\lambda \rightarrow \infty} J_\lambda(x) = P_{A^{-1}(0)}(x), \quad \forall x \in \bigcap_{\lambda>0} \mathcal{D}(J_\lambda).$$

The following theorem describes the behavior of the resolvent J_λ as $\lambda \rightarrow 0$.

Theorem 6 *Let $A \in \mathcal{X}(M)$ be monotone. Then, if $x \in \mathcal{D}(A) \cap \mathcal{D}(J_\lambda)$, for all $\lambda > 0$,*

$$\lim_{\lambda \rightarrow 0} J_\lambda(x) = x.$$

Proof By the definition of Yosida approximation A_λ and inequality (43), we have that

$$d(J_\lambda x, x) = \|\exp_x^{-1} J_\lambda x\| = \lambda \|A_\lambda x\| \leq \lambda |Ax|.$$

Since $|Ax| = \inf\{\|v\| \mid v \in Ax\}$, there exists $M > 0$ such that $|A(x)| \leq M$. Thus, letting $\lambda \rightarrow 0$ the statement of the theorem is proved. □

The existence of singularities of a strongly maximal monotone vector field was proved in [28] as Theorem 3 establishes. From Lemma 6 and Theorem 4 existence results for monotone vector fields under certain boundary condition are next proved. A similar result was obtained in Banach spaces in [20, 41].

Let $x_0 \in M$ and $\epsilon > 0$. We denote

$$\mathbf{B}(x_0, \epsilon) := \{y \in M \mid d(x, y) \leq \epsilon\}$$

and

$$\partial \mathbf{B}(x_0, \epsilon) := \{y \in M \mid d(x, y) = \epsilon\}.$$

Theorem 7 *Let $A \in \mathcal{X}(M)$ be an upper Kuratowski semicontinuous monotone vector field. Suppose that there exist $x_0 \in \bigcap_{\lambda > 0} \mathcal{D}(J_\lambda)$ and $\epsilon > 0$ such that*

$$|A(x_0)| < |A(x)|, \quad \forall x \in \partial \mathbf{B}(x_0, \epsilon). \tag{51}$$

Then there exists a singularity $y \in \mathbf{B}(x_0, \epsilon)$.

Proof Let $x_0 \in \bigcap_{\lambda > 0} \mathcal{D}(J_\lambda)$ and $\epsilon > 0$ be such that (51) is satisfied. By Theorem 5, it suffices to verify that $J_\lambda(x_0) \in \mathbf{B}(x_0, \epsilon)$ for all $\lambda \geq 0$. To do this, let $\lambda > 0$. From Proposition 5, we know that $P_{J_\lambda(x_0), x_0} A_\lambda(x_0) \in AJ_\lambda(x_0)$ and

$$|A(J_\lambda(x_0))| = |P_{x_0, J_\lambda(x_0)} A(J_\lambda(x_0))| \leq \|A_\lambda(x_0)\| \leq |A(x_0)|. \tag{52}$$

So our boundary condition implies the fact that $J_\lambda(x_0) \notin \partial \mathbf{B}(x_0, \epsilon)$ for each $\lambda > 0$. Below we show that the function $\lambda \mapsto J_\lambda(x_0)$ is continuous on $[0 + \infty)$. Granting this, one sees that $J_\lambda(x_0) \in \mathbf{B}(x_0, \epsilon)$ for any $\lambda > 0$ because $J_0(x_0) = x_0 \in \mathbf{B}(x_0, \epsilon)$; hence $\{J_\lambda(x_0)\}$ is bounded and the proof is complete. Let $\lambda_0 \in [0 + \infty)$. By Theorem 6, we only need to consider the case when $\lambda_0 > 0$. Since

$$d(J_{\lambda_n}(x_0), x_0) = \|\exp_{x_0}^{-1} J_{\lambda_n}(x_0)\| = \lambda_n |A_\lambda(x_0)| \leq \lambda_n \|A(x_0)\|, \tag{53}$$

it follows that $\{J_\lambda(x_0)\}$ is bounded as $\lambda \rightarrow \lambda_0$. Suppose that there exists subsequence $\{\lambda_n\}$ with $\lambda_n \rightarrow \lambda_0$ such that $J_{\lambda_n}(x_0) \rightarrow z_0$. By the definition of J_{λ_n} , $x_0 \in \exp_{J_{\lambda_n}(x_0)}^{-1} \lambda_n A(z_n)$; that is,

$$x_0 = \exp_{J_{\lambda_n}(x_0)}^{-1} \lambda_n y_n \tag{54}$$

with $y_n \in A(J_{\lambda_n}(x_0))$ for each n . This implies that $\{y_n\}$ is bounded as $\lambda_n \rightarrow \lambda_0 > 0$ and $J_{\lambda_n}(x_0) \rightarrow z_0$. Thus, without loss of generality, we assume that $y_n \rightarrow y_0$. By the assumed upper semicontinuity, $y_0 \in A(z_0)$. Taking limit in the equality (54) we obtain that

$$x_0 = \exp_{z_0}^{-1} \lambda_0 y_0 \in \exp_{z_0}^{-1} \lambda_0 A(z_0). \tag{55}$$

This means that $z_0 = J_{\lambda_0}(x_0)$ by definition and the function $\lambda \mapsto J_\lambda(x_0)$ is continuous at λ_0 . □

Corollary 5 *Let $A \in \mathcal{X}(M)$ be maximal monotone with full domain $\mathcal{D}(A) = M$. Suppose that there exist $x_0 \in M$ and $\epsilon > 0$ such that*

$$|A(x_0)| < |A(x)|, \quad \forall x \in \partial \mathbf{B}(x_0, \epsilon). \tag{56}$$

Then there exists a singularity $y \in \mathbf{B}(x_0, \epsilon)$.

In the setting of Hilbert spaces there is a class of nonexpansive type mappings, the so called pseudo-contractive mappings, which are closely related to the class of monotone operators. Recall that a set-valued mapping $T : H \rightarrow 2^H$ is said to be pseudo-contractive if, for any $r > 0$, for all $x, y \in X$ and for any $u \in Tx, v \in T(y)$,

$$\|x - y\| \leq \|(1 + r)(x - y) - r(u - v)\| \tag{57}$$

holds. This concept was introduced by Browder and Petryshyn, in [5], and they proved that a mapping T is pseudo-contractive if and only if the operator $I - T$ is monotone. This means that the problem of solving an equation involving monotone

operators may be formulated as a fixed point problem for a pseudo-contractive mapping on a Hilbert space.

Recall that, in a Hadamard manifold, the complementary vector field establishes a connection between nonexpansive mappings and monotone vector fields; see Theorem 2. Thanks to this vector field, we can define the concept of pseudo-contractive mappings as follows.

Definition 8 A set-valued mapping $T : K \subseteq M \rightarrow 2^M$ is said to be pseudo-contractive if its complementary vector field is accretive; that is, given $x, y \in K$, for any $u \in T(x)$, $v \in T(y)$ and $r \geq 0$ we have that

$$d(x, y) \leq d(\exp_x(-r \exp_x^{-1} u), \exp_y(-r \exp_y^{-1} v)). \quad (58)$$

Remark 6 Note that, indeed, when M is an Euclidean space, inequalities (57) and (58) coincide. On the other hand, by Theorem 1, a mapping T is pseudo-contractive if and only if its complementary vector field is monotone. This together with Theorem 2 implies that any nonexpansive mapping is pseudo-contractive.

The notion of pseudo-contractive mappings was previously introduced in hyperbolic spaces in [40] but defined as co-accretive operators.

The following result about the existence of fixed points of continuous pseudo-contractive mappings is the counterpart of Theorem 1 in [19], proved by Kirk and Schöneberg in the setting of Hilbert spaces.

Corollary 6 Let $T : M \rightarrow M$ be a continuous pseudo-contractive mapping. Let $x_0 \in M$ and $\epsilon > 0$ such that

$$d(x_0, T(x_0)) < d(x, T(x)), \quad (59)$$

for any $x \in \partial \mathbf{B}(x_0, \epsilon)$. Then there exists a fixed point of T in $\mathbf{B}(x_0, \epsilon)$.

Proof Let A be the complementary vector field of T ; that is, $A(x) = -\exp_x^{-1} T(x)$, for any $x \in M$. Since T is single-valued, so is A , then this means that $|A(x)| = \|\exp_x^{-1} T(x)\| = d(x, T(x))$, for any $x \in M$. Therefore, by hypothesis (59), condition (51) in Theorem 7 is satisfied. On the other hand, the fact that T is a continuous pseudo-contractive mapping implies that A is maximal monotone. Thus Theorem 7 assures the existence of a singularity of A in $\mathbf{B}(x_0, \epsilon)$ which can be proved to be a fixed point of T . \square

The following example illustrates an application of the previous results to approximate the solution of a convex minimization problem on M .

Example 1 Let $f : M \rightarrow M$ be a convex function. The Moreau–Yosida regularization $R_\lambda^f : M \rightarrow M$ of f is defined by

$$R_\lambda^f(x) := \operatorname{argmin}_{y \in M} \left\{ \lambda f(y) + \frac{1}{2} d^2(x, y) \right\}. \quad (60)$$

It was proved in [14] that R_λ^f is a single-valued mapping with $\mathcal{D}(R_\lambda^f) = M$ for each $\lambda > 0$, which is characterized by

$$\exp_{y_\lambda}^{-1} x \in \partial f(y_\lambda), \quad \forall x \in M, \quad (61)$$

where ∂f denotes the subdifferential vector field of f defined by

$$\partial f(x) = \{u \in T_x M \mid \langle u, \exp_x^{-1} y \rangle \leq f(y) - f(x), \quad \forall y \in M\}.$$

Consider the set-valued vector field $A = \partial f$. It was proved in [28] that A is maximal monotone with full domain. By the characterization (61) we can readily check that the resolvent of A coincides with the Moreau–Yosida regularization, that is, $R_\lambda^f = J_\lambda^A$, and therefore Theorem 4 recovers the fact that R_λ^f has full domain, and implies its firm nonexpansivity.

Furthermore, consider the minimization problem

$$\min_{x \in M} f(x) \quad (62)$$

and denote its solution set by S . Assume that $S \neq \emptyset$. Then by the asymptotic behavior of the resolvent stated in Theorem 4 we obtain an iterative method for approximating a minimizer of f . More precisely, for any $x \in M$, the net $\{R_\lambda^f(x)\}$ converges to $P_S(x)$ as $\lambda \rightarrow \infty$.

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