Precise asymptotics in the law of logarithm under dependence assumptions

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\begin{abstract}
In a recent paper by Spătaru \cite{Precise asymptotics for a series of T.L. Lai, Proc. Amer. Math. Soc. 132 (11) (2004) 3387–3395} a precise asymptotics in the law of the logarithm for sequence of i.i.d. random variables has been established. In this paper we show that there is an analogous result for strictly stationary \( \varphi \)-mixing sequence. To prove this result, we have to use a different method. One of our main tools is the Gaussian approximation technique.
\end{abstract}

\section{Introduction and results}

Let \( \{X_n; n \geq 1\} \) be a sequence of independent and identically distributed (i.i.d.) random variables with common distribution \( F \). Let \( S_n = \sum_{k=1}^{n} X_k \) and suppose that \( \mathbb{E}X = 0 \) and \( 0 < \mathbb{E}X^2 = \sigma^2 < \infty \). There is a lot of literature concerning precise asymptotic behavior of the partial sum \( S_n \). The first such result is due to Heyde \cite{1}, who proved that \( \lim_{n \rightarrow 0} \frac{\sigma^2 \sum_{k=1}^{n} P(|S_n| \geq \varepsilon n^\gamma)}{n} \rightarrow \sigma^2. \) This result was extended in \cite{2}, wherein the author proved a more general theorem. In a recent paper, assuming the distribution of \( X \) is attracted to a stable distribution with exponent \( \alpha > 1 \), Spătaru \cite{3} proved \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} P(|S_n| \geq \varepsilon n^\gamma) \sim \frac{\sigma^2}{\gamma - 1} (-\log \varepsilon) \) as \( \varepsilon \searrow 0 \). After this interesting contribution, more and more authors have devoted their efforts to the work of precise asymptotics; see \cite{4-8}, for example.

There already exist some classical methods to deal with the precise asymptotics for the case of \( \varepsilon \searrow 0 \). Such results are usually not too difficult to derive. However, for the case of \( \varepsilon \searrow \varepsilon_0 \) with some positive number \( \varepsilon_0 \), powerful tools and finer arguments are needed. For example, by using a non-uniform estimate in the normal approximation, Li, Wang and Rao \cite{9} obtained much more general results on precise asymptotics in the law of the iterated logarithm for the i.i.d. case. Using the Berry–Esseen inequality, Spătaru \cite{7} obtained the precise asymptotics in the law of logarithm. His result is as follows.

\textbf{Theorem A.} Let \( \{X, X_1, X_2, \ldots\} \) be a sequence of i.i.d random variables, \( 1 < r < 3/2, \mathbb{E}X^2 = \sigma^2 \) and \( \mathbb{E}(X^2/ \log^+ |X|)^2 < \infty \). Then

\[ \lim_{\varepsilon \searrow \varepsilon_0} \epsilon \sqrt{2(r-1)} \sqrt{\epsilon^2 - 2\sigma^2(r-1)} \sum_{n=2}^{\infty} \sigma^2 n^{r-2} P \left( |S_n| \geq \varepsilon \sqrt{n \log n} + a_n \right) = \sigma \sqrt{\frac{2}{r-1}}, \]

where \( a_n = o(n \sqrt{\log n}) \).
(Theorem A is exactly the same as Theorem 1 of [7], in which \( a_n = o(\sqrt{n} / \log n) \); see the proof of [7].)

It is well-known that the rate in central limit theorem is of the order of \( O(n^{-1/2}) \) and can not be improved. Therefore, by examining the proof in [7], one can find that it is difficult to extend Theorem A to the case of \( r \geq 3/2 \) by his method. Moreover, the rate of the central limit theorem for mixing random variables is not as sharp as that of the i.i.d. case. Hence we shall develop a different method to extend Theorem A to the case of \( r \geq 3/2 \). Meanwhile an analogous result to Theorem A will also be derived for mixing random variables. Our method is based on a coupling lemma (see Lemma 2.1) and the Gaussian approximation technique due to Sakhaneko [10]. The latter approximation was used by Zhang [11] for obtaining the sufficient and necessary conditions of the precise rates in the law of iterated logarithm for the i.i.d. random variables.

Now we give some definitions of mixing random variables. Let \( \mathcal{A}_n \) denote the \( \sigma \)-field generated by \( X_0, X_{n+1}, \ldots, X_k \) and define

\[
\varphi(\mathcal{A}_1, \mathcal{A}_k) := \sup\{|P(A) - P(B)|; A \in \mathcal{A}_1, B \in \mathcal{A}_k\},
\]

\[
\varphi(\mathcal{A}_1, \mathcal{A}_{k+n}) := \sup\{|\text{corr}(U, V)|; U \in L^2(\mathcal{A}_1), V \in L^2(\mathcal{A}_{k+n})\},
\]

\[
\varphi(n) := \sup_{k \geq 1} \varphi(\mathcal{A}_k, \mathcal{A}_{k+n}), \quad \varphi(n) := \sup_{k \geq 1} \varphi(\mathcal{A}_k, \mathcal{A}_{k+n}).
\]

A sequence \( \{X_n\}_{n \geq 1} \) of random variables is called \( \varphi \)-mixing if \( \varphi(n) \to 0 \) and \( \rho \)-mixing if \( \varphi(n) \to 0 \). It is known that \( \varphi(n) \leq 2^{\varphi(n/2)} \) and hence a \( \varphi \)-mixing sequence is \( \rho \)-mixing.

We state our results as follows.

**Theorem 1.1.** Let \( 1 < r < 3/2 \). Let \( \{X_n; n \geq 1\} \) be a strictly stationary \( \varphi \)-mixing sequence such that

\[
\varphi(n) = O\left(\frac{1}{n^2}\right) \quad \text{for some } T > 2,
\]

and

\[
EX = 0, \quad E(X^{2r}/(\log^+ |X|)^r) < \infty.
\]

Then

\[
\lim_{\epsilon \searrow \sqrt{2(r-1)}} \epsilon^2 - 2(r-1) \sum_{n=1}^{\infty} n^{r-2} P(|S_n| \geq \epsilon \sqrt{E S_n^2} \log n + a_n) = \frac{2}{\sqrt{r-1}},
\]

whenever \( a_n = O(\sqrt{n}/(\log n)^{\gamma}) \) for some \( \gamma > 1/2 \).

**Theorem 1.2.** Let \( r \geq 3/2 \). Let \( \{X_n; n \geq 1\} \) be a strictly stationary \( \varphi \)-mixing sequence such that

\[
\varphi(n) = O\left(\frac{1}{n^2}\right) \quad \text{for some } T > 2r - 1,
\]

and

\[
EX = 0, \quad E(X^{2r}/(\log^+ |X|)^r) < \infty.
\]

Then

\[
\lim_{\epsilon \searrow \sqrt{2(r-1)}} \epsilon^2 - 2(r-1) \sum_{n=1}^{\infty} n^{r-2} P(|S_n| \geq \epsilon \sqrt{E S_n^2} \log n + a_n) = \frac{2}{\sqrt{r-1}},
\]

whenever \( a_n = O(\sqrt{n}/(\log n)^{\gamma}) \) for some \( \gamma > 1/2 \).

The paper is organized as follows. Throughout the paper, \( C \) denotes a positive constant and may be different in every line. Some lemmas are collected in Section 2. The proofs of the main results are given in Section 3.

## 2. Preliminary lemmas

In this section, we state some lemmas, which will be used in the proof of our main result. The first one comes from [12].

**Lemma 2.1.** Let \( \{(\mathbb{B}_k, \| \cdot \|_k), k \geq 1\} \) be a sequence of complete separable metric spaces. Let \( \{X_k, k \geq 1\} \) be a sequence random variables with values in \( \mathbb{B}_k \) and let \( \{\mathcal{A}_k, k \geq 1\} \) be a sequence of \( \sigma \)-fields such that \( X_k \) is \( \mathcal{A}_k \)-measurable. Suppose that for some \( \phi_k \geq 0 \)

\[
|P(AB) - P(A)P(B)| \leq \phi_k P(A)
\]

for all \( A \in \bigvee_{j \leq k} \mathcal{A}_j \) and \( B \in \mathcal{A}_k \). Then, without changing its distribution we can redefine the sequence \( \{X_k, k \geq 1\} \) on a richer probability space together with a sequence \( \{Y_k, k \geq 1\} \) of independent random variables such that \( Y_k \) has the same distribution as \( X_k \) and

\[
P(\|X_k - Y_k\| \geq 6\phi_k) \leq 6\phi_k \quad k = 1, 2, \ldots.
\]
Lemma 2.2. For any sequence of independent random variables \( \{\xi_i, n \geq 1\} \) with mean zero and finite variance, there exists a sequence of independent normal variables \( \{\eta_n, n \geq 1\} \) with \( \mathbb{E}\eta_n = 0 \) and \( \mathbb{E}\eta_n^2 = \mathbb{E}\xi_i^2 \) such that for all \( Q > 2 \) and \( y > 0 \),

\[
P\left( \max_{k \leq n} \left| \sum_{i=1}^{k} \xi_i - \sum_{i=1}^{k} \eta_i \right| \geq y \right) \leq (AQ)^y y^{-Q} \sum_{i=1}^{n} \mathbb{E}|\xi_i|^Q,
\]

whenever \( \mathbb{E}|\xi_i|^Q < \infty, i = 1, 2, \ldots, n \). Here \( A \) is a universal constant.

Proof. See [13–15]. \( \square \)

Lemma 2.3. For some \( \tau \in \mathbb{R} \), let \( a_n' \) satisfy \( \sqrt{\log n} a_n' \to \tau \) as \( n \to \infty \). Then we have

\[
\lim_{\varepsilon \to 0} \sqrt{2} \exp(-\varepsilon^2/2) \leq \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{(\varepsilon \sqrt{\log n} + a_n')^2}{2}\right) \leq \frac{2}{\sqrt{2\pi}} \exp(-\varepsilon^2/2) \exp(-\varepsilon n' \sqrt{\log n})
\]

where \( N \) is a standard normal distributed random variable.

Proof. Since \( \mathbb{P}(|N| \geq x) \sim \frac{2}{\sqrt{2\pi}} \exp(-x^2/2) \) as \( x \to \infty \), we have

\[
P\left( |N| \geq \varepsilon \sqrt{\log n} + a_n' \right) \sim \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{(\varepsilon \sqrt{\log n} + a_n')^2}{2}\right)
\]

as \( n \to \infty \), uniformly in \( \varepsilon \in (\sqrt{2}, \sqrt{2} + \delta) \) for some \( \delta > 0 \). So, for any \( 0 < \theta < 1 \), there exist \( \delta > 0 \) and \( n_0 \) such that for all \( n \geq n_0 \) and \( \varepsilon \in (\sqrt{2}, \sqrt{2} + \delta) \),

\[
\frac{\sqrt{2}}{\varepsilon \sqrt{\pi} \log n} \exp(-\varepsilon^2 \log n/2) \exp(-\sqrt{2} \tau - \theta) \leq \frac{\sqrt{2}}{\varepsilon \sqrt{\pi} \log n} \exp(-\varepsilon^2 \log n/2) \exp(-\sqrt{2} \tau + \theta)
\]

since \( \sqrt{\log n} a_n' \to \tau \). Also,

\[
\lim_{\varepsilon \to 0} \sqrt{\frac{2}{2\pi}} \exp(-\varepsilon^2 \log n/2) = \sqrt{\frac{2}{\pi (r-1)}} (\frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sqrt{2n-2}}{\sqrt{\pi} \log n} \exp(-\varepsilon^2 \log n/2))
\]

\[
= \lim_{\varepsilon \to 0} \sqrt{\frac{2}{\pi (r-1)}} \left( \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sqrt{2n-2}}{\sqrt{\pi} \log n} \exp(-\varepsilon^2 \log n/2) \right)
\]

\[
= \lim_{\varepsilon \to 0} \sqrt{\frac{2}{\pi (r-1)}} \left( \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sqrt{2n-2}}{\sqrt{\pi} \log n} \right) \exp(-\varepsilon^2 \log n/2)
\]

\[
= \sqrt{\frac{2}{\pi (r-1)}} \lim_{\varepsilon \to 0} \frac{1}{2\sqrt{\pi} \sqrt{2(r-1)}} \int_{-\infty}^{\infty} t^{-\frac{1}{2}} \exp(-t) dt
\]

Therefore, Lemma 2.3 can be concluded at once. \( \square \)

3. Proofs

Proof of Theorem 1.1. Let \( a_n = O(\sqrt{n}/(\log n)^\gamma) \), but might be different in every line in the proof. Let \( H_i, l_i \) be the long and short blocks, respectively, with

\[
|H_i| = [i^\alpha], \quad |l_i| = [i^{\rho\alpha}],
\]

where \( 0 < \alpha, \rho < 1 \) and \( \alpha, \rho \) are close to one enough, such that \( \alpha \rho > \frac{3}{2} \). Denote \( N_m = \sum_{i=1}^{m} \text{card}(H_i \cup l_i) \sim \frac{1}{\rho-1} m^{\rho+1} \). Clearly, for each \( n \) there exists a unique \( m_n \) such that \( N_{m_n} \leq n < N_{m_n+1} \). By simple calculation, we have \( m_n \sim ((\rho+1)n)^{1/\rho+1} \). Let

\[
u_i = \sum_{i \in H_i} X_i, \quad \nu_i = \sum_{i \in l_i} X_i.
\]
Hence, $S_n = \sum_{i=1}^{m_n} u_i + \sum_{i=1}^{m_n} v_i + \sum_{i=N_{m_n}+1}^{n} X_i$. Write

\[ A := \sum_{n=1}^{\infty} n^{r-2} P \left( |S_n| \geq \varepsilon \sqrt{\mathbb{E}S_n^2 \log n} + a_n \right); \]
\[ A_1 := \sum_{n=1}^{\infty} n^{r-2} P \left( \sum_{i=1}^{m_n} u_i \geq \varepsilon \sqrt{\mathbb{E}S_n^2 \log n} + a_n \right); \]
\[ A_2 := \sum_{n=1}^{\infty} n^{r-2} P \left( \sum_{i=1}^{m_n} v_i \geq 2 \sqrt{n} \left( \log n \right)^{\gamma} \right); \]
\[ A_3 := \sum_{n=1}^{\infty} n^{r-2} P \left( \sum_{i=N_{m_n}+1}^{n} X_i \geq 2 \sqrt{n} \left( \log n \right)^{\gamma} \right). \]

Then, we have $A_1 - A_2 - A_3 \leq A = A_1 + A_2 + A_3$. ($A_1$ on the left hand side of $A$ is different from that on the right hand side. But to simplify, we use a common notation.) We will complete the proof by showing that $A_2 < C, A_3 < C$ and

\[ \lim_{\varepsilon \to \sqrt{2(r-1)}} 2(r-1)A_1 = \frac{2}{1}. \quad \square \]

To check $A_2 < C, A_3 < C$, we need the following lemma.

**Lemma 3.1.** Under the assumption of Theorem 1.1, we have $A_2 < C$ and $A_3 < C$.

**Proof.** Let $v'_i = \sum_{j \in P_i} |X_j||X_i| \leq \sqrt{n \log n} - \mathbb{E}X_i|X_i| \leq \sqrt{n \log n}$. By noticing that

\[ \sum_{i=1}^{m_n} |I| \mathbb{E}I|I| \mathbb{E}|X_i| \geq \sqrt{n \log n} \]

\[ \sqrt{\mathbb{E}I} \mathbb{E}I \mathbb{E}|X_i| \to 0 \quad \text{as} \quad n \to \infty, \]

and by the Rosenthal’s inequality for mixing sequence (see [16]), take $q$ large enough, we have

\[ A_2 \leq C \sum_{n=1}^{\infty} n^{r-2} P \left( \sum_{i=1}^{m_n} v'_i \geq \frac{\sqrt{n}}{\left( \log n \right)^{\gamma}} \right) + C \sum_{n=1}^{\infty} n^{r-2} \frac{a_{n+1}}{\left( \log n \right)^{\gamma}} P \left( |X| \geq \sqrt{n \log n} \right) \]

\[ \leq C \sum_{n=1}^{\infty} n^{r-2} \frac{a_{n+1}}{\left( \log n \right)^{\gamma}} \mathbb{E}|X|^q \left( |X| \leq \sqrt{n \log n} \right) + C \sum_{n=1}^{\infty} n^{r-2} \frac{\left( \log n \right)^{\gamma}}{\left( \log n \right)^{2\gamma}} \sum_{i=1}^{m_n} |I|^{q/2} + C \]

\[ \leq C \sum_{n=1}^{\infty} \left( k \log k \right)^{q/2} P \left( k \log k \leq |X| < \sqrt{(k+1) \log (k+1)} \right) \sum_{n=k}^{\infty} n^{r-2} \frac{a_{n+1}}{\left( \log n \right)^{\gamma}} + C \sum_{n=1}^{\infty} \left( \log n \right)^{\gamma,n^{r-2} \frac{a_{n+1}}{\left( \log n \right)^{\gamma}}} + C \]

\[ \leq C. \]

Since $m_n \leq (\rho + 1)n^{1/\gamma}$, we have $N_{m_n} \leq C(\rho + 1)n^{1/\gamma}$ and

\[ \sum_{i=1}^{m_n} |I| \mathbb{E}I|I| \mathbb{E}|X_i| \to 0 \quad \text{as} \quad n \to \infty. \]

Define $X'_i = X_i|I| \mathbb{E}|X_i| \leq \sqrt{n \log n} - \mathbb{E}X_i|I| \mathbb{E}|X_i| \leq \sqrt{n \log n}$. Therefore, by the Rosenthal’s inequality for mixing sequence again,

\[ A_3 \leq C \sum_{n=1}^{\infty} n^{r-2} P \left( \sum_{i=N_{m_n}+1}^{n} X'_i \geq \frac{\sqrt{n}}{\left( \log n \right)^{\gamma}} \right) + C \sum_{n=1}^{\infty} n^{r-2} \frac{a_{n+1}}{\left( \log n \right)^{\gamma}} P \left( |X| \geq \sqrt{n \log n} \right) \]

\[ \leq C \sum_{n=1}^{\infty} n^{r-2} \frac{a_{n+1}}{\left( \log n \right)^{\gamma}} \mathbb{E}|X|^q \left( |X| \leq \sqrt{n \log n} \right) + C \sum_{n=1}^{\infty} n^{r-2} \frac{\left( \log n \right)^{\gamma}}{\left( \log n \right)^{2\gamma}} \mathbb{E}|X|^q + C \]

\[ =: A_{31} + A_{32} + C. \]
Choosing \( q > 2r \), we have
\[
A_{31} \leq C \sum_{k=1}^{\infty} (k \log k)^{\frac{q}{2}} P \left( \frac{k \log k}{\log n} \leq \frac{\sqrt{\log n}}{\sqrt{(k + 1) \log (k + 1)}} \right) \sum_{n=k}^{\infty} n^{-r - \frac{1}{2} + \frac{\rho}{2}, r} (\log n)^{\theta}
\]
\[
\leq C \sum_{k=1}^{\infty} k^r P \left( \frac{k \log k}{\log n} \leq \frac{\sqrt{\log n}}{\sqrt{(k + 1) \log (k + 1)}} \right) \leq C
\]
and
\[
A_{32} \leq C \sum_{n=1}^{\infty} n^{-r - \frac{1}{2} + \frac{\rho}{2}, r} (\log n)^{\theta} \leq C.
\]

Therefore, it holds that \( A_3 \leq C \), and we finish the proof of Lemma 3.1.

In the rest of the paper, we only need to show that
\[
\lim_{\varepsilon \to 2(r - 1)} 2 = \sqrt{\frac{2}{r - 1}}.
\]

By Lemma 2.1, we can construct independent random variables \( Y_i \), \( 1 \leq i \leq m_n \) such that \( Y_i \) has the same distribution as \( u_i \) for \( 1 \leq i \leq m_n \). Write
\[
A_{11} = \sum_{n=1}^{\infty} n^{-r - 2} P \left( \sum_{i=1}^{m_n} (u_i - Y_i) \geq \frac{2 \sqrt{n}}{\log (n)} \right),
\]
\[
A_{12} = \sum_{n=1}^{\infty} n^{-r - 2} P \left( \sum_{i=1}^{m_n} Y_i \geq \varepsilon \frac{\log n}{\log n} + a_n \right),
\]
where \( a_n^* = a_n + \frac{2 \sqrt{n}}{\log (n)} \frac{\log (n)}{\log (n)} = O \left( \frac{2 \sqrt{n}}{\log (n)} \right) \). While \( A_{12} - A_{11} \leq A_1 \leq A_{11} + A_{11} \) (with \( a_n^* \) taking different values on both sides of \( A_1 \)). It suffices to show that \( A_{11} \leq C \) and
\[
\lim_{\varepsilon \to 2(r - 1)} 2 = \sqrt{\frac{2}{r - 1}}.
\]

Let \( k_n = \left( m_n \right)^{1/(1+r)} \) and \( a' > 0 \). We have
\[
A_{11} \leq \sum_{n=1}^{\infty} n^{-r - 2} P \left( \sum_{i=1}^{k_n} (u_i - Y_i) \geq \frac{\sqrt{n}}{\log (n)} + a_n \right) + \sum_{n=1}^{\infty} n^{-r - 2} P \left( \sum_{i=1}^{m_n} (u_i - Y_i) \geq \frac{\sqrt{n}}{\log (n)} + a_n \right)
\]
\[
= A_{111} + A_{112}.
\]

We first consider \( A_{112} \). Since \( \alpha \rho > 2/2, T > 2 \) for \( 1 < r < 3/2 \), and \( T > 2r - 1 \) for \( r \geq 3/2 \), according to Lemma 2.1, we have
\[
A_{112} \leq C \sum_{n=1}^{\infty} n^{-r - 2} \sum_{i=k_n}^{m_n} P \left( (u_i - Y_i) \geq 6 \varphi(|l_i|) \right) \leq C \sum_{n=1}^{\infty} n^{-r - 2} \sum_{i=k_n}^{m_n} \varphi(|l_i|)
\]
\[
\leq C \sum_{n=1}^{\infty} n^{-r - 2} \sum_{i=k_n}^{m_n} \frac{1}{\varphi(p)} \leq C \sum_{n=1}^{\infty} n^{-r - 2} \frac{1}{\varphi(p)} \leq C.
\]

Using a similar method for proving \( A_2 \leq C \), we can get \( A_{111} \leq C \). Therefore, we have \( A_{11} \leq C \).

To estimate \( A_{12} \), we need the following lemma.

**Lemma 3.2.** Under the assumption of Theorem 1.1, we have
\[
\lim_{\varepsilon \to 2(r - 1)} 2 = \sqrt{\frac{2}{r - 1}}.
\]

**Proof.** Let \( \{X_j, j \in H_i\} \) be an independent of \( \{X_j, j \in H_i\} \), and \( \{X_j, j \in H_i\} \), \( 1 \leq i \leq m_n \), are independent. Let
\[
Y_i = \sum_{j \in H_i} X_j \quad \text{and} \quad Y_i' = \sum_{j \in H_i} \left( X_j \mathbf{1} \left( |X_j| \leq \frac{n}{\log n} \right) - \mathbb{E} X_j \mathbf{1} \left( |X_j| \leq \frac{n}{\log n} \right) \right)
\]
for \( 1 \leq i \leq m_n \). Obviously, \( \sum_{i=1}^{m_n} Y_i \) and \( \sum_{i=1}^{m_n} Y_i' \) have the same distribution. So
\[
A_{12} = \sum_{n=3}^{\infty} n^{-r - 2} P \left( \sum_{i=1}^{m_n} Y_i \geq \varepsilon \sqrt{\frac{\log n}{n}} + a_n \right).
\]
And
\[ \sum_{i=1}^{m_n} |H_i| \mathbb{E}|X_i|(|X_i| \geq \sqrt{n \log n}) \bigg/ \sqrt{n \log n} \to 0 \]
as \( n \to \infty \). Hence
\[ A_{12} \leq \sum_{n=1}^{\infty} n^{-2} P \left( \left\{ \sum_{i=1}^{m_n} Y_i \right\} \geq \varepsilon \sqrt{\mathbb{E}_n^2 \log n + a_n} + o \left( \sqrt{\frac{n}{\log n}} \right) \right) + \sum_{n=3}^{\infty} n^{-1} P \left( |X| \geq \sqrt{n \log n} \right) \]
\[ =: A_{121} + A_{122}. \]

Also, \( A_{12} \geq A_{121} - A_{122} \). Note that \( A_{122} \leq C \varepsilon (X r^2 + (|X|)^r) \to \infty \), we only need to prove
\[ \lim_{\varepsilon \to \sqrt{2(r-2)}} \sqrt{\varepsilon^2 - 2(r-2)} A_{121} = \sqrt{\frac{2}{r-1}}. \]

For \( 1 < r < \frac{3}{2} \), we have
\[ A_{121} \leq \sum_{n=1}^{\infty} n^{-2} \left| \left\{ \sum_{i=1}^{m_n} Y_i \right\} \geq \varepsilon \sqrt{\mathbb{E}_n^2 \log n + a_n} + \frac{a_n}{\sqrt{\mathbb{E}_n^2 \log n}} + o \left( \sqrt{\frac{n}{\log n}} \right) \right| + \sum_{n=3}^{\infty} n^{-2} \varepsilon \left( \varepsilon \sqrt{\mathbb{E}_n^2 \log n + a_n} + \frac{a_n}{\sqrt{\mathbb{E}_n^2 \log n}} + o \left( \sqrt{\frac{n}{\log n}} \right) \right) \]
\[ =: A_{121a} + A_{121b}. \]

Also, \( A_{121} \geq A_{121b} - A_{121a} \). By the non-uniform Berry–Esseen bound together with the Rosenthal’s inequality for mixing sequence, we easily show that
\[ A_{121a} \leq C \sum_{n=1}^{\infty} n^{-2} m_n \mathbb{E}|Y_i|^3 \leq C. \]

And by Lemma 2.3, we have
\[ \lim_{\varepsilon \to \sqrt{2(r-2)}} \sqrt{\varepsilon^2 - 2(r-2)} A_{121b} = \sqrt{\frac{2}{r-1}}. \]

Thus, (3.2) is concluded. Theorem 1.1 can be concluded immediately from Lemma 3.1 and (3.1).

\[ \text{Proof of Theorem 1.2.} \] Since the proof of Theorem 1.2 is analogous to the proof of Theorem 1.1, we simply outline the main steps and illustrate the proof for the case \( r \geq 3/2 \), which is distinguished from the case \( 1 < r < 3/2 \) in Theorem 1.1.

Notice that the moment condition of \( X \) is slightly strengthened in Theorem 1.2, hence the results in Lemma 3.1 still hold under the assumptions of Theorem 1.2. Following the approach in the proof of Theorem 1.1, we only need to prove the lemma below.

**Lemma 3.3.** Under the assumption of Theorem 1.2, we have
\[ \lim_{\varepsilon \to \sqrt{2(r-1)}} \sqrt{\varepsilon^2 - 2(r-1)} A_{12} = \sqrt{\frac{2}{r-1}}. \]

**Proof.** The essential differences between Lemmas 3.2 and 3.3 is that when \( r \geq \frac{3}{2} \), we still need (3.2) hold. So we set
\[ B_1(\beta) = \sum_{n=1}^{\infty} n^{-2} P \left( \left\{ \sum_{i=1}^{m_n} Y_i \right\} \geq \varepsilon \sqrt{\mathbb{E}_n^2 \log n + a_n} + \beta \sqrt{\frac{n}{\log n}} \right) \]
and
\[ B_2 = \sum_{n=1}^{\infty} n^{-2} P \left( \left\{ \sum_{i=1}^{m_n} (\hat{Y}_i - \hat{Y}_i^*) \right\} \geq \beta \sqrt{\frac{n}{\log n}} \right) . \]
Lemma 2.2 will be concluded if we prove (3.3)

\[ B_2 \leq C \beta^{-q} \sum_{n=1}^{\infty} (\log n)^{q/2} n^{-q/2} \sum_{i=1}^{m_n} E|Y_i|^q \]

\[ \leq C \beta^{-q} \sum_{n=1}^{\infty} (\log n)^{q/2} n^{-q/2} \sum_{i=1}^{m_n} |H_i|^{q/2} + C \beta^{-q} \sum_{n=1}^{\infty} (\log n)^{q/2} n^{-q/2} \sum_{i=1}^{m_n} |H_i| |X_i|^q \]

\[ \{ |X| \leq \sqrt{\frac{n}{\log n}} \} \]

\[ =: B_{21} + B_{22}. \]

For \( B_{21} \), take \( q \) large enough, we have

\[ B_{21} \leq C \beta \sum_{n=1}^{\infty} (\log n)^{q/2} n^{-q/2} \sum_{i=1}^{m_n} \beta^q \]

\[ \leq C \beta \sum_{n=1}^{\infty} (\log n)^{q/2} n^{-q/2} \sum_{i=1}^{m_n} \leq C \beta. \] (3.3)

For \( B_{22} \), we have

\[ B_{22} \leq C \beta \sum_{n=1}^{\infty} (\log n)^{q/2} n^{-1-q/2} E|X|^{q/2} \]

\[ \leq C \beta \sum_{n=1}^{\infty} (\log n)^{q/2} n^{-1-q/2} \sum_{k=1}^{n} \frac{k^{q/2}}{(\log k)^{q/2}} \log \left( \frac{k}{\log k} \right) \leq \frac{k+1}{\log k} \log \left( \frac{k+1}{\log k} \right) \]

\[ \leq C \beta \sum_{n=1}^{\infty} k \rho \left( \frac{k}{\log k} \right) \log \left( \frac{k+1}{\log (k+1)} \right) \leq C \beta \rho \log \left( \frac{n}{\log n} \right). \] (3.4)

So, by (3.3) and (3.4), in order to prove (3.2), we only need to show that

\[ \lim_{\beta \to 0} \sup_{r \in \sqrt{2}} \sqrt{e^2 - 2(r-1)B_1(-\beta)} = \frac{2}{r - 1}, \] (3.5)

and

\[ \lim_{\beta \to 0} \inf_{r \in \sqrt{2}} \sqrt{e^2 - 2(r-1)B_1(\beta)} = \frac{2}{r - 1}. \] (3.6)

First we show that

\[ \left| E S_n^2 - \sum_{i=1}^{m_n} \operatorname{Var} Y_i \right| = o \left( \frac{n}{\log n} \right). \] (3.7)

Denote that

\[ S_n = \sum_{i=1}^{n} X_i \left| \left| X_i \right| \leq \sqrt{\frac{n}{\log n}} \right. - \operatorname{EX} \left\{ \left| X_i \right| \leq \sqrt{\frac{n}{\log n}} \right\}. \]

Since \( |ES_n^2 - ES_n^2| = o \left( \frac{n}{\log n} \right) \) as \( n \to \infty \), (3.7) will be concluded if we prove \( |ES_n^2 - \sum_{i=1}^{m_n} \operatorname{Var} Y_i| = o \left( \frac{n}{\log n} \right) \). Notice that

\[ E S_n^2 = E \left( \sum_{i=1}^{m_n} \pi_i \right)^2 + E \left( \sum_{i=1}^{m_n} \sum_{k=nm_n+1}^{n} X_k \right)^2 + 2E \left( \sum_{i=1}^{m_n} \sum_{j \in H_i} \pi_i + \sum_{k=nm_n+1}^{n} X_k \right), \]

where

\[ X_i = X_i \left| \left| X_i \right| \leq \sqrt{\frac{n}{\log n}} \right. - \operatorname{EX} \left\{ \left| X_i \right| \leq \sqrt{\frac{n}{\log n}} \right\}, \quad \pi_i = \sum_{j \in H_i} \tilde{X}_j, \quad \nu_i = \sum_{j \in H_i} \tilde{X}_j. \]

Since

\[ \sum_{i=1}^{m_n} |H_i| + n - N_{m_n} \leq C \left( n^{\frac{1}{m_n+1}} + n^{\frac{1}{m_n}} \right), \]
we can get
\[ E \left( \sum_{i=1}^{m_u} Y_i + \sum_{k=m_u+1}^{n} X_k \right)^2 = O(n^{m_u+1}), \]
which implies
\[ \left| E S_n^2 - E \left( \sum_{i=1}^{m_u} V_i \right)^2 \right| = o \left( \frac{n}{\log n} \right) \quad \text{as} \quad n \to \infty. \]
So it suffice to check that
\[ E \left| \sum_{i=1}^{m_u} Y_i \right|^2 = o \left( \frac{n}{\log n} \right) \quad \text{as} \quad n \to \infty. \]
But this follows from \( \alpha \rho > 2/T \) and
\[ \left| E \left( \sum_{i=1}^{m_u} Y_i \right)^2 - \sum_{i=1}^{m_u} E|Y_i|^2 \right| \leq C \sum_{i=1}^{m_u} |Y_i|^2 \left( |Y_i|^2 + |X_i|^2 \right) \]
\[ \leq C \sum_{i=1}^{m_u} |X_i|^2 \left( |X_i|^2 + |Y_i|^2 \right) \leq C \sum_{i=1}^{m_u} |X_i|^2 \]
\[ \leq \frac{\alpha \rho}{\rho^{1-\alpha} \rho^{1}}. \]
Thus, (3.7) is proved.

Now, Following from
\[ \sqrt{\frac{\log n}{n}} \left| \sqrt{E S_n^2} \log n - \sqrt{\sum_{i=1}^{m_u} E|Y_i|^2 \log n} \right| \leq C \frac{\log n}{n} \left| \sum_{i=1}^{m_u} E|Y_i|^2 \right| = C \frac{\log n}{n} o \left( \frac{n}{\log n} \right) = o(1), \]
we have
\[ \sqrt{\frac{\log n}{n}} \left| \sqrt{E S_n^2} \log n - \sqrt{\sum_{i=1}^{m_u} E|Y_i|^2 \log n} \right| = o \left( \frac{n}{\log n} \right) \quad \text{uniformly in} \quad \epsilon \in (\sqrt{2}, 2). \]
So
\[ B_1(-\beta) = \sum_{n=1}^{\infty} n^{-2} P \left( \sum_{i=1}^{m_u} Y_i \geq \epsilon \sqrt{\sum_{i=1}^{m_u} E|Y_i|^2 \log n} + o \left( \sqrt{\frac{n}{\log n}} \right) \right) \]
\[ = \sum_{n=1}^{\infty} n^{-2} P \left( |N| \geq \epsilon \sqrt{\log n} + a_n^\epsilon \right), \]
where
\[ a_n^\epsilon = o \left( \frac{1}{\sqrt{\log n}} \right) - \beta \left( \frac{n}{\sqrt{\sum_{i=1}^{m_u} E|Y_i|^2 \log n}} \right). \]
Obviously \( \sqrt{\log n} a_n^\epsilon \to -\beta/\sigma \quad \text{as} \quad n \to \infty, \)
where \( \sigma^2 = \lim_{n \to \infty} ES_n^2/n. \) Hence, by Lemma 2.3, we have
\[ \lim_{\beta \to 0} \limsup_{r \to \sqrt{2}} \sqrt{r^2 - 2(r-1)B_1(-\beta)} = \sqrt{\frac{2}{r-1}}. \]
Similarly, we can get
\[ \lim_{\beta \to 0} \liminf_{r \to \sqrt{2}} \sqrt{r^2 - 2(r-1)B_1(\beta)} = \sqrt{\frac{2}{r-1}}. \]
Consequently, for the case of \( r \geq 3/2, \) (3.2) follows from (3.3) to (3.6). Therefore, Lemma 3.3 holds as desired. Thus we have finished the proof of Theorem 1.2. □
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