1.5 Markov Property of Brownian motion

1.5.1 Markov Property

Definition

Let $X(t), t \geq 0$ be a stochastic process on filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. The process is called a Markov process if for any $t$ and $s > 0$, the conditional distribution of $X(t+s)$ given $\mathcal{F}_t$ is the same as the conditional distribution of $X(t+s)$ given $X(t)$.
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1.5.1 Markov Property

**Definition**

that is,

\[
P(X(t + s) \leq y | \mathcal{F}_t) = P(X(t + s) \leq y | X(t));
\]

or equivalently, if for any \( t \) and \( s > 0 \) and every nonnegative Borel-measurable function \( f \), there is another Borel-measurable function \( g \) such that

\[
E[f(X(t + s)) | \mathcal{F}_t] = g(X(t)).
\]
Theorem

Brownian motion $B(t)$ process Markov property.
Theorem

*Brownian motion $B(t)$ process Markov property.*

**Proof.** It is easy to see by using the moment generating function that the conditional distribution of $B(t + s)$ given $\mathcal{F}_t$ is the same as that given $B(t)$. 
Indeed,

\[ \mathbb{E} \left[ e^{u B(t+s)} \mid \mathcal{F}_t \right] \]
\[ = e^{u B(t)} \mathbb{E} \left[ e^{u \{B(t+s) - B(t)\}} \mid \mathcal{F}_t \right] \]
\[ = e^{u B(t)} \mathbb{E} \left[ e^{u \{B(t+s) - B(t)\}} \right] \]

since \( e^{u \{B(t+s) - B(t)\}} \) is independent of \( \mathcal{F}_t \)

\[ = e^{u B(t)} \mathbb{E} \left[ e^{u \{B(t+s) - B(t)\}} \mid B(t) \right] \]

since \( e^{u \{B(t+s) - B(t)\}} \) is independent of \( B(t) \)

\[ = \mathbb{E} \left[ e^{u B(t+s)} \mid B(t) \right] . \]
Remark: With the conditional characteristic function taking the place of conditional moment generating function, one can show that every stochastic process with independent increments is a Markov process.
Transition probability function of a Markov process $X$ is defined as

\[ P(y, t; x, s) = P(X(t) \leq y | X(s) = x) \]

the conditional distribution function of the process at time $t$, given that it is at point $x$ at time $s$. 
In the case of Brownian motion the transition probability function is given by the distribution of the normal \( N(x, t - s) \) distribution

\[
P(y, t; x, s) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi(t - s)}} \exp \left\{ -\frac{(u - x)^2}{2(t - s)} \right\} \, du
\]

\[
= \int_{-\infty}^{y} p_{t-s}(x, u) \, du.
\]
It is easy seen that, for Brownian motion $B(t)$,

$$P(y, t; x, s) = P(y, t - s; x, 0).$$

In other words,

$$P(B(t) \leq y | B(s) = x) = P(B(t - s) \leq y | B(0) = x).$$

The above property states that Brownian motion is time-homogeneous, that is, its distributions do not change with a shift in time.
Now, we calculate $E[f(B(t))|\mathcal{F}_s]$.

$$E[f(B(t))|B(s) = x] = \int f(y) P(dy, t; x, s) = \int f(y)p_{t-s}(x, y)dy.$$  

So,

$$E[f(B(t))|\mathcal{F}_s] = E[f(B(t))|B(s)] = \int f(y)p_{t-s}(B(s), y)dy.$$  

1.5.2 Stopping time

Definition

A random time $T$ is called a stopping for $B(t)$, $t \geq 0$, if for any $t$ it is possible to decide whether $T$ has occurred or not by observing $B(s), 0 \leq s \leq t$. More rigorously, for any $t$ the sets $\{T \leq t\} \in \mathcal{F}_t = \sigma(B(s), 0 \leq s \leq t)$, the $\sigma$-field generated by $B(s), 0 \leq s \leq t$.

If $T$ is a stopping time, events observed before or at time $T$ are described by $\sigma$-field $\mathcal{F}_T$, defined as the collection of sets

$$\{A \in \mathcal{F} : \text{ for any } t, A \cap \{T \leq t\} \in \mathcal{F}_t\}.$$
Example

1. Any nonrandom time $T$ is a stopping time. Formally, \( \{T \leq t\} \) is either the $\emptyset$ or $\Omega$, which are members of $\mathcal{F}_t$ for any $t$. 
Example

The first passage time of level $a$,

$$T_a = \inf\{t > 0 : B(t) = a\}$$

is a stopping time. Clearly, if we know $B(s)$ for all $s \leq t$ then we know whether the Brownian motion took the value $a$ before or at $t$ or not. Thus we know that $\{T_a \leq t\}$ has occurred or not just by observing the past of the process prior to $t$. Formally,

$$\{T_a \leq t\} = \{\max_{u \leq t} B(u) \geq a\} \in \mathcal{F}_t.$$
Example

Let $T$ be the time when Brownian motion reaches its maximum on the interval $[0, 1]$. Then clearly, to decide whether $\{T \leq t\}$ has occurred or not, it is not enough to know the values of the process prior to time $t$, one needs to know all the values on the interval $[0, 1]$. So that $T$ is not a stopping time.
1.5.3 Strong Markov property

Strong Markov property is similar to the Markov property, except that in the definition a fixed time $t$ is replaced by a stopping time.

**Theorem**

Brownian motion $B(t)$ has the strong Markov property: for a finite stopping time $T$ the regular conditional distribution of $B(T + t), t \geq 0$ given $\mathcal{F}_T$ is $P_{B(T)}$, that is,

$$P(B(T + t) \leq y | \mathcal{F}_T) = P(B(T + t) \leq y | B(T)).$$
Corollary

Let $T$ be a finite stopping time. Define the new process in $t \geq 0$ by

$$\hat{B}(t) = B(T + t) - B(T).$$

Then $\hat{B}(t)$ is a Brownian motion started at zero and independent of $\mathcal{F}_T$. 
1.6 Functions of Brownian motion

1.6.1 The first passage time

Let $x$ be a real number, the first passage time of Brownian motion $B(t)$ is $T_x = \inf \{t > 0 : B(t) = x \}$ where $\inf \emptyset = \infty$. 
1.6 Functions of Brownian motion

1.6.1 The first passage time

Let \( x \) be a real number, the first passage time of Brownian motion \( B(t) \) is

\[
T_x = \inf\{t > 0 : B(t) = x\}
\]

where \( \inf \emptyset = \infty \).
Theorem

\[ P_a(T_b < \infty) = 1 \text{ for all } a \text{ and } b. \]
To prove this theorem, we need a property of martingale, the proof is omitted.

Lemma
(i) (Stopped martingale) If $M(t)$ is a martingale with filtration $\{\mathcal{F}_t\}$ and $\tau$ is a stopping time, then the stopped process $M(t \wedge \tau)$ is a martingale, in particular for any $t$, $EM(t \wedge \tau) = EM(0)$.
(ii) (Optional stopping theorem) If $M(t)$ is a right-continuous martingale with filtration $\{\mathcal{F}_t\}$ and $\tau, \sigma$ are bounded stopping times such that $P(\tau \leq \sigma) = 1$. Then

$$E[M(\sigma)|\mathcal{F}_\tau] = M(\tau).$$
Proof of the Theorem. Notice

\[ P_a(T_b < \infty) \]

\[ = P( \inf \{ t > 0 : B(t) = b \} < \infty | B(0) = a \} \]

\[ = P( \inf \{ t > 0 : B(t) - B(0) = b - a \} < \infty | B(0) = a \} \]

\[ = P_0( T_{b-a} < \infty ) \]
and

\[ P_0(T_{b-a} < \infty) \]

\[ = \mathbb{P}(\inf\{t > 0 : B(t) = b - a\} < \infty | B(0) = 0) \]

\[ = \mathbb{P}(\inf\{t > 0 : -B(t) = a - b\} < \infty | -B(0) = 0) \]

\[ = P_0(T_{a-b} < \infty) \]

by the symmetry of Brownian motion. So, without loss of generality we assume \( a = 0, b \geq 0 \) and that the Brownian motion starts at 0.
For $u > 0$, let

$$Z(t) = \exp \left\{ uB(t) - \frac{u^2}{2} t \right\}.$$  

Then $Z(t)$ is a martingale. By the theorem for stopped martingale, $\mathbb{E}Z(0) = \mathbb{E}Z(t \wedge T_b)$, that is

$$1 = \mathbb{E} \exp \left\{ uB(t \wedge T_b) - \frac{u^2}{2} (t \wedge T_b) \right\}.$$  \hspace{1cm} (1.6.1)

We will take $t \to \infty$ in (1.6.1).
First, notice that the Brownian motion is always at or below level $b$ for $t \leq T_b$ and so

$$0 \leq \exp \left\{ uB(t \wedge T_b) - \frac{u^2}{2}(t \wedge T_b) \right\} \leq \exp \{ uB(t \wedge T_b) \} \leq e^{ub},$$

that is, the random variables in (1.6.1) is bounded by $e^{ub}$. 
Next, on the event \( \{T_b = \infty\} \),

\[
\exp \left\{ uB(t \wedge T_b) - \frac{u^2}{2} (t \wedge T_b) \right\} \\
\leq e^{ub} \exp \left\{ -\frac{u^2}{2} t \right\} \to 0, \quad t \to \infty,
\]

and on the event \( \{T_b < \infty\} \),

\[
\exp \left\{ uB(t \wedge T_b) - \frac{u^2}{2} (t \wedge T_b) \right\} \\
= \exp \left\{ ub - \frac{u^2}{2} (t \wedge T_b) \right\} \\
\to \exp \left\{ ub - \frac{u^2}{2} T_b \right\}, \quad t \to \infty.
\]
Taking the limits in (1.6.1) yields

\[
E \left[ \mathbb{I}\{T_b < \infty\} \exp \left\{ u b - \frac{u^2}{2} T_b \right\} \right] = 1
\] (1.6.2)

by the Dominated Convergence Theorem.
Taking the limits in (1.6.1) yields

\[ E \left[ \mathbb{I}\{T_b < \infty\} \exp \left\{ ub - \frac{u^2}{2} T_b \right\} \right] = 1 \]  

(1.6.2)

by the Dominated Convergence Theorem.

Again, random variables in (1.6.2) are bounded by 1. Taking \( \mu \to 0 \) yields

\[ P(T_b < \infty) = E \left[ \mathbb{I}\{T_b < \infty\} \right] = 1. \]

The proof is now completed.
And also from (1.6.2), it also holds that

$$E \left[ \exp \left\{ -\frac{u^2}{2} T_b \right\} \right] = e^{-ub}.$$

Replacing $u^2/2$ by $\alpha$ yields

$$E e^{-\alpha T_b} = e^{-b\sqrt{2\alpha}}.$$ 

This is the Laplace transform of $T_b$. 
Theorem

For real number $b$, let the first passage time of Brownian motion $B(t)$ be $T_b$. Then the Laplace transform of the distribution of $T_b$ is given by

$$E e^{-\alpha T_b} = e^{-|b|\sqrt{2\alpha}} \text{ for all } \alpha > 0,$$

(1.6.3)

and the density of $T_b$ is

$$f_{T_b}(t) = \frac{|b|}{\sqrt{2\pi}} t^{-3/2} \exp\left\{-\frac{b^2}{2t}\right\}, \quad t \geq 0$$

(1.6.4)

for $b \neq 0$. 
Proof. For non-negative level $b$, (1.6.3) is proved. If $b$ is negative, then because Brownian motion is symmetric, the first passage time $T_b$ and $T_{|b|}$ have the same distribution.
Proof. For non-negative level $b$, (1.6.3) is proved. If $b$ is negative, then because Brownian motion is symmetric, the first passage time $T_b$ and $T_{|b|}$ have the same distribution. Equation (1.6.3) for negative $b$ follows. (1.6.4) follows because

$$\int_0^{\infty} e^{-\alpha t} f_{T_b}(t) dt = e^{-|b|\sqrt{2\alpha}} \text{ for all } \alpha > 0.$$ 

Checking this equality is another hard work. We omit it.
Differentiation of (1.6.3) with respect to $\alpha$ results in

$$E \left[ T_b e^{-\alpha T_b} \right] = \frac{|b|}{\sqrt{2\alpha}} e^{-|b|\sqrt{2\alpha}} \text{ for all } \alpha > 0.$$ 

Letting $\alpha \downarrow 0$, we obtain $E T_b = \infty$ so long as $b \neq 0$. 

Corollary $E T_b = \infty$ for any $b \neq 0$. 
Differentiation of (1.6.3) with respect to $\alpha$ results in

$$E \left[ T_b e^{-\alpha T_b} \right] = \frac{|b|}{\sqrt{2\alpha}} e^{-|b|\sqrt{2\alpha}} \quad \text{for all } \alpha > 0.$$

Letting $\alpha \downarrow 0$, we obtain $ET_b = \infty$ so long as $b \neq 0$.

**Corollary**

$ET_b = \infty$ for any $b \neq 0$. 
The next result looks at the first passage time $T_x$ as a process in $x$.

**Theorem**

The process of the first passage times $\{T_x, x > 0\}$, has increments independent of the past, that is, for any $0 < a < b$, $T_b - T_a$ is independent of $B(t), t \leq T_a$, and the distribution of the increment $T_b - T_a$ is the same as that of $T_{b-a}$.
Proof. By the continuity, the Brownian motion must pass $a$ before it passes $b$. So, $T_a \leq T_b$. By the strong Markov property $\hat{B}(t) = B(T_a + t) - B(T_a)$ is Brownian motion started at zero, and independent of the past $B(t), t \leq T_a$.

$$T_b - T_a = \inf\{t \geq 0 : B(t) = b\} - T_a$$

$$= \inf\{t \geq T_a : B(t) = b\} - T_a$$

$$= \inf\{t \geq T_a : B(t - T_a + T_a) - B(T_a) = b - B(T_a)\} - T_a$$

$$= \inf\{t \geq 0 : \hat{B}(t) = b - a\}.$$

Hence $T_b - T_a$ is the same as first hitting time of $b - a$ by $\hat{B}$. The conclusion follows.
1.6.2 Maximum and Minimum

Let $B(t)$ be Brownian motion (which starts at zero). Define

$$M(t) = \max_{0 \leq s \leq t} B(s) \quad \text{and} \quad m(t) = \min_{0 \leq s \leq t} B(s).$$

**Theorem**

For any $x > 0$,

$$P(M(t) \geq x) = 2P(B(t) \geq x) = 2 \left(1 - \Phi\left(\frac{x}{\sqrt{t}}\right)\right),$$

where $\Phi(x)$ stands for the standard normal distribution function.
Proof. The second equation is obvious. For the first one, notice

\[ P(M(t) \geq x) = P(T_x \leq t) \]

\[ = \int_0^t \frac{x}{\sqrt{2\pi}} u^{-3/2} \exp \left\{ -\frac{x^2}{2u} \right\} \, du \]

(letting \( \frac{x^2}{u} = \frac{y^2}{t}, y > 0 \), then \( u^{-3/2} \, du = -2u^{-1/2} = -\frac{1}{xt^{1/2}} \, dy \))

\[ = \int_x^\infty \sqrt{\frac{2}{\pi t}} \exp \left\{ -\frac{y^2}{2t} \right\} \, dy. \]

The proof is now completed.
Another proof. Observe that the event \( \{M(t) \geq x\} \) and \( \{T_x \leq t\} \) are the same. Since

\[
\{B(t) \geq x\} \subset \{T_x \leq t\},
\]

\[
P(B(t) \geq x) = P(B(t) \geq x, T_x \leq t).
\]

As \( B(T_x) = x \),

\[
P(B(t) \geq x) = P(T_x \leq t, B(T_x + (t - T_x)) - B(T_x) \geq 0).
\]
Since $T_x$ is a finite stopping time. By the strong Markov property, $\hat{B}(t) = B(T_x + s) - B(T_x)$ is a Brownian motion which is independent of $\mathcal{F}_{T_x}$. So when $t \geq T_x$,

$$P(\hat{B}(t - T_x) \geq 0 | \mathcal{F}_{T_x}) = \frac{1}{2}.$$  

So,

$$P(B(t) \geq x) = E \left[ \mathbb{I}\{T_x \leq t\} P(\hat{B}(t - T_x) \geq 0 | \mathcal{F}_{T_x}) \right]$$

$$= E \left[ \mathbb{I}\{T_x \leq t\} \frac{1}{2} \right]$$

$$= \frac{1}{2} P(T_x \leq t) = \frac{1}{2} P(M(t) \geq x).$$
To find the distribution of the minimum of Brownian motion \( m(t) = \min_{0 \leq s \leq t} B(s) \) we use the symmetry argument, and that
\[
- \min_{0 \leq s \leq t} B(s) = \max_{0 \leq s \leq t} (-B(s)).
\]

Notice \(-B(t)\) is also a Brownian motion which has the property as \( B(t) \). It follows that for \( x < 0 \),
\[
\mathbb{P}(\min_{0 \leq s \leq t} B(s) \leq x) = \mathbb{P}(\max_{0 \leq s \leq t} (-B(s)) \geq -x)
\]
\[
= 2 \mathbb{P}(-B(t) \geq -x) = 2 \mathbb{P}(B(t) \leq x).
\]
Theorem

For any $x < 0$,

\[ P\left( \min_{0 \leq s \leq t} B(s) \leq x \right) = 2P(B(t) \leq x) = 2P(B(t) \geq -x). \]
1.6.3 Reflection principle and joint distribution

Theorem

(Reflection principle) Let $T$ be a stopping time. Define

$$\hat{B}(t) = B(t) \text{ for } t \leq T,$$

and

$$\hat{B}(t) = 2B(T) - B(t) \text{ for } t \geq T.$$  

Then $\hat{B}$ is also Brownian motion.
1.6.3 Reflection principle and joint distribution

Theorem

(Reflection principle) Let $T$ be a stopping time. Define

\[ \hat{B}(t) = B(t) \text{ for } t \leq T, \text{ and } \hat{B}(t) = 2B(T) - B(t) \text{ for } t \geq T. \]

Then $\hat{B}$ is also Brownian motion.

Note that $\hat{B}$ defined above coincides with $B(t)$ for $t \leq T$, and then for $t \geq T$ it is the reflected path about the horizontal line passing though $(T, B(T))$, that

\[ \hat{B}(t) - B(T) = -(B(t) - B(T)), \]

which gives the name to the result.
Corollary

For any \( y > 0 \), let \( \hat{B}(t) \) be \( B(t) \) reflected at \( T_y \), that is, \( \hat{B}(t) \) equals \( B(t) \) before \( B(t) \) hits \( y \), and is the reflection of \( B(t) \) after the first hitting time. Then \( \hat{B}(t) \) is also a Brownian motion.
Theorem

The joint distribution of \((B(t), M(t))\) has the density

\[
f_{B,M}(x, y) = \sqrt{\frac{2}{\pi}} \frac{2y - x}{t^{3/2}} \exp \left\{ - \frac{(2y - x)^2}{2t} \right\}, \quad \text{for } y \geq 0, x \leq y.
\]
Proof. Let $y > 0$ and $y > x$. Let $\hat{B}(t)$ be $B(t)$ reflected $T_y$.

Then

$$P(B(t) \leq x, M(t) \geq y) = P(B(t) \leq x, T_y \leq t)$$

$$= P(T_y \leq t, \hat{B}(t) \geq 2y - x) \quad (\text{on } \{T_y \leq t\}, \hat{B}(t) = 2y - B(t))$$

$$= P(T_y \leq t, B(t) \geq 2y - x)$$

(since $T_y$ is the same for $\hat{B}$ and $B$)

$$= P(B(t) \geq 2y - x)$$

(since $y - x > 0$, and $\{B(t) \geq 2y - x\} \subset \{T_y \leq t\}$)

$$= 1 - \Phi\left(\frac{2y - x}{\sqrt{t}}\right).$$
That is

\[ \int_{-\infty}^{x} \int_{y}^{\infty} f_{B,M}(u,v) \, du \, dv = 1 - \Phi\left(\frac{2y - x}{\sqrt{t}}\right). \]

The density is obtained by differentiation.
Corollary

The conditional distribution of $M(t)$ given $B(t) = x$ is

$$f_{M|B}(y|x) = \frac{2(2y - x)}{t} \exp \left\{ -\frac{2y(y - x)}{t} \right\}, \quad y > 0, x \leq y.$$
Proof. The density of $B(t)$ is

$$f_B(x) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\}.$$ 

So for $y > 0$ and $x \leq y$,

$$f_M|_B(y|x) = \frac{f_{B,M}(x,y)}{f_B(x)} = \frac{\sqrt{2} \frac{2y-x}{\pi t^{3/2}}} {\sqrt{2\pi t}} \exp\left\{-\frac{(2y-x)^2}{2t} + \frac{x^2}{2t}\right\} = \frac{2(2y-x)}{t} \exp\left\{-\frac{2y(y-x)}{t}\right\}.$$
1.6.4 Arcsine law

A time point $\tau$ is called a zero of Brownian motion if $B(\tau) = 0$. Let $\{B^x(t)\}$ denotes Brownian motion started at $x$.

**Theorem**

For any $x \neq 0$, the probability that $\{B^x(t)\}$ has at least one zero in the interval $(0, t)$, is given by

$$\frac{|x|}{\sqrt{2\pi}} \int_0^t u^{-3/2} \exp \left\{ -\frac{x^2}{2u} \right\} du = \sqrt{\frac{2}{\pi t}} \int_{|x|}^{\infty} \exp \left\{ -\frac{y^2}{2t} \right\} dy,$$

that is the same probability of $P_0(T_{|x|} \leq t)$. 

Proof. If \( x < 0 \), then due to continuity of \( B^x(t) \), the events
\[
\{ B^x \text{ has at least one zero between } 0 \text{ and } t \} \quad \text{and}
\{ \max_{0 \leq s \leq t} B^x(t) \geq 0 \}
\]
are the same. Since \( B^x(t) = B(t) + x \).
So
\[
P(B^x \text{ has at least a zero between } 0 \text{ and } t) = P(\max_{0 \leq s \leq t} B^x(t) \geq 0)
= P_0(\max_{0 \leq s \leq t} B(t) + x \geq 0) = P_0(\max_{0 \leq s \leq t} B(t) \geq -x)
= 2P_0(B(t) \geq -x) = P_0(T_{\{|x| \leq t\}}).
\]

If \( x > 0 \), then \(-B^x(t)\) is a Brownian motion started at \(-x\) by the symmetry of Brownian motion. The result follows.
Theorem

The probability that Brownian motion $B(t)$ has no zeros in the time interval $(a, b)$ is given by

$$\frac{2}{\pi} \arcsin \sqrt{\frac{a}{b}}.$$
Proof. Denote by

\[ h(x) = P(B \text{ has at least one zero in } (a, b) | B_a = x). \]

By the Markov property

\[ P(B \text{ has at least one zero in } (a, b) | B_a = x) \text{ is the same as } P(B^x \text{ has at least one zero in } (0, b - a)). \]

So

\[ h(x) = \frac{|x|}{\sqrt{2\pi}} \int_0^{b-a} u^{-3/2} \exp \left\{ -\frac{x^2}{2u} \right\} \, du. \]
By conditioning

$$P(B \text{ has at least one zero in } (a, b))$$

$$= P(B \text{ has at least one zero in } (a, b) | B_a = x) P(B_a \in dx)$$

$$= \int_{-\infty}^{\infty} h(x) P(B_a \in dx)$$

$$= \sqrt{\frac{2}{\pi a}} \int_0^{\infty} h(x) \exp \left\{ -\frac{x^2}{2a} \right\} dx.$$
By conditioning

\[ P(B \text{ has at least one zero in } (a, b)) = P(B \text{ has at least one zero in } (a, b) | B_a = x) P(B_a \in dx) \]

\[ = \int_{-\infty}^{\infty} h(x) P(B_a \in dx) \]

\[ = \sqrt{\frac{2}{\pi a}} \int_{0}^{\infty} h(x) \exp \left\{ -\frac{x^2}{2a} \right\} dx. \]

Putting in the expression for \( h(x) \) and performing the necessary calculations we obtain

\[ P(B \text{ has at least one zero in } (a, b)) = \frac{2}{\pi} \arccos \sqrt{\frac{a}{b}}. \]

The result follows.
Let

\[ \gamma_t = \sup\{s \leq t : B(s) = 0\} = \text{last zero before } t. \]

\[ \beta_t = \inf\{s \geq t : B(s) = 0\} = \text{first zero after } t. \]

Note that \( \beta_t \) is a stopping time but \( \gamma_t \) is not.
Let

\[ \gamma_t = \sup\{s \leq t : B(s) = 0\} = \text{last zero before } t. \]

\[ \beta_t = \inf\{s \geq t : B(s) = 0\} = \text{first zero after } t. \]

Note that \( \beta_t \) is a stopping time but \( \gamma_t \) is not.

**Corollary**

\[ P(\gamma_t \leq x) = \frac{2}{\pi} \arcsin \sqrt{\frac{x}{t}}, \quad x \leq t, \]

\[ P(\beta_t \geq y) = \frac{2}{\pi} \arcsin \sqrt{\frac{t}{y}}, \quad y \geq t, \]

\[ P(\gamma_t \leq x, \beta_t \geq y) = \frac{2}{\pi} \arcsin \sqrt{\frac{x}{y}}, \quad x \leq t \leq y. \]
Proof.

\[ \{ \gamma_t \leq x \} = \{ B \text{ has no zeros in } (x, t) \}, \]

\[ \{ \beta_t \geq y \} = \{ B \text{ has no zeros in } (t, y) \}, \]

\[ \{ \gamma_t \leq x, \beta_t \geq y \} = \{ B \text{ has no zeros in } (x, y) \}. \]