Section 6. Laplacian, volume and Hessian comparison theorems

Weimin Sheng

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Two fundamental results in Riemannian geometry are the Laplacian and Hessian comparison theorems for the distance function. They are directly related to the volume comparison theorem and a special case of the Rauch comparison theorem. The Hessian comparison theorem may also be used to prove the Toponogov triangle comparison theorem.

1 Laplacian comparison theorem.

The idea of comparison theorems is to compare a geometric quantity on a Riemannian manifold with the corresponding quantity on a model space. Typically, in Riemannian geometry, model spaces have constant sectional curvature.

Theorem 6.1 (Laplacian comparison). If \((M^n, g)\) is a complete Riemannian manifold with \(Rc \geq (n - 1) K\), where \(K \in R\), and if \(p \in M^n\), then for any \(x \in M^n\) where \(d_p(x)\) is smooth, we have

\[
\Delta d_p(x) \leq \begin{cases} 
(n - 1) \sqrt{K} \cot \left( \sqrt{K} d_p(x) \right) & \text{if } K > 0 \\
\frac{n - 1}{d_p(x)} & \text{if } K = 0 \\
(n - 1) \sqrt{|K|} \coth \left( \sqrt{|K|} d_p(x) \right) & \text{if } K < 0.
\end{cases}
\]

(6.1)

On the whole manifold, the Laplacian comparison theorem holds in the sense of distributions.
In general, we say that $\Delta f \leq F$ in the sense of distributions if for any nonnegative $C^\infty$ function $\varphi$ on $M^n$ with compact support, we have
\[ \int_{M^n} f \Delta \varphi d\mu \leq \int_{M^n} F \varphi d\mu. \]

Form Theorem 6.1 we can derive the following

**Corollary 6.1.** If $K \leq 0$, then
\[ \Delta d_p \leq \frac{n-1}{d_p} + (n-1) \sqrt{|K|} \tag{6.2} \]

in the sense of distributions. In particular, as above, if $(M^n, g)$ is a complete Riemannian manifold with $\text{Ric} \geq 0$, then for any $p \in M^n$
\[ \Delta d_p \leq \frac{n-1}{d_p} \tag{6.3} \]
in the sense of distributions.

**Remark.** Estimate (6.1) is sharp as can be seen from considering space forms of constant curvature $-K$. If $K = 0$, then (6.3) is sharp since on Euclidean space $\Delta |x| = \frac{n-1}{|x|}$.

## 2 Volume comparison theorem.

A consequence of the Laplacian comparison theorem is the following

**Theorem 6.2** (Bishop volume comparison). If $(M^n, g)$ is a complete Riemannian manifold with $\text{Rc} \geq (n-1)K$, where $K \in \mathbb{R}$, then for any $p \in M^n$, the volume ratio
\[ \frac{\text{Vol} \left( B (p, r) \right)}{\text{Vol}_K \left( B (p_K, r) \right)} \]
is a nonincreasing function of $r$, where $p_K$ is a point in the $n$-dimensional simply connected space form of constant curvature $K$ and $\text{Vol}_K$ denotes the volume in the space form. In particular
\[ \text{Vol} \left( B (p, r) \right) \leq \text{Vol}_K \left( B (p_K, r) \right) \tag{6.4} \]
for all $r > 0$. Given $p$ and $r > 0$, equality holds in (6.4) if and only if $B (p, r)$ is isometric to $B (p_K, r)$.
In the case of nonnegative Ricci curvature we have the following

**Corollary 6.2.** If \((M^n, g)\) is a complete Riemannian manifold with \(Ric \geq 0\), then for any \(p \in M^n\), the volume ratio \(\frac{Vol(B(p, r))}{r^n}\) is a nonincreasing function of \(r\). Since \(\lim_{r \to 0} \frac{Vol(B(p, r))}{r^n} = \omega_n\), we have \(\frac{Vol(B(p, r))}{r^n} \leq \omega_n\) for all \(r > 0\), where \(\omega_n\) is the volume of the Euclidean unit \(n\)-ball.

One of the many useful consequences of this is the following characterization of Euclidean space.

**Corollary 6.3.** (Volume characterization of \(\mathbb{R}^n\)). If \((M^n, g)\) is a complete noncompact Riemannian manifold with \(Rc \geq 0\) and if for some \(p \in M^n\)

\[
\lim_{r \to 0} \frac{Vol(B(p, r))}{r^n} = \omega_n,
\]

then \((M^n, g)\) is isometric to Euclidean space.

**Proof.** By the Bishop-Gromov volume comparison theorem, we actually have \(\frac{Vol(B(p, r))}{r^n} \equiv \omega_n\) for all \(r > 0\). The result now follows from the equality case. QED

The Bishop-Gromov volume comparison theorem has been generalized to the relative volume comparison theorem. Let \((M^n, g)\) be a complete Riemannian manifold and \(p \in M^n\). Given a measurable subset \(\Gamma\) of the unit sphere \(S^{n-1}_p \subset T_p M\) and \(0 < r \leq R < \infty\), define the annular-type region:

\[
A^r_{r,R}(p) := \left\{ x \in M^n : r \leq d(x, p) \leq R \& \text{there exists a unit speed minimal geodesic } \gamma \text{ from } \gamma(0) = p \text{ to } x \text{ satisfying } \gamma'(0) \in \Gamma \right\} 
\subset B(p, R) \setminus B(p, r).
\]

Note that if \(\Gamma = S^{n-1}_p\), then \(A^r_{r,R}(p) = B(p, R) \setminus B(p, r)\).

**Theorem 6.3.** Suppose that \((M^n, g)\) is a complete Riemannian manifold with \(Rc(g) \geq (n-1)K\). If \(0 \leq r \leq R \leq S, r \leq s \leq S\) and if \(\Gamma \subset S^{n-1}_p\) is a measurable subset, then

\[
\frac{Vol \left( A^r_{r,S}(p) \right)}{Vol_K \left( A^r_{s,S}(p_K) \right)} \leq \frac{Vol \left( A^r_{r,R}(p) \right)}{Vol_K \left( A^r_{r,R}(p_K) \right)}
\]

Taking \(r = s = 0\) and \(\Gamma = S^{n-1}_p\) yields Theorem 6.2. In particular, taking the limit as \(R \to 0\) gives (6.4).
As a consequence, we have the following result of Yau about the volume growth of a complete noncompact manifold with nonnegative Ricci curvature.

**Corollary 6.4** \((Rc \geq 0\) has at least linear volume growth).  Let \((M^n, g)\) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. For any point \(p \in M^n\), there exists a constant \(C > 0\) such that for any \(r \geq 1\)

\[
\text{Vol} (B(p, r)) \geq C r.
\]

**Proof.** Let \(x \in M^n\) be a point with \(d(x, p) = r \geq 2\). By the Bishop-Gromov relative volume comparison theorem, we have

\[
\frac{\text{Vol} (B(x, r + 1)) - \text{Vol} (B(x, r - 1))}{\text{Vol} (B(x, r - 1))} \leq \frac{(r + 1)^n - (r - 1)^n}{(r - 1)^n} \leq \frac{C(n)}{r}.
\]

(6.5)

Since \(B(p, 1) \subset B(x, r + 1) \setminus B(x, r - 1)\) and \(B(x, r - 1) \subset B(p, 2r - 1)\) by (6.5) we have

\[
\text{Vol} (B(p, 2r - 1)) \geq \text{Vol} (B(x, r - 1)) \geq \frac{\text{Vol} (B(p, 1))}{C(n)} r.
\]

We have proved the corollary for \(r \geq 3\). Clearly it is then true for \(r \geq 1\) (or any other positive constant). QED

### 3 Hessian comparison theorem.

The following roughly says that the larger the curvature, the smaller the Hessian of the distance function.

**Proposition 6.1** (Hessian comparison theorem-General version).  Let \(i = 1, 2\). Let \((M^n_i, g_i)\) be complete Riemannian \(n\)-manifolds, let \(\gamma_i : [0, L] \to M^n_i\) be geodesics parametrized by arc length such that \(\dot{\gamma}_i\) does not intersect the cut locus of \(\gamma_i(0)\), and let \(d_i := d(\cdot, \gamma_i(0))\). If for all \(t \in [0, L]\) we have

\[
K_{g_1} \left( V_1 \land \dot{\gamma}_1(t) \right) \geq K_{g_2} \left( V_2 \land \dot{\gamma}_2(t) \right)
\]

for all unit vectors \(V_i \in T_{\gamma_i(t)}M^n_i\) perpendicular to \(\dot{\gamma}_i(t)\), then

\[
\nabla^2 d_1 (X_1, X_1) \leq \nabla^2 d_2 (X_2, X_2)
\]
for all $X_i \in T_{\gamma_i(t)}M^n_i$ perpendicular to $\gamma_i(t)$ and $t \in (0, L]$.

Following theorem is the special case of the above result, namely comparing to constant curvature spaces.

**Theorem 6.4** (Hessian comparison theorem – special case). Let $(M^n, g)$ be a complete Riemannian manifold with $\text{Sect} \geq K$. For any point $p \in M$ the distance function $r(x) := d(x, p)$ satisfies

$$\nabla_i \nabla_j r = h_{ij} \geq \frac{1}{n - 1} H_K(r) g_{ij}$$

at all points where $r$ is smooth (i.e. away from $p$ and the cut locus). On all of $M$ the above inequality holds in the sense of support functions.

## 4 Mean value inequalities.

The following mean value inequality, which follows from the Laplacian comparison theorem, has an application in the proof of the splitting theorem.

**Proposition 6.1** (Mean value inequality for $\text{Ric} \geq 0$). If $(M^n, g)$ is a complete Riemannian manifold with $\text{Ric} \geq 0$ and if $f \leq 0$ is a Lipschitz function with $\Delta f \geq 0$ in the sense of distributions (subharmonic), then for any $x \in M^n$ and $0 < r < \text{inj}(x)$

$$f(x) \leq \frac{1}{\omega_n r^n} \int_{B(x, r)} f d\mu,$$

where $\omega_n$ is the volume of the unit Euclidean $n$-ball.

**Proof.** By the divergence theorem, we have

$$0 \leq \frac{1}{r^{n-1}} \int_{B(x, r)} \Delta f d\mu = \int_{\partial B(x, r)} \frac{\partial f}{\partial r} \frac{\sqrt{\det(g)}}{r^{n-1}} d\Theta,$$

where $d\Theta := d\theta^1 \wedge \ldots \wedge d\theta^{n-1}$. Since $\frac{\partial}{\partial r} \frac{\sqrt{\det(g)}}{r^{n-1}} \leq 0$ from $H = \frac{\partial}{\partial r} \log J \leq \frac{n-1}{r}$ and $f \leq 0$, we have

$$0 \leq \int_{\partial B(x, r)} \left( \frac{\partial f}{\partial r} \frac{\sqrt{\det(g)}}{r^{n-1}} + f \frac{\partial}{\partial r} \frac{\sqrt{\det(g)}}{r^{n-1}} \right) d\Theta$$

$$= \frac{d}{dr} \left( \frac{1}{r^{n-1}} \int_{\partial B(x, r)} f d\sigma \right)$$

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where we used \(d\sigma = \sqrt{\det (g)}d\Theta\). Since \(\lim_{r \to 0} \frac{1}{r^{n-1}} \int_{\partial B(x,r)} f d\sigma = n\omega_n f(x)\), integrating the above inequality over \([0, s]\) yields

\[
s^{n-1} f(x) \leq \frac{1}{n\omega_n} \int_{\partial B(x,s)} f d\sigma.
\]

Integrating this again, now over \([0, r]\) implies

\[
f(x) \leq \frac{1}{\omega_n r^n} \int_{B(x,r)} f d\mu.
\]

QED

In the case where the sectional curvature is bounded from above, we have

**Proposition 6.2** (Mean value inequality for \(\text{Sect} \leq H\)). Suppose that \((M^n, g)\) is a complete Riemannian manifold with \(\text{Sect} (g) \leq H\) in a ball \(B(x, r)\) where \(r < \text{inj}(g)\). If \(f \in C^\infty (M^n)\) is subharmonic, i.e., if \(\Delta f \geq 0\), and if \(f \geq 0\) on \(M^n\), then

\[
f(x) \leq \frac{1}{V_H(r)} \int_{B(x,r)} f d\mu,
\]

where \(V_H(r)\) is the volume of a ball of radius \(r\) in the complete simply connected manifold of constant sectional curvature \(H\).