Matrix Analysis and Relative Problems

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Contents

1 Elementary notations .......................................................... 2
   1.1 Groups ........................................................................... 2
   1.2 Conjugacy classes .......................................................... 5
   1.3 Semidirect product groups ............................................... 6
   1.4 Action and $F[G]$-module ............................................... 8
   1.5 Representations ............................................................ 11
   1.6 Group algebras .............................................................. 15
   1.7 Schur Lemma .................................................................. 18
   1.8 Irreducible decomposition of group algebras ...................... 19
   1.9 Characters ...................................................................... 25
   1.10 Projection formulas from $C[G]$-module onto its submodule .. 37

2 Symmetry classes .................................................................... 40
   2.1 Antisymmetric subspace .................................................. 40
   2.2 Symmetric subspace ....................................................... 42

3 Schur-Weyl duality ................................................................ 54
   3.1 Some elementary theorems of matrix algebras .................... 54
   3.2 Schur-Weyl duality ......................................................... 62

4 Schur-Weyl duality of symmetric and unitary groups ............... 67
   4.1 Some elementary notations of the symmetric groups .......... 67
   4.2 The irreducible submodules of $C[S_k]$ ............................. 69
   4.3 Combinatorics of Young tableaux .................................... 76
   4.4 Applications of Schur-Weyl duality ................................. 77
1 Elementary notations

1.1 Groups

A group consists of a set $G$, together with a rule for combining any two elements $g, h$ of $G$ to form another element of $G$, written as $gh$. This rule must satisfy the following axioms:

(i) for all $g, h, k$ in $G$, 

\[(gh)k = g(hk);\]

(ii) there exists an element $e$ in $G$ such that for all $g$ in $G$, 

\[eg = ge = g;\]

(iii) for all $g$ in $G$, there exists an element $g^{-1}$ in $G$ such that 

\[gg^{-1} = g^{-1}g = e.\]

We refer to the rule for combining elements of $G$ as the product operation on $G$. A group $G$ is said to be abelian if $gh = hg$ for all $g$ and $h$ in $G$.

If the number of elements in $G$ is finite, then we call $G$ a finite group. The number of elements in $G$ is called the order of $G$, written as $|G|$.

Example 1.1. Let $\mathbb{F}$ be either $\mathbb{R}$, the set of real numbers, or $\mathbb{C}$, the set of complex numbers. The set of all invertible $n \times n$ matrices with entries in $\mathbb{F}$, under matrix multiplication, forms a group. This group is called the general linear group of degree $n$ over $\mathbb{F}$, and is denoted by $\text{GL}(n, \mathbb{F})$. It is an infinite group. The identity of $\text{GL}(n, \mathbb{F})$ is of course the identity matrix, which is denoted by $1_n$ or just $1$.

Example 1.2. For a positive integer $n$, the set of all permutations of $\{1, 2, \ldots, n\}$ under the product operation of composition is a group. It is called the symmetric group of degree $n$, and is written as $S_n$. The order of $S_n$ is $n!$. 
A subset $H$ of a group $G$ is said to be a subgroup if $H$ is itself a group under the product operation inherited from $G$.

It is easy to see that a subset $H$ of a group $G$ is a subgroup if and only if the following two conditions hold:

(i) $e \in H$,

(ii) $hk^{-1} \in H$ for any $h, k \in H$.

**Example 1.3.** A transposition in the symmetric group $S_n$ is a permutation which interchanges two of the numbers $1, 2, \ldots, n$ and fixes the other $n - 2$ numbers. Every permutation $\pi$ in $S_n$ can be expressed as a product of transpositions. It can be shown that either all such expressions for $\pi$ have an even number of transpositions, or they all have an odd number of transpositions. We call $\pi$ an even or an odd permutation, accordingly.

The subset 

$$A_n = \{\pi \in S_n : \pi \text{ is an even permutation}\}$$

is a subgroup of $S_n$, called the alternating group of degree $n$.

Let $\sigma \in S_n$. We define $\text{sign}(\sigma) = 1$ if $\sigma$ is an even permutation, and $\text{sign}(\sigma) = -1$ if $\sigma$ is an odd permutation.

There is an important identity:

$$\text{sign}(\sigma \pi) = \text{sign}(\sigma) \text{sign}(\pi)$$

for all $\sigma, \pi \in S_n$.

If $G$ and $H$ are groups, then a homomorphism from $G$ to $H$ is a functions $\varphi : G \to H$ which satisfies

$$\varphi(g_1 g_2) = \varphi(g_1) \varphi(g_2) \quad \text{for all } g_1, g_2 \in G.$$ 

An invertible (bijective) homomorphism is called an isomorphism.

Let $G$ be a group and $H$ be a subgroup of $G$. For $x$ in $G$, the subset 

$$Hx = \{hx : h \in H\}$$

of $G$ is called a right coset of $H$ in $G$. The distinct right cosets of $H$ in $G$ form a partition of $G$, which every element of $G$ is in precisely one of the cosets.

Suppose now that $G$ is finite. Let $Hx_1, \ldots, Hx_r$ be all the distinct right cosets of $H$ in $G$. For all $i$, the function

$$h \to hx_i \quad (h \in H)$$
is a bijection from $H$ to $Hx_i$. So $|Hx_i| = |H|$. Since

$$G = Hx_1 \cup \cdots \cup Hx_r,$$

and $Hx_i \cap Hx_j$ is empty if $i \neq j$, we deduce that $|G| = r|H|$. In particular, we have

**Theorem 1.4** (Lagrange’s theorem). *If $G$ is a finite group and $H$ is a subgroup of $G$, then $|H|$ divides $|G|$.*

A subgroup $N$ of a group $G$ is said to be a *normal subgroup* of $G$ if $g^{-1}Ng = N$, i.e., for all $g \in G$, $g^{-1}Ng \overset{\text{def}}{=} \{g^{-1}ng : n \in N\} = N$. It is written as $N \triangleleft G$. Let $G/N$ be the set of left or right cosets of $N$ in $G$.

A group $G$ is said to be *simple* if $G \neq \{e\}$ and the only normal subgroups of $G$ are $\{e\}$ and $G$.

Let $G$ and $H$ be groups and suppose that $\varphi : G \to H$ is a homomorphism. We define the *kernel* of $\varphi$ by

$$\ker(\varphi) = \{g \in G : \varphi(g) = e\}.$$  \hspace{1cm} (1.1)

Then $\ker(\varphi)$ is a normal subgroup of $G$. In addition, the *image* of $\varphi$ is defined as

$$\im(\varphi) = \{\varphi(g) : g \in G\},$$  \hspace{1cm} (1.2)

and $\im(\varphi)$ is a subgroup of $H$. The following result describes the way in which the kernel and the image of $\varphi$ are related.

**Theorem 1.5.** *Suppose that $G$ and $H$ are groups and let $\varphi : G \to H$ be a homomorphism. Then $G/\ker(\varphi) \cong \im(\varphi)$. An isomorphism is given by the function

$$Ng \to \varphi(g)(g \in G),$$

where $N = \ker(\varphi)$.\*

A linear transformation from a vector space $V$ over $F$ to itself is called an *endomorphism* of $V$. All endomorphisms of $V$ is denoted by $\text{End}_F(V)$ or $\text{End}(V)$ or $L(V)$.

**Theorem 1.6** (Rank-Nullity theorem). *Let $V$ and $W$ be vector spaces over $F$. Suppose that $\varphi : V \to W$ is a linear transformation. We have:

$$\dim(V) = \dim(\ker(\varphi)) + \dim(\im(\varphi)).$$  \hspace{1cm} (1.3)
1.2 Conjugacy classes

Definition 1.7. Let $G$ be a finite group and $x, y \in G$. We say that $x$ is conjugate to $y$ if
\[ y = g^{-1}xg \text{ for some } g \in G. \]
The set of all elements conjugate to $x$ in $G$ is
\[ x^G \stackrel{\text{def}}{=} \{ g^{-1}xg : g \in G \}, \]
which is called the conjugacy class of $x$ in $G$.

Proposition 1.8. The conjugacy relationship is an equivalence relationship, and every group is a union of conjugacy classes, and distinct conjugacy classes are disjoint.

Definition 1.9. Let $G = x_1^G \cup \cdots \cup x_l^G$. If the conjugacy classes $x_1^G, \ldots, x_l^G$ are distinct, then $x_1, \ldots, x_l$ are called representatives of the conjugacy classes of $G$.

Proposition 1.10. Let $x, y \in G$. If $x$ is conjugate to $y$ in $G$, then $x^n$ is conjugate to $y^n$ in $G$ for every integer $n$. In addition, $x$ and $y$ have the same order.

Definition 1.11. Let $x \in G$. The centralizer of $x$ in $G$, written as $C_G(x)$, is the set of elements of $G$ which commute with $x$, that is,
\[ C_G(x) = \{ g \in G : xg = gx \}. \]

Theorem 1.12. Let $x \in G$. Then the size of the conjugacy class $x^G$ is given by
\[ |x^G| = |G| / |C_G(x)|. \]
In particular, $|x^G|$ divides $|G|$.

Proof. In fact, $C_G(x)$ is a subgroup of $G$. For every $k \in G$, $kxk^{-1} \in x^G$ corresponding to the left coset $kC_G(x)$ of $C_G(x)$. It is easy to show that the correspondence is a bijection and so the conclusion is true.

Definition 1.13. The center of $G$, written as $Z(G)$, is defined by
\[ Z(G) \stackrel{\text{def}}{=} \{ z \in G : zg = gz \text{ for all } g \in G \}. \]
Clearly $Z(G)$ is a normal subgroup of $G$.

We make the observation that
\[ |x^G| = 1 \iff g^{-1}xg = x \text{ for all } g \in G \iff x \in Z(G). \]
**Proposition 1.14.** Let $H$ be a subgroup of $G$. Then $H \triangleleft G$ if and only if $H$ is a union of some conjugacy classes of $G$.

*Proof.* If $H$ is a union of conjugacy classes, then

$$h \in H, g \in G \implies g^{-1}hg \in H.$$ 

So $g^{-1}Hg \subseteq H$. Thus $H \triangleleft G$.

Conversely, if $H \triangleleft G$, then for all $h \in H, g \in G$, we have $g^{-1}hg \in H$, and so $h^G \subseteq H$.

Therefore

$$H = \bigcup_{h \in H} h^G,$$

and $H$ is a union of conjugacy classes of $G$. \qed

### 1.3 Semidirect product groups

Given groups $H$ and $K$, the *direct product* $H \times K$ is defined as follows:

The elements of $H \times K$ are ordered pairs $(h, k)$, where $h \in H$ and $k \in K$.

If we introduce the binary operation on $H \times K$ by

$$(h_1, k_1)(h_2, k_2) = (h_1h_2, k_1k_2).$$

It is easy to check that $H \times K$ is a group.

The subgroup $H \times \{e\}$ of $H \times K$ is isomorphism to $H$ through the mapping $(h, e) \to h$, where $e$ is the unit of $K$. Similarly, $\{e\}' \times K$ is isomorphism to $K$.

The following statements are satisfied:

(i). $H \times \{e\}$ and $\{e\}' \times K$ are normal subgroups of $H \times K$;

(ii). Every element of $H \times \{e\}$ commutes with every member of $\{e\}' \times K$;

(iii). Every element of $H \times K$ can be represented uniquely by the product of elements in $H \times \{e\}$ and $\{e\}' \times K$;

Thus, we introduce the following definition:

**Definition 1.15.** Given a group $G$, if there exist subgroups $H$ and $K$ such that

(i). every element of $H$ commutes with every member of $K$;

(ii). every element of $G$ can be represented uniquely by the product of elements in $H$ and $K$, then $G$ is said to be the *direct product group* of $H$ and $K$. 

6
It follows from (ii) that $H \cap K = \{e\}$. In fact, if $g \in H \cap K$, then $g = ge = eg$. Thus, $g = e$. Moreover, we can prove easily that $H$ and $K$ are normal subgroups of $G$.

Now, we introduce another important concept.

**Definition 1.16.** Given a group $G$, if there exist normal subgroup $H$ and subgroup $K$ such that $G = HK$ and $H \cap K = \{e\}$, then $G$ is said to be the semi-direct product group of $H$ and $K$, and denote it by $H \rtimes K$.

Let $H$ and $K$ be two groups, $Aut(H)$ be the group of all the automorphism of $H$. Then the homomorphism $\rho : K \to Aut(H)$ is called an action of $K$ on $H$.

Moreover, let $G$ be a group, $H$ be normal subgroup of $G$, $K$ be subgroups of $G$. For $k \in K$, the mapping $\rho_k : H \to H, x \to kxk^{-1}$ is said to be a conjugate action of $K$ on $H$.

Given two groups $H$ and $K$, let $\nu$ be an action of $K$ on $H$, for $k \in K$, denote $\nu(k) \triangleq \nu_k$. For $h, h' \in H$ and $k, k' \in K$, the binary operation is defined as

$$(h, k)(h', k') = (h\nu_k(h'), kk').$$

We denote $H \times K$ with the above binary operation by $H \rtimes_\nu K$.

Now, we prove $H \rtimes_\nu K$ is a group.

It is clear that $\nu_e = I$. For any $k, k' \in K, h \in H$, $\nu_{kk'}(h) = \nu(k'k)(h) = \nu_k(\nu_{k'}(h))$. Thus,

$$(h_1, k_1)(h_2, k_2)(h_3, k_3) = (h_1\nu_{k_1}(h_2), k_1k_2(h_3, k_3)) = (h_1\nu_{k_1}(h_2)\nu_{k_1}(h_3), k_1k_2k_3),$$

while

$$(h_1, k_1)(h_2, k_2)(h_3, k_3) = (h_1, k_1)(h_2\nu_{k_2}(h_3), k_2k_3) = (h_1\nu_{k_1}(h_2)\nu_{k_2}(h_3), k_1k_2k_3) = (h_1\nu_{k_1}(h_2)\nu_{k_1}(h_3), k_1k_2k_3).$$

Hence the product is associative. It is not difficult to see that the unit of $H \rtimes_\nu K$ is $(e', e)$ and the inverse of $(h, k)$ is $(\nu_{k^{-1}}(h^{-1}), k^{-1})$. Therefore, $H \rtimes_\nu K$ is a group.

If $\nu_k = I$ for all $k \in K$, then $H \rtimes_\nu K$ is the direct product group of $H$ and $K$.

Now, it is easy to show that if identity $(h, e)$ with $h$, $(e', k)$ with $k$, then every element $g$ of $H \rtimes_\nu K$ can be written as $g = (h, k) = (h, e)(e', k) = hk$. Moreover, $H \cap K = (e', e)$, and $K$ is a subgroup, $H$ is a normal subgroup of $H \rtimes_\nu K$.

In fact, suppose $(h, k) \in H \cap K$. Since $H$ identity with $H \times \{e\}$ and $K$ identity with $\{e'\} \times K$, $(h, k) \in H \times \{e\} \cap \{e'\} \times K$, so $h = e', k = e$. Thus $(h, k) = (e', e)$, that is, $H \cap K = (e', e)$.

Next, we show that $H$ is a normal subgroup of $H \rtimes_\nu K$. Indeed, if $k \in K, h \in H$, then $k(hk^{-1} = (e, k)(h, k^{-1}) = \nu_k(h)$. So for $g = hk \in H \rtimes_\nu K, h' \in H$, we have $gh'g^{-1} = (hk)h'(k^{-1}h^{-1}) = h(kh'k^{-1})h^{-1} \in H$. Thus, $H$ is a normal subgroup of $H \rtimes_\nu K$. 

7
Moreover, $H \rtimes_v K/H \cong K$, and the action $v$ of $K$ on $H$ is just the conjugate action of the elements of $K$, that is, for every $k \in K$ and $h \in H$, we have $v(k)h = khk^{-1}$.

In fact, consider the mapping $\rho : G = H \rtimes_v K \to K, \rho(hk) = k$. Note that $\rho((h_1k_1h_2k_2)(h_1k_1)^{-1}h_1k_1k_2) = \rho(h_1k_1h_2h_1k_1^{-1}h_1k_1k_2) = k_1k_2$, thus, $\rho$ is a surjective homomorphism and $\ker(\rho) = H$, and so $H \rtimes_v K/H \cong K$. Therefore, $K$ is a subgroup of $H \rtimes_v K$.

Now, for every $h \in H$ and $k \in K$,

$$khk^{-1} = (e', k)(h, e)k^{-1} = (v_k(h), k)k^{-1} = v_k(h)kk^{-1} = v_k(h) = v(k)h,$$

the action $v$ of $K$ on $H$ is just the conjugate action of the elements of $K$.

**Theorem 1.17.** Let $H$ and $K$ be two groups, above showed that if we identity $(H, e)$ with $H$, $(e', K)$ with $K$, then $H \rtimes_v K$ is the semidirect product group of $H$ and $K$.

### 1.4 Action and $\mathbb{F}[G]$-module

An action of a finite group $G$ on a finite-dimensional vector space $V$ over field $\mathbb{F}$ is a homomorphism $\pi : G \to \text{GL}(V)$ of $G$ to the general linear group of $V$, i.e., the invertible elements in $\text{End}(V)$.

The vector space $V$ will be referred to as an action space of $G$.

**Definition 1.18.** Let $V$ be a vector space over field $\mathbb{F}$ and $G$ a finite group. Then $V$ is said to be an $\mathbb{F}[G]$-module if a multiplication $gv$ of $g \in G, v \in V$ is defined, satisfying the following conditions for all $\lambda \in \mathbb{F}, g, h \in G$ and $u, v \in V$:

- (i) $gv \in V$;
- (ii) $(gh)v = g(hv)$;
- (iii) $ev = v$;
- (iv) $g(\lambda v) = \lambda(gv)$;
- (v) $g(u + v) = gu + gv$.

It is easily to show that every action of $G$ on the vector space $V$ over field $\mathbb{F}$ induces an $\mathbb{F}[G]$-module $V$, and every $\mathbb{F}[G]$-module $V$ induces an action of $G$ on vector space $V$ by the following form: for $v \in V$ and $g \in G$, let

$$\pi(g) : v \to gv.$$
Henceforth, we will identify an $\mathbb{F}[G]$-module with the action $\pi$ of $G$ on vector space $V$.

**Definition 1.19.** Let $V$ be an $\mathbb{F}[G]$-module and $V_0$ be a vector subspace of $V$. If for every $v \in V_0$ and $g \in G$, $gv \in V_0$, then $V_0$ is said to be a submodule of $V$.

**Definition 1.20.** An $\mathbb{F}[G]$-module $V$ is said to be irreducible if it has no non-trivial submodules, which means the only submodules are $\{0\}$ or $V$.

**Definition 1.21.** Let $V$ and $W$ be $\mathbb{F}[G]$-modules. A function $\phi : V \rightarrow W$ is said to be an $\mathbb{F}[G]$-homomorphism if $\phi$ is a linear transformation and

$$\phi(gv) = g\phi(v) \text{ for all } v \in V, g \in G.$$ 

In other words, if $\phi$ sends $v$ to $w$ then it sends $gv$ to $gw$. Similarly, we can define the $\mathbb{F}[G]$-isomorphism

**Proposition 1.22.** Let $V$ and $W$ be $\mathbb{F}[G]$-modules and $\phi : V \rightarrow W$ be an $\mathbb{F}[G]$-homomorphism. Then $\ker(\phi)$ is an $\mathbb{F}[G]$-submodule of $V$, and $\im(\phi)$ is an $\mathbb{F}[G]$-submodule of $W$.

**Proof.** Note that $\ker(\phi)$ is a subspace of $V$ and $\im(\phi)$ is a subspace of $W$ since $\phi$ is a linear transformation.

Let $v \in \ker(\phi)$ and $g \in G$, then

$$\phi(gv) = g\phi(v) = g(0) = 0.$$ 

So $gv \in \ker(\phi)$. Therefore $\ker(\phi)$ is an $\mathbb{F}[G]$-submodule of $V$.

Now let $w \in \im(\phi)$, then we have $w = \phi(v)$ for some $v \in V$. For all $g \in G$,

$$gw = g\phi(v) = \phi(gv) \in \im(\phi).$$

Thus, $\im(\phi)$ is an $\mathbb{F}[G]$-submodule of $W$. \qed

**Theorem 1.23** (Maschke’s theorem). Let $G$ be a finite group, $V$ be an $\mathbb{F}[G]$-module. If $U$ is an $\mathbb{F}[G]$-submodule of $V$, then there is an $\mathbb{F}[G]$-submodule $W$ of $V$ such that

$$V = U \bigoplus W.$$ 

**Proof.** Choose any subspace $W_0$ of $V$ such that

$$V = U \bigoplus W_0.$$ 

9
In fact, there are many choices for $W_0$— simply take a basis $v_1, \ldots, v_m$ of $U$, extend it to a basis $v_1, \ldots, v_n$ of $V$. Then let $W_0 = \text{span}\{v_{m+1}, \ldots, v_n\}$.

For $v \in V$, we have $v = u + w$ for unique vectors $u \in U$ and $w \in W_0$. We define $\phi : V \to V$ by setting $\phi(v) = u$. Thus $\phi$ is a projection of $V$ with kernel $W_0$ and image $U$.

We will modify the projection $\phi$ to create an $F[G]$-homomorphism from $V$ to $V$ with image $U$. To this end, define $\hat{\phi} : V \to V$ as

$$\hat{\phi}(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \phi(gv), \quad v \in V. \quad (1.4)$$

It is clear that $\hat{\phi}$ is an endomorphism of $V$ and $\text{im}(\hat{\phi}) \subseteq U$.

At first, we are going to show that $\hat{\phi}$ is an $F[G]$-homomorphism. For $v \in V$ and $x \in G$,

$$\hat{\phi}(xv) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \phi(g(xv)).$$

As $g$ runs over the elements of $G$, so does $h = gx$. Hence

$$\hat{\phi}(xv) = \frac{1}{|G|} \sum_{h \in G} xh^{-1} \phi(hv) = x \left( \frac{1}{|G|} \sum_{h \in G} h^{-1} \phi(hv) \right) = x\hat{\phi}(v).$$

Thus $\hat{\phi}$ is an $F[G]$-homomorphism.

Next, we prove that $\hat{\phi}^2 = \hat{\phi}$. First note that for $u \in U, g \in G$, we have $gu \in U$ and $\phi(gu) = gu$. So we get

$$\hat{\phi}(u) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \phi(gu) = \frac{1}{|G|} \sum_{g \in G} g^{-1} gu = u. \quad (1.5)$$

If $v \in V$, then $\hat{\phi}(v) \in U$. By the above equation, we have $\hat{\phi}(\hat{\phi}(v)) = \hat{\phi}(v)$. Consequently $\hat{\phi}^2 = \hat{\phi}$, as claimed.

We have now established that $\hat{\phi} : V \to V$ is a projection and an $F[G]$-homomorphism. Moreover, $\text{im}(\hat{\phi}) = U$. Let $W = \ker(\hat{\phi})$. Then $W$ is an $F[G]$-submodule of $V$, and $V = U \oplus W$.

This completes the proof of Maschke’s Theorem. \hfill \Box

From the proof of above theorem and the induction methods, we have

**Corollary 1.24.** Let $V$ and $W$ be $F[G]$-modules and $\phi : V \to W$ an $F[G]$-homomorphism. Then there is an $F[G]$-submodule $U$ of $V$ such that

$$V = \ker(\phi) \bigoplus U \text{ and } U \cong \text{im}(\phi).$$
Corollary 1.25. Let \( W \) be an \( \mathbb{F}[G] \)-module. Then \( W \) is a direct sum of irreducible \( \mathbb{F}[G] \)-submodules \( V_i \), i.e.

\[
W = \bigoplus_i V_i.
\]

Now, we are going to define the tensor product of \( \mathbb{F}[G] \)-modules.

Definition 1.26. Let \( G \) be a finite group, \( V \) and \( W \) be \( \mathbb{F}[G] \)-modules. For all \( v \in V \), \( w \in W \) and \( g \in G \), define

\[
g(v \otimes w) = gv \otimes gw.
\]

Then \( V \otimes W \) is an \( \mathbb{F}[G] \)-module, which is called the tensor product module of \( V \) and \( W \).

Theorem 1.27. Let \( G \) be a finite group, \( V \) be an \( \mathbb{F}[G] \)-module. For every \( g \in G \) and \( v \in \mathbb{F}^k \), define \( gv = v \). Then \( \mathbb{F}^k \) is an \( \mathbb{F}[G] \)-module. There exists an \( \mathbb{F}[G] \)-module isomorphism \( U : V^\oplus k \to V \otimes \mathbb{F}^k \). Moreover, if \( A \in \text{End}(V) \), then \( U^{-1}(A \otimes I_{\mathbb{F}^k})U = A^\oplus k \).

Proof. In fact, take a linear basis \( \{e_j\}_{j=1}^k \) of \( \mathbb{F}^k \), and define

\[
U(\bigoplus_{i=1}^k x_i) = \sum_{i=1}^k x_i \otimes e_i,
\]

then \( U \) is an \( \mathbb{F}[G] \)-module isomorphism of \( V^\oplus k \) and \( V \otimes \mathbb{F}^k \), satisfying \( U^{-1}(A \otimes I_{\mathbb{F}^k})U = A^\oplus k \). \qed

Henceforth, we will identity \( V^\oplus k \) with \( V \otimes \mathbb{F}^k \), \( A^\oplus k \) with \( A \otimes I_{\mathbb{F}^k} \).

1.5 Representations

Definition 1.28. A representation of a finite group \( G \) over \( \mathbb{F} \) is a homomorphism \( \phi \) from \( G \) to \( \text{GL}(n, \mathbb{F}) \), for some \( n \).

We refer to \( n \) as its order or dimension, denoted by \( \text{dim}(\phi) \).

Definition 1.29. Let \( \phi : G \to \text{GL}(n, \mathbb{F}) \) and \( \phi' : G \to \text{GL}(n, \mathbb{F}) \) be representations of \( G \) over \( \mathbb{F} \). We say that \( \phi \) is equivalent to \( \phi' \) if there exists an invertible \( n \times n \) matrix \( T \) such that for all \( g \in G \),

\[
\phi' = T^{-1}\phi(g)T.
\]

Definition 1.30. Let \( V \) be an \( \mathbb{F}[G] \)-module, \( \mathcal{B} \) be a basis of \( V \). For every \( g \in G \), let \( [g]_\mathcal{B} \) denote the matrix of the endomorphism \( v \mapsto gv \) of \( V \), relative to the basis \( \mathcal{B} \).
By the Jordan theory of matrices, we have

**Proposition 1.31.** Let $G$ be a finite group and $V$ be a $\mathbb{C}[G]$-module. If $g \in G$, then there is a basis $\mathcal{B}$ of $V$ such that the matrix $[g]_{\mathcal{B}}$ is diagonal. If $g$ has order $n$, then the entries on the diagonal of $[g]_{\mathcal{B}}$ are $n$th roots of unity.

The connection between $\mathbb{F}[G]$-modules and representations of $G$ over $\mathbb{F}$ is revealed in the following basic result.

**Theorem 1.32.** (i) If $\phi : G \to \text{GL}(n, \mathbb{F})$ is a representation of $G$ over $\mathbb{F}$, and $V = \mathbb{F}^n$, then $V$ becomes an $\mathbb{F}[G]$-module by defining the multiplication $gv$ as

$$gv \overset{\text{def}}{=} \phi(g)v \quad g \in G, v \in V.$$  

Moreover, there is a basis $\mathcal{B}$ of $V$ such that

$$\phi(g) = [g]_{\mathcal{B}} \quad \text{for all } g \in G.$$  

(ii) Assume that $V$ is an $\mathbb{F}[G]$-module and let $\mathcal{B}$ be a basis of $V$. Then the function

$$g \mapsto [g]_{\mathcal{B}} \quad g \in G$$  

is a representation of $G$ over $\mathbb{F}$.

Every $\mathbb{F}[G]$-module gives us many representations

$$g \mapsto [g]_{\mathcal{B}} \quad g \in G$$  

for all basis $\mathcal{B}$ of $V$.

The next result shows that all these representations are equivalent to each other.

**Theorem 1.33.** Suppose that $V$ is an $\mathbb{F}[G]$-module with basis $\mathcal{B}$, and let $\phi$ be the representation of $G$ over $\mathbb{F}$ defined by

$$\phi : g \mapsto [g]_{\mathcal{B}} \quad g \in G.$$  

(i) If $\mathcal{B}'$ is another a basis of $V$, then the representation

$$\phi' : g \mapsto [g]_{\mathcal{B}'} \quad g \in G.$$  

of $G$ is equivalent to $\phi$. 

12
(ii) If $\phi''$ is a representation of $G$ which is equivalent to $\phi$, then there is a basis $B''$ of $V$ such that

$$\phi'' : g \mapsto [g]_{B''} \quad g \in G.$$  

**Definition 1.34.** A representation $\phi : G \to \text{GL}(n, \mathbb{F})$ is **irreducible** if the corresponding $\mathbb{F}[G]$-module $\mathbb{F}^n$ given by

$$gv = \phi(g)v \quad v \in \mathbb{F}^n, g \in G$$

is irreducible.

It follows from the modular decomposition theorem that

**Proposition 1.35.** Let $G$ be a finite group and $X$ a representation of $G$ of order $n$. Then there is a fixed matrix $T$ such that every matrix $X(g), g \in G$, has the form

$$TX(g)T^{-1} = \begin{bmatrix} X_1(g) & 0 & \cdots & 0 \\ 0 & X_2(g) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_k(g) \end{bmatrix},$$

where every $X_i$ is an irreducible representation of $G$.

The following theorem is very important in quantum theory.

**Theorem 1.36.** Let $V$ be a representation of order $n$ of $G$. Then there is a unitary representation $X$ which is equivalent to $V$.

**Proof.** To the end, we define a Hermitian matrix as follows:

$$H \overset{\text{def}}{=} \sum_{g \in G} V(g)V(g)^\dagger.$$  \hspace{1cm} (1.6)

Clearly $H = H^\dagger$. By the spectral decomposition theorem, there is a unitary matrix $U$ such that $H = UDU^\dagger$, where

$$D = \begin{bmatrix} \lambda_1 \\ & \ddots \\ & & \lambda_n \end{bmatrix}.$$
is diagonal matrix with the eigenvalues \(\lambda_1, \ldots, \lambda_n\) of \(H\) as its diagonal entries. It is clear that \(H\) is positive definite with all eigenvalues \(\lambda_i > 0\). Note that

\[
D = U^\dagger H U = U^\dagger \left( \sum_{g \in G} V(g) V(g)^\dagger \right) U
\]

(1.7)

\[
= \sum_{g \in G} \left( U^\dagger V(g) U \right) \left( U^\dagger V(g)^\dagger U \right)
\]

(1.8)

\[
= \sum_{g \in G} \left( U^\dagger V(g) U \right) (U^\dagger V(g) U)^\dagger = \sum_{g \in G} W(g) W(g)^\dagger,
\]

(1.9)

where \(W(g) := U^\dagger V(g) U\).

Define \(T := U \sqrt{D}\) and \(X(g) := T^{-1} V(g) T\) for all \(g \in G\). Now we claim that \(X\) is a unitary representation which is equivalent to \(V\). Clearly \(X(g) = D^{-1/2} W(g) D^{1/2}\). Now, we how that \(X\) is a unitary representation. Since

\[
X(g) X(g)^\dagger = \left( D^{-1/2} W(g) D^{1/2} \right) \left( D^{-1/2} W(g) D^{1/2} \right)^\dagger
\]

\[
= D^{-1/2} W(g) D W(g)^\dagger D^{-1/2}
\]

\[
= D^{-1/2} W(g) \left( \sum_{g' \in G} W(g') W(g')^\dagger \right) W(g)^\dagger D^{-1/2}
\]

\[
= D^{-1/2} \left( \sum_{g' \in G} W(g) W(g') W(g')^\dagger W(g)^\dagger \right) D^{-1/2}
\]

\[
= D^{-1/2} \left( \sum_{g' \in G} W(gg') W(gg')^\dagger \right) D^{-1/2}
\]

\[
= D^{-1/2} \left( \sum_{g' \in G} W(g') W(g')^\dagger \right) D^{-1/2} = D^{-1/2} D D^{-1/2} = 1_n.
\]

This completes the proof. \(\square\)

**Remark 1.37.** Because of this theorem, all attention will be focused on unitary representations. Given a finite group \(G = \{g_1, \ldots, g_m\}\) of order \(|G| = m\). Let \(X\) be a unitary representation of \(G\). Denote

\[
X(G) := \{X(g_1), \ldots, X(g_m)\}.
\]

Since \(g, g^{-1} \in G\) and \(X(g^{-1}) = X(g)^{-1} = X(g)^\dagger\), it follows that \(X(g), X(g)^\dagger \in X(G)\). To find the irreducible representations of \(G\), we need to find common proper invariant subspaces of all unitary matrices from the set \(\{X(g_1), \ldots, X(g_m)\}\).
If $N$ is a common proper invariant subspace of all unitary matrices from the set \{\(X(g_i)\)\}, then \(N^\perp\) is a common proper invariant subspace of all unitary matrices from the set \{\(X(g_i)\)\}. This statement can follow from the discussion below.

If $N$ is a common proper invariant subspace of both matrices $A$ and $A^\dagger$ in $M(\mathbb{C}^d)$, then we have $N^\perp$ is a common proper invariant subspace of both matrices $A$ and $A^\dagger$. Indeed, let $P_N$ be the projector from $\mathbb{C}^d$ onto $N$, then we have

\[
AP_N = P_NAP_N \quad \text{and} \quad A^\dagger P_N = P_NA^\dagger P_N.
\]

Therefore $AP_N = P_NA$. Moreover $AP_{N^\perp} = P_{N^\perp}A$. Thus $\mathbb{C}^d = N \oplus N^\perp$ and $A = P_NAP_N \oplus P_{N^\perp}AP_{N^\perp}$. From this, we see that if $N$ is a common proper invariant subspace of all unitary matrices from the set \{\(X(g_1), \ldots, X(g_m)\)\}, i.e. $N$ is a common proper invariant subspace of all pairs \((X(g_i), X(g_i)\dagger)\) and $N^\perp$ is a common proper invariant subspace of all pairs \((X(g_i), X(g_i)\dagger)\) and $X(g_i)P_N = P_NX(g_i)$. Finally we get that

\[
X(g_i) = P_NX(g_i)P_N \oplus P_{N^\perp}X(g_i)P_{N^\perp} := \begin{bmatrix}
    P_NX(g_i)P_N & 0 \\
    0 & P_{N^\perp}X(g_i)P_{N^\perp}
\end{bmatrix},
\]

where $i = 1, \ldots, |G|$.

**Definition 1.38.** The representation $\phi : G \to GL(1, \mathbb{F})$ with

\[
\phi(g) = (1) \quad \text{for all} \quad g \in G
\]

is called the **trivial representation** of $G$.

**Definition 1.39.** A representation $\phi : G \to GL(n, \mathbb{F})$ is said to be **faithful** if $\ker(\phi) = \{e\}$.

### 1.6 Group algebras

Let $G$ be a finite group, whose elements are $g_1, \ldots, g_n$. We define a vector space over $\mathbb{F}$ with $g_1, \ldots, g_n$ as a basis, which is written as $\mathbb{F}[G]$. The elements of $\mathbb{F}[G]$ have the form

\[
\lambda_1g_1 + \cdots + \lambda_ng_n \quad \forall \lambda_i \in \mathbb{F}.
\]

The rules for addition and scalar multiplication in $\mathbb{F}[G]$ are the natural ones

\[
u = \sum_{i=1}^{n} \lambda_i g_i \quad \text{and} \quad v = \sum_{i=1}^{n} \mu_i g_i
\]
for \( \lambda_i, \mu_i \in \mathbb{F} \). Then
\[
   u + v = \sum_{i=1}^{n} (\lambda_i + \mu_i) g_i \quad \text{and} \quad \lambda u = \sum_{i=1}^{n} (\lambda \lambda_i) g_i.
\]

With these rules, \( \mathbb{F}[G] \) is a vector space over \( \mathbb{F} \) of dimension \( n \) with basis \( g_1, \ldots, g_n \). The basis \( g_1, \ldots, g_n \) is called the natural basis of \( \mathbb{F}[G] \).

Sometimes the elements of \( \mathbb{F}[G] \) are written in the form
\[
   \sum_{g \in G} \lambda_g g \quad \lambda_g \in \mathbb{F}.
\]

**Definition 1.40.** The vector space \( \mathbb{F}[G] \), with multiplication defined by
\[
   \left( \sum_{g \in G} \lambda_g g \right) \left( \sum_{h \in G} \mu_h h \right) = \sum_{g, h \in G} \lambda_g \mu_h (gh) \quad \lambda_g, \mu_h \in \mathbb{F},
\]
is called the group algebra of \( G \) over \( \mathbb{F} \).

The group algebra \( \mathbb{F}[G] \) contains an identity for multiplication, namely the element \( 1e \), where \( 1 \) is the identity of \( \mathbb{F} \) and \( e \) is the identity of \( G \). Usually, we write this element simply as \( e \).

**Proposition 1.41.** Multiplication in \( \mathbb{F}[G] \) satisfies the following properties, for all \( r, s, t \in \mathbb{F}[G] \) and \( \lambda \in \mathbb{F} \)
\[
   \begin{cases}
      rs \in \mathbb{F}[G], \\
      r(st) = (rs)t, \\
      r1 = 1r = r, \\
      (\lambda r)s = \lambda(rs) = r(\lambda s), \\
      (r + s)t = rt + st, \\
      r(s + t) = rs + rt, \\
      r0 = 0r = 0.
   \end{cases}
\]

We now use the group algebra to define an important \( \mathbb{F}[G] \)-module. Let \( V = \mathbb{F}[G] \), so that \( V \) is a vector space of dimension \( n \) over \( \mathbb{F} \), where \( n = |G| \). For all \( u, v \in V, \lambda \in \mathbb{F} \)
and \(g, h \in G\), we have

\[
\begin{align*}
gv & \in V, \\
(gh)v &= g(hv), \\
ev &= v, \\
g(\lambda v) &= \lambda g(v), \\
g(u + v) &= gu + gv.
\end{align*}
\]

Therefore \(V\) is an \(\mathbb{F}[G]\)-module.

**Definition 1.42.** Let \(G\) be a finite group. The vector space \(\mathbb{F}[G]\) with the natural multiplication \(gv, g \in G, v \in \mathbb{F}[G]\), is called the *regular* \(\mathbb{F}[G]\)-module.

Henceforth, we use \(R\) to denote \(\mathbb{C}[G]\).

The representation \(g \mapsto [g]_B\) by taking \(B\) to be the natural basis of \(\mathbb{F}[G]\) is called the *regular representation* of \(G\) over \(\mathbb{F}\).

**Definition 1.43.** Suppose that \(V\) is an \(\mathbb{F}[G]\)-module, and that \(v \in V\) and \(r \in \mathbb{F}[G]\); say 
\[r = \sum_{g \in G} \mu_g g.\]
Define \(rv\) by
\[rv \overset{\text{def}}{=} \sum_{g \in G} \mu_g (gv).\]

**Theorem 1.44.** Suppose that \(V\) is an \(\mathbb{F}[G]\)-module with basis \(B\), and \(W\) is an \(\mathbb{F}[G]\)-module with basis \(B'\). Then \(V\) and \(W\) are \(\mathbb{F}[G]\)-isomorphic if and only if the representations 
\[g \mapsto [g]_B\text{ and } g \mapsto [g]_{B'}\]
are equivalent.

The elements of the group algebra can be regarded as \(\mathbb{F}\)-valued functions \(a : G \to \mathbb{F}\). For two such functions \(a(g)\) and \(b(g)\), their product is
\[(a \star b)(g) := \sum_{z \in G} a(gz^{-1})b(z).\]

And the adjoint of \(a(g)\) is given by
\[a^*(g) = \overline{a(g^{-1})}.\]

Finally, we introduce a new operation on \(\mathbb{C}[G]\) as following
\[
\left(\sum_{g \in G} a(g)g\right)^\wedge = \sum_{g \in G} a(g)g^{-1}.
\]
**Proposition 1.45.** For a finite group $G$, if $a, b \in \mathbb{C}[G]$, then

$$(ab)^\wedge = b\hat{a}.$$  

*Proof.* It is easy to see that

$$\hat{b}\hat{a} = \left(\sum_g b(g)g^{-1}\right)\left(\sum_h a(h)h^{-1}\right) = \sum_{g,h} a(h)b(g)g^{-1}h^{-1}$$

$$= \sum_z \left(\sum_g a(zg^{-1})b(g)\right)z^{-1} = \sum_z a \star b(z)z^{-1} = (ab)^\wedge.$$  

We are done. \hfill \Box

### 1.7 Schur Lemma

Schur Lemma is a basic result concerning irreducible modules, which deals with $\mathbb{C}[G]$-modules rather than $\mathbb{R}[G]$-modules.

**Lemma 1.46 (Schur).** Let $V$ and $W$ be irreducible $\mathbb{C}[G]$-modules.

(i) If $\phi : V \to W$ is a $\mathbb{C}[G]$-homomorphism, then either $\phi$ is a $\mathbb{C}[G]$-isomorphism or $\phi = 0$.

(ii) If $\phi : V \to V$ is a $\mathbb{C}[G]$-isomorphism, then $\phi$ is a scalar multiple of the identity endomorphism $1$.

*Proof.* Suppose that $\phi(v) \neq 0$ for some $v \in V$. Then $\text{im}(\phi) \neq \{0\}$. As $\text{im}(\phi)$ is a $\mathbb{C}[G]$-submodule of $W$, and $W$ is irreducible, we have $\text{im}(\phi) = W$. In addition, $\ker(\phi)$ is a $\mathbb{C}[G]$-submodule of $V$; as $\ker(\phi) \neq V$ and $V$ is irreducible, $\ker(\phi) = \{0\}$. Thus $\phi$ is invertible, and hence is a $\mathbb{C}[G]$-isomorphism.

If $\phi : V \to V$ is a $\mathbb{C}[G]$-isomorphism, then $\phi$ is also a linear isomorphism. It follows from linear algebra theory that $\phi$ has an eigenvalue $\lambda_0 \in \mathbb{C}$. Note that $\phi - \lambda_0 1$ is also a $\mathbb{C}[G]$-homomorphism and $\ker(\phi - \lambda_0 1) \neq \{0\}$, $V$ is irreducible, so $\ker(\phi - \lambda_0 1) = V$. Thus,

$$\phi - \lambda_0 1 = 0.$$  

That is, $\phi = \lambda_0 1$, as required. \hfill \Box

**Lemma 1.47 (Matrix version of Schur Lemma).** Let $\varphi : G \to \text{GL}(n, \mathbb{C})$ be a irreducible representation of $G$. If $n \times n$ matrix $A$ satisfies

$$A(\varphi(g)) = (\varphi(g))A$$

for all $g \in G$,

then $A = \lambda 1_n$ with $\lambda \in \mathbb{C}$. 

18
Proof. Regard $\mathbb{C}^n$ as a $\mathbb{C}[G]$-module by defining $gv = \varphi(g)v$ for all $g \in G, v \in \mathbb{C}^n$. Let $A$ be an $n \times n$ matrix over $\mathbb{C}$. The endomorphism $v \mapsto Av$ of $\mathbb{C}^n$ is a $\mathbb{C}[G]$-homomorphism if and only if

$$A(gv) = g(Av) \text{ for all } g \in G, v \in \mathbb{C}^n,$$

that is, if and only if

$$A(\varphi(g)) = (\varphi(g))A \text{ for all } g \in G.$$

By Schur’s Lemma, we have that the $\mathbb{C}[G]$-homomorphism $v \mapsto Av$ must be a scalar multiple of the identity homomorphism, i.e., $\lambda \mathbb{1}$. □

**Proposition 1.48.** If $G$ is a finite abelian group, then every irreducible non-zero $\mathbb{C}[G]$-module has dimension 1.

**Proof.** Let $G$ be a finite abelian group, $V$ an irreducible non-zero $\mathbb{C}[G]$-module. Pick $x \in G$. Since $G$ is abelian, we have

$$x(gv) = g(xv) \text{ for all } g \in G.$$

Hence the endomorphism $v \mapsto xv$ of $V$ is a $\mathbb{C}[G]$-homomorphism. By Schur’s Lemma, this endomorphism is a scalar multiple of the identity, say $\lambda_x \mathbb{1}$. Thus

$$xv = \lambda_x v \text{ for all } v \in V.$$

This implies that every subspace of $V$ is a $\mathbb{C}[G]$-submodule. As $V$ is irreducible, we deduce that $\dim(V) = 1$. □

### 1.8 Irreducible decomposition of group algebras

Let $G$ be a finite group, $R = \mathbb{C}[G]$ be the group algebra of $G$ over $\mathbb{C}$, which is also the regular $\mathbb{C}[G]$-module. By the module decomposition theorem, we have

$$R = U_1 \oplus \cdots \oplus U_r,$$

where every $U_i$ is an irreducible $\mathbb{C}[G]$-module.

We shall show that every irreducible $\mathbb{C}[G]$-module is a $\mathbb{C}[G]$ isomorphic to one of the $\mathbb{C}[G]$-modules $U_1, \ldots, U_r$. As a consequence, there are only finitely many non-isomorphic irreducible $\mathbb{C}[G]$-modules.
Proposition 1.49. Let $V$ be a $\mathbb{C}[G]$-module, and write
$$V = V_1 \bigoplus \cdots \bigoplus V_s,$$
as a direct sum of irreducible $\mathbb{C}[G]$-submodules $V_i$. If $W$ is any irreducible $\mathbb{C}[G]$-submodule of $V$, then $W \cong V_i$ for some $i$.

Proof. For $u \in W$, we have $u = v_1 + \cdots + v_s$ for unique vectors $v_i \in V_i (1 \leq i \leq s)$. Define $\pi_i : W \to V_i$ by setting $\pi_i u = v_i$. Choosing $i_0$ such that $v_{i_0} \neq 0$ for some $u \in W$, we have $\pi_{i_0} \neq 0$.

Now $\pi_{i_0}$ is a $\mathbb{C}[G]$-homomorphism. As $W$ and $V_{i_0}$ are irreducible and $\pi_{i_0} \neq 0$, Schur’s Lemma implies that $\pi_{i_0}$ is a $\mathbb{C}[G]$-isomorphism. Therefore $W \cong V_{i_0}$. \hfill \Box

Definition 1.50. (1) If $V$ is a $\mathbb{C}[G]$-module and $U$ is an irreducible $\mathbb{C}[G]$-module, then we say that $U$ is a composition factor of $V$ if $V$ has a $\mathbb{C}[G]$-submodule which is isomorphic to $U$.

(2) Two $\mathbb{C}[G]$-modules $V$ and $W$ are said to have a common composition factor if there is an irreducible $\mathbb{C}[G]$-module which is a composition factor of both $V$ and $W$.

Theorem 1.51. If
$$R = U_1 \bigoplus \cdots \bigoplus U_r,$$
a direct sum of irreducible $\mathbb{C}[G]$-submodules, then every irreducible $\mathbb{C}[G]$-module $V_0$ is isomorphic to one of the $\mathbb{C}[G]$-modules $U_i$.

Proof. Choosing a non-zero $v_0 \in V_0$, we define $\phi : R \to V_0$ by
$$\phi(r) = rv_0 \quad r \in R.$$Then $\phi$ is a $\mathbb{C}[G]$-homomorphism since for all $r, s \in R$
$$\phi(sr) = (sr)v_0 = s(rv_0) = s\phi(r).$$Note that $V_0$ is an irreducible $\mathbb{C}[G]$-module and $v_0 \in \phi(R)$ and $\phi(R)$ is a module. So $\phi(R) = V_0$. It follows from the Corollary of Maschke’s theorem that $V_0$ is isomorphic to a submodule of $R$. Thus, we can consider it as a submodule of $R$. Therefore, by above theorem we have proved the conclusion. \hfill \Box

Corollary 1.52. If $G$ is a finite group, then there are only finitely many non-isomorphic irreducible $\mathbb{C}[G]$-modules.
We now explore further into the structure of $R$ of a finite group $G$. We write

$$R = U_1 \oplus \cdots \oplus U_r,$$

a direct sum of irreducible $C[G]$-modules $U_i$. We have proved that every irreducible $C[G]$-module $U$ is isomorphic to one of the $U_i$. The question arises: how many of the $U_i$ are isomorphic to $U$? There is an elegant and significant answer to this question: the number is precisely $\dim(U)$.

**Definition 1.53.** Let $V$ and $W$ be $C[G]$-modules. We write $\text{Hom}_{C[G]}(V, W)$ for the set of all $C[G]$-homomorphisms from $V$ to $W$.

Define addition and scalar multiplication on $\text{Hom}_{C[G]}(V, W)$ as follows: for $\varphi, \phi \in \text{Hom}_{C[G]}(V, W)$ and $\lambda \in C$, define $\varphi + \phi$ and $\lambda \varphi$ by

$$\begin{align*}
(\varphi + \phi)(v) &= \varphi(v) + \phi(v), \\
(\lambda \varphi)(v) &= \lambda(\varphi(v))
\end{align*}$$

for all $v \in V$. Then $\varphi + \phi, \lambda \varphi \in \text{Hom}_{C[G]}(V, W)$. With these definitions, it is easily checked that $\text{Hom}_{C[G]}(V, W)$ is a vector space over $C$.

**Proposition 1.54.** Suppose that $V$ and $W$ are irreducible $C[G]$-modules. Then

$$\dim(\text{Hom}_{C[G]}(V, W)) = \begin{cases} 1, & \text{if } V \cong W, \\ 0, & \text{if } V \not\cong W \end{cases}$$

**Proof.** If $V \not\cong W$, then this is immediate from Schur Lemma. Now suppose that $V \cong W$, and let $\varphi : V \to W$ be a $C[G]$-isomorphism. If $\phi \in \text{Hom}_{C[G]}(V, W)$ and $\phi \neq 0$, then $\phi$ is a $C[G]$-isomorphism. So $\varphi^{-1}\phi$ is a $C[G]$-isomorphism from $V$ to $V$. By Schur Lemma, there exists a $\lambda \in C$ such that

$$\varphi^{-1}\phi = \lambda \mathbb{1}.$$ 

Then $\phi = \lambda \varphi$, and so $\text{Hom}_{C[G]}(V, W) = \{ \lambda \varphi : \lambda \in C \}$, which is a 1-dimensional space.

**Proposition 1.55.** Let $V$ and $W$ be $C[G]$-modules, and suppose that $\text{Hom}_{C[G]}(V, W) \neq \{0\}$. Then $V$ and $W$ have a common composition factor.

The next few results show how to calculate the dimension of $\text{Hom}_{C[G]}(V, W)$ in general. The key step is the following proposition.
Proposition 1.56. Let \( V, V_1, V_2 \) and \( W, W_1, W_2 \) be \( \mathbb{C}[G] \)-modules. Then

(i) \( \dim(\text{Hom}_{\mathbb{C}[G]}(V, W_1 \oplus W_2)) = \dim(\text{Hom}_{\mathbb{C}[G]}(V, W_1)) + \dim(\text{Hom}_{\mathbb{C}[G]}(V, W_2)). \)

(ii) \( \dim(\text{Hom}_{\mathbb{C}[G]}(V_1 \oplus V_2, W)) = \dim(\text{Hom}_{\mathbb{C}[G]}(V_1, W)) + \dim(\text{Hom}_{\mathbb{C}[G]}(V_2, W)). \)

Now suppose that we have \( \mathbb{C}[G] \)-modules \( V, W, V_i, W_j (1 \leq i \leq r, 1 \leq j \leq s) \). By an obvious induction, we have

\[
\begin{align*}
\dim(\text{Hom}_{\mathbb{C}[G]}(V_1 \oplus \cdots \oplus V_r, W_1 \oplus \cdots \oplus W_s)) &= \sum_{i=1}^{r} \sum_{j=1}^{s} \dim(\text{Hom}_{\mathbb{C}[G]}(V_i, W_j)).
\end{align*}
\]

Applying the above equation when all \( V_i \) and \( W_j \) are irreducible, we can find the dimension of \( \text{Hom}_{\mathbb{C}[G]}(V, W) \) in general.

Proposition 1.57. Let \( V \) be a \( \mathbb{C}[G] \)-module with

\[
V = U_1 \oplus \cdots \oplus U_s,
\]

where each \( U_i \) is an irreducible \( \mathbb{C}[G] \)-module. Let \( W \) be any irreducible \( \mathbb{C}[G] \)-module. Then the dimensions of \( \text{Hom}_{\mathbb{C}[G]}(V, W) \) and \( \text{Hom}_{\mathbb{C}[G]}(W, V) \) are both equal to the number of \( \mathbb{C}[G] \)-modules \( U_i \) such that \( U_i \cong W \).

Proposition 1.58. If \( U \) is a \( \mathbb{C}[G] \)-module, then

\[
\dim(\text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U)) = \dim(U).
\]

Proof. Let \( d = \dim(U) \) and choose a basis \( u_1, \ldots, u_d \) of \( U \). For \( 1 \leq i \leq d \), define \( \phi_i : \mathbb{C}[G] \rightarrow U \) as

\[
\phi_i(r) = ru_i \quad r \in \mathbb{C}[G].
\]

Then \( \phi_i \in \text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U) \) since for all \( r, s \in \mathbb{C}[G] \),

\[
\phi_i(sr) = (sr)u_i = s(ru_i) = s\phi_i(r).
\]

We will prove that \( \phi_1, \ldots, \phi_d \) is a basis of \( \text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U) \).
Suppose that $\phi \in \text{Hom}_{C[G]}(C[G], U)$. Then
\[
\phi(e) = \lambda_1 u_1 + \cdots + \lambda_d u_d
\]
for some $\lambda_i \in C$. Since $\phi$ is a $C[G]$-homomorphism, for all $r \in C[G]$ we have
\[
\phi(r) = r\phi(e) = \lambda_1 ru_1 + \cdots + \lambda_d ru_d = (\lambda_1 \phi_1 + \cdots + \lambda_d \phi_d)(r).
\]
Hence $\phi = \lambda_1 \phi_1 + \cdots + \lambda_d \phi_d$. Therefore $\phi_1, \ldots, \phi_d$ span $\text{Hom}_{C[G]}(C[G], U)$.

Let
\[
\lambda_1 \phi_1 + \cdots + \lambda_d \phi_d = 0 \quad (\lambda_i \in C).
\]
Evaluating both sides at the identity $e$, we have
\[
0 = (\lambda_1 \phi_1 + \cdots + \lambda_d \phi_d)(e) = \lambda_1 u_1 + \cdots + \lambda_d u_d,
\]
which forces $\lambda_i = 0$ for all $i$. Hence $\phi_1, \ldots, \phi_d$ is a basis of $\text{Hom}_{C[G]}(C[G], U)$, which has dimension $d$. \hfill \Box

**Theorem 1.59.** Suppose that
\[
R = U_1 \oplus \cdots \oplus U_r,
\]
a direct sum of irreducible $C[G]$-submodules. If $U$ is any irreducible $C[G]$-module, then the number of $C[G]$-modules $U_i$ with $U_i \cong U$ is equal to $\dim(U)$.

**Proof.** Note that
\[
\dim(U) = \dim(\text{Hom}_{C[G]}(C[G], U)) = \dim(\text{Hom}_{C[G]}(U_1 \oplus \cdots \oplus U_r, U))
\]
\[
= \sum_{i=1}^r \dim(\text{Hom}_{C[G]}(U_i, U))
\]
and
\[
\dim(\text{Hom}_{C[G]}(U_i, U)) = \begin{cases} 1, & \text{if } U_i \cong U, \\ 0, & \text{if } U_i \ncong U. \end{cases}
\]
This is equal to the number of $U_i$ with $U_i \cong U$. \hfill \Box

**Definition 1.60.** We say that the irreducible $C[G]$-modules $V_1, \ldots, V_k$ form a complete set of non-isomorphic irreducible $C[G]$-modules if every irreducible $C[G]$-module is isomorphic to some $V_i$, and no two of $V_1, \ldots, V_k$ are isomorphic.
Theorem 1.61. Let $V_1, \ldots, V_k$ form a complete set of non-isomorphic irreducible $C[G]$-modules. Then
\[
C[G] = \bigoplus_{i=1}^{k} V_i^{\oplus \dim(V_i)},
\]
\[
\dim(C[G]) = \sum_{i=1}^{k} \dim^2(V_i) = |G|.
\]

Proposition 1.62 (Module version). Let $V$ be a $C[G]$-module and
\[
V = V_1^{\oplus m_1} \bigoplus V_2^{\oplus m_2} \bigoplus \cdots \bigoplus V_k^{\oplus m_k},
\]
where $V_i$ are pairwise inequivalent irreducibles and $\dim(V_i) = d_i$.

If we denote $Z(\text{Hom}_{C[G]}(V, V)) = \{ \phi : \phi \in \text{Hom}_{C[G]}(V, V), \phi \phi = \phi \phi \text{ for each } \phi \in \text{Hom}_{C[G]}(V, V) \}$.

Then
(i) $\dim(V) = m_1 d_1 + m_2 d_2 + \cdots + m_k d_k$;
(ii) $\text{Hom}_{C[G]}(V, V) \cong \bigoplus_{i=1}^{k} M_{m_i}(C)$;
(iii) $\dim(\text{Hom}_{C[G]}(V, V)) = m_1^2 + m_2^2 + \cdots + m_k^2$;
(iv) $Z(\text{Hom}_{C[G]}(V, V))$ is isomorphic to the algebra of diagonal matrices of degree $k$.
(v) $\dim(Z(\text{Hom}_{C[G]}(V, V))) = k$.

Proposition 1.63 (Matrix version). Let $X$ be a representation of a finite group $G$ such that
\[
X = X_1^{\oplus m_1} \bigoplus X_2^{\oplus m_2} \bigoplus \cdots \bigoplus X_k^{\oplus m_k},
\]
where $X_i$ are pairwise inequivalent irreducible representations and $\dim(X_i) = d_i$. Then
(i) $\dim(X) = m_1 d_1 + m_2 d_2 + \cdots + m_k d_k$;
(ii) $\{X(g) : g \in G\}' := G'_X = \{ Y \in M_{\dim(X)}(C), X(g)Y = YX(g) \text{ for each } g \in G \} = \bigoplus_{i=1}^{k} (M_{m_i}(C) \otimes 1_{d_i})$;
(iii) $\dim(G'_X) = m_1^2 + m_2^2 + \cdots + m_k^2$;
(iv) $Z(G'_X) = \{ Y : Y \in G'_X, WY = YW, W \in G'_X \} = \{ \bigoplus_{i=1}^{k} c_i 1_{m_i} \otimes 1_{d_i} : c_i \in C \}$;
(v) $\dim(Z(G'_X)) = k$. 

24
In fact, let $C^n$ as a $C[G]$-module by defining $gv = X(g)v$ for all $g \in G, v \in C^n$, where $n = \dim(X)$. Let $A$ be an $n \times n$ matrix over $C$. The endomorphism $v \mapsto Av$ of $C^n$ is a $C[G]$-homomorphism if and only if

$$A(gv) = g(Av) \text{ for all } g \in G, v \in C^n,$$

that is, if and only if

$$A(X(g)) = (X(g))A \text{ for all } g \in G.$$

This showed that $A \in \text{Hom}_{C[G]}(C^n, C^n)$ if and only if $A \in G'_X$. Therefore, $G'_X = \text{Hom}_{C[G]}(C^n, C^n)$. Thus, it is easy to follow from above proposition that $\text{Hom}_{C[G]}(C^n, C^n) = \bigoplus_{i=1}^{k} (M_{m_i}(C) \otimes 1_{d_i})$.

### 1.9 Characters

**Definition 1.64.** Suppose that $V$ is a $C[G]$-module with a basis $B$. Then the character of $V$ is the function $\chi : G \to C$ defined by

$$\chi(g) \overset{\text{def}}{=} \text{Tr}([g]_B) \quad g \in G.$$

The character of $V$ does not depend on the basis $B$. If $B$ and $B'$ are bases of $V$, then

$$[g]_{B'} = T^{-1}[g]_B T$$

for some invertible matrix $T$. Thus

$$\text{Tr}([g]_{B'}) = \text{Tr}([g]_B) \text{ for all } g \in G.$$

Naturally enough, we define the character of a representation $\varphi : G \to \text{GL}(n, C)$ to be the character $\chi$ of the corresponding $C[G]$-module $C^n$, namely,

$$\chi(g) \overset{\text{def}}{=} \text{Tr}(\varphi(g)) \quad g \in G.$$

**Definition 1.65.** We say that $\chi$ is a character of $G$ if $\chi$ is the character of some $C[G]$-module. Further, $\chi$ is an irreducible character of $G$ if $\chi$ is the character of an irreducible $C[G]$-module.

Thus, every character $\chi$ of $G$ can be considered as an element of $C[G]$, that is, $\chi = \sum_{g \in G} \chi(g)g$. 

25
Proposition 1.66. (i) Isomorphic \( \mathbb{C}[G] \)-modules have the same character.

(ii) If \( x \) and \( y \) are conjugate elements of the group \( G \), then

\[
\chi(x) = \chi(y)
\]

for all characters \( \chi \) of \( G \).

Proof. (1) Let \( \phi \) be the \( \mathbb{C}[G] \)-modules isomorphic of \( V \) and \( W \). Take a basis \( B_V = \{ v_i \} \) of \( V \), then \( B_W = \{ w_i = \phi(v_i) \} \) is a basis of \( W \). Note that \( gw_i = g\phi(v_i) = \phi(gv_i) \),

\[
(gw_1, gw_2, \cdots, gw_n) = (w_1, w_2, \cdots, w_n)[g]_{B_W},
\]

\[
(gv_1, gv_2, \cdots, gv_n) = (v_1, v_2, \cdots, v_n)[g]_{B_V},
\]

thus, we have

\[
[g]_{B_V} = [g]_{B_W}
\]

for all \( g \in G \).

Consequently \( \text{Tr} \left([g]_{B_V}\right) = \text{Tr} \left([g]_{B_W}\right) \) for all \( g \in G \), and so \( V \) and \( W \) have the same character.

(2) Assume that \( x \) and \( y \) are conjugate elements of \( G \), so that \( x = g^{-1}yg \) for some \( g \in G \). Let \( V \) be a \( \mathbb{C}[G] \)-module, and \( B \) be a basis of \( V \). Then

\[
[x]_B = [g^{-1}yg]_B = [g]^{-1}_B[y]_B[g]_B.
\]

We have \( \text{Tr} \left([x]_B\right) = \text{Tr} \left([y]_B\right) \). Therefore \( \chi(x) = \chi(y) \), where \( \chi \) is the character of \( V \). \( \square \)

Definition 1.67. If \( \chi \) is the character of the \( \mathbb{C}[G] \)-module \( V \), then the dimension of \( V \) is called the degree of \( \chi \).

Proposition 1.68. Let \( \chi \) be a character of the \( \mathbb{C}[G] \)-module \( V \). Suppose that \( g \in G \) and \( g \) has order \( m \). Then

(i) \( \chi(e) = \dim(V) \);

(ii) \( \chi(g) \) is a sum of \( m \)-th roots of unity;

(iii) \( \chi(g^{-1}) = \overline{\chi(g)} \);

(iv) \( \chi(g) \) is a real number if \( g \) is conjugate to \( g^{-1} \).

In fact, (ii) can be followed from the Proposition in Section 1.4 and (iii) can be followed from every representation is equivalent to unitary representation.
Theorem 1.69. Let $G$ be a finite group and $\varphi : G \to \text{GL}(n, \mathbb{C})$ a representation of $G$, and $\chi$ be the character of $\varphi$.

(i) For $g \in G$,

$$|\chi(g)| = \chi(e) \iff \varphi(g) = \lambda I_n \text{ for some } \lambda \in \mathbb{C}, |\lambda| = 1.$$

(ii) $\ker(\varphi) = \{g \in G : \chi(g) = \chi(e)\}$.

In fact, since $G$ is a finite group, it follows from above theorem that $\chi(g)$ is a sum of $n$ numbers $\lambda_i$ and $|\lambda_i| = 1$. From $|\chi(g)| = \chi(e)$ we have $\varphi(g) = \lambda I_n$ for some $\lambda \in \mathbb{C}, |\lambda| = 1$.

(ii) can be proved easily by (i).

Definition 1.70. If $\chi$ is a character of $G$, then the kernel of $\chi$, written as $\ker(\chi)$, is defined by

$$\ker(\chi) = \{g \in G : \chi(g) = \chi(e)\}.$$

If $\varphi$ is a representation of $G$ with character $\chi$, then $\ker(\varphi) = \ker(\chi)$. In particular, $\ker(\chi) \triangleleft G$. We call $\chi$ a faithful character if $\ker(\chi) = \{e\}$.

We next prove a result which is sometimes useful for constructing a new character from a given one. For a character $\chi$ of $G$, define $\overline{\chi} : G \to \mathbb{C}$ as

$$\overline{\chi}(g) \overset{\text{def}}{=} \overline{\chi(g)} \quad g \in G.$$

Proposition 1.71. Let $\chi$ be a character of $G$. Then $\overline{\chi}$ is a character of $G$. If $\chi$ is irreducible, then so is $\overline{\chi}$.

In fact, it only need to take the new representation matrices for the conjugation matrices.

Definition 1.72. Let $V$ and $W$ be $\mathbb{C}[G]$-modules with characters $\chi$ and $\psi$, respectively. Then

$$\chi \psi(g) = \chi(g) \psi(g) \text{ for all } g \in G$$

is a character of the $\mathbb{C}[G]$-module $V \otimes W$, which is called the product character of $\chi$ and $\psi$.

Definition 1.73. The regular character of $G$ is the character of the regular $\mathbb{C}[G]$-module $R$. We write the regular character as $\chi_R$. 

27
Proposition 1.74.\[\chi_R(g) = \begin{cases} |G|, & \text{if } g = e, \\ 0, & \text{if } g \neq e. \end{cases}\]

Proposition 1.75. Let $V$ be a $\mathbb{C}[G]$-module, and suppose that $V = U_1 \oplus \cdots \oplus U_r$, a direct sum of $\mathbb{C}[G]$-modules $U_i$. Then the character of $V$ is equal to the sum of the characters of the $\mathbb{C}[G]$-modules $U_1, \ldots, U_r$.

Theorem 1.76. Let $V_1, \ldots, V_k$ be a complete set of non-isomorphic irreducible $\mathbb{C}[G]$-modules, $\chi_i$ the character of $V_i$, $d_i = \chi_i(e)$. Then

$$\chi_R = d_1\chi_1 + \cdots + d_k\chi_k.$$ 

Next, we are going to introduce inner product on the group algebra $\mathbb{C}[G]$, and use it to prove a series of nice properties.

Definition 1.77. Suppose that $\varphi, \phi \in \mathbb{C}[G]$. The inner product of $\varphi$ and $\phi$ is defined by

$$\left(\varphi, \phi\right) \overset{\text{def}}{=} \frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\phi(g)}.$$ 

Let $V_1, \ldots, V_k$ be a complete set of non-isomorphic irreducible $\mathbb{C}[G]$-modules. For every $V_\alpha \in \{V_1, \ldots, V_k\}$, without lose generality, we can assume $V_\alpha$ is an irreducible unitary representation and denote its matrix entries by $V_{\alpha,ij}$, where $i, j \in \{1, \ldots, d_\alpha = \dim V_\alpha\}$. We have the following important theorem:

Theorem 1.78 (Orthogonality Relations). The functions $\{\sqrt{d_\alpha}V_{\alpha,ij} : \alpha = 1, 2, \ldots, k; i, j = 1, \ldots, d_\alpha\}$ are an orthonormal basis of $\mathbb{C}[G]$.

Proof. Note that the numbers of all these functions is $\sum_\alpha d_\alpha^2$, and the dimensional of $\mathbb{C}[G]$ is also $\sum_\alpha d_\alpha^2$. So, it is enough to show that

$$\left(\sqrt{d_\beta}V_{\beta,kl}, \sqrt{d_\alpha}V_{\alpha,ij}\right) := \frac{1}{|G|} \sum_{g \in G} \sqrt{d_\alpha}V_{\alpha,ij}(g)\sqrt{d_\beta}V_{\beta,kl}(g)$$

$$= \frac{\sqrt{d_\alpha d_\beta}}{|G|} \sum_{g \in G} V_{\alpha,ij}(g)V_{\beta,kl}(g) = \delta_{(\beta,kl),(\alpha,ij)}.$$ 

28
Let $X = [x_{ik}]$ be a $d_\alpha \times d_\beta$ matrix of indeterminates $x_{ik}$ and consider the matrix

$$Y = \frac{1}{|G|} \sum_{g \in G} V_\alpha(g^{-1}) XV_\beta(g).$$

We claim that

$$V_\alpha(h) Y = Y V_\beta(h), \quad \forall h \in G.$$

Indeed,

$$V_\alpha(h^{-1}) Y V_\beta(h) = \frac{1}{|G|} \sum_{g \in G} V_\alpha(h^{-1}g^{-1}) XV_\beta(g h) = Y,$$

and our assertion is proved. Thus by Schur’s lemma,

$$Y = \begin{cases} 0, & \text{if } \alpha \neq \beta, \\ c\mathbf{1}_\alpha, & \text{if } \alpha = \beta. \end{cases}$$

(1) If $\alpha \neq \beta$, since $Y = 0$, we can take the $(j, l)$ entry in the definition of $Y$ to obtain

$$0 = y_{jl} = \frac{1}{|G|} \sum_{i,k} \sum_{g \in G} V_{a,ji}(g^{-1}) x_{ik} V_{\beta,kl}(g) = \frac{1}{|G|} \sum_{i,k} \sum_{g \in G} \overline{V_{a,ji}(g)} x_{ik} V_{\beta,kl}(g), \quad \forall j, l.$$

Here we have used the fact that if $V$ is a unitary representation, i.e.

$$V(g^{-1}) = V(g)^{-1} = [V(g)]^* := [\overline{V(g)}]^T,$$

then $V_{ji}(g^{-1}) = \overline{V_{ij}(g)}$.

If this polynomial is zero, the coefficient of each $x_{ik}$ must also be zero. So

$$0 = \sum_{i,k} \left( \frac{1}{|G|} \sum_{g \in G} \overline{V_{a,ji}(g)} V_{\beta,kl}(g) \right) x_{ik},$$

implying that

$$\frac{1}{|G|} \sum_{g \in G} V_{a,ji}(g) V_{\beta,kl}(g) = 0, \quad \forall i,j,k,l.$$

Notice that the above equation can be more simply stated as

$$(V_{\beta,kl}, V_{a,ji}) = 0, \quad \forall i,j,k,l.$$

(2) Now suppose that $\alpha = \beta$. Consider

$$\frac{1}{|G|} \sum_{g \in G} V_{a}(g^{-1}) XV_{a}(g) = c\mathbf{1}_\alpha = Y.$$

29
and take the trace on both sides to get \( y_{jj} = c = \frac{1}{d_\alpha} \text{Tr} (X) \). Thus \( y_{jl} = \frac{1}{d_\alpha} \text{Tr} (X) \delta_{jl} \).

Therefore, we have

\[
\frac{1}{d_\alpha} \text{Tr} (X) \delta_{jl} = y_{jl} = \frac{1}{d_\alpha} \sum_{i,k} x_{ik} \delta_{ik} \delta_{jl} = \sum_{i,k} \left( \frac{1}{d_\alpha} \delta_{ik} \delta_{jl} \right) x_{ik} = \frac{1}{|G|} \sum_{i,k} \sum_{g \in G} V_{a,ij}(g) V_{a,kl}(g) x_{ik} = \sum_{i,k} \left( \frac{1}{|G|} \sum_{g \in G} V_{a,ij}(g) V_{a,kl}(g) \right) x_{ik}.
\]

From the identical polynomial’s theorem, it follows that

\[
\frac{1}{|G|} \sum_{g \in G} V_{a,ij}(g) V_{a,kl}(g) = \frac{1}{d_\alpha} \delta_{ik} \delta_{jl} = \frac{1}{d_\alpha} \delta_{(ij),(kl)}.
\]

In summary, we can conclude that

\[
(V_{\beta,kl}, V_{a,ij}) = \frac{1}{d_\alpha} \delta_{(\beta,kl),(aij)}.
\]

\(\square\)

**Corollary 1.79.** The irreducible characters of group \( G \) satisfy

\[
(\chi_\alpha, \chi_\beta) = \delta_{\alpha\beta}.
\]

**Proof.** By the above theorem, we have

\[
(\chi_\beta, \chi_\alpha) = \frac{1}{|G|} \sum_{g \in G} \chi_\alpha(g) \chi_\beta(g) = \frac{1}{|G|} \sum_{g \in G} \text{Tr} (V_\alpha(g)) \text{Tr} (V_\beta(g))
\]

\[
= \frac{1}{|G|} \sum_{g \in G} \sum_{i,k} V_{a,ii}(g) V_{\beta,kk}(g) = \sum_{i,k} \left( \frac{1}{|G|} \sum_{g \in G} V_{a,ii}(g) V_{\beta,kk}(g) \right)
\]

\[
= \frac{1}{\sqrt{d_\alpha d_\beta}} \sum_{i,k} \left( \sqrt{d_\alpha} V_{\beta,kk} \sqrt{d_\alpha} V_{a,ii} \right) = \frac{1}{\sqrt{d_\alpha d_\beta}} \sum_{i,k} \delta_{\alpha\beta} \delta_{ik} = \delta_{\beta\alpha}.
\]

The proof is complete. \(\square\)

**Definition 1.80.** Let \( G \) be a finite group. The **center** of the group algebra \( \mathbb{C}[G] \) is defined by

\[
Z(\mathbb{C}[G]) \overset{\text{def}}{=} \{ z \in \mathbb{C}[G] : zr = rz \text{ for all } r \in \mathbb{C}[G] \}.
\]
For abelian groups $G$, the center $Z(C[G])$ is the whole group algebra. For arbitrary groups $G$, we shall see that $Z(C[G])$ plays a crucial role in the study of representations of $G$.

**Example 1.81.** The elements $e$ and $\sum_{g \in G} g$ lie in $Z(C[G])$. Moreover, if $N$ is any normal subgroup of $G$, then

$$\sum_{h \in N} h \in Z(C[G]).$$

To see this, write $z = \sum_{h \in N} h$. Then for all $g \in G$,

$$g^{-1}zg = \sum_{h \in N} g^{-1}hg = \sum_{h \in N} h = z,$$

and so $zg = gz$. Consequently $zr = rz$ for all $r \in C[G]$.

We use Schur Lemma to prove the following interesting property of the elements of $Z(C[G])$.

**Proposition 1.82.** Let $V$ be an irreducible $C[G]$-module, $z \in Z(C[G])$. Then there exists a $\lambda \in C$ such that

$$zv = \lambda v \text{ for all } v \in V.$$

**Proof.** For all $r \in C[G]$ and $v \in V$, we have

$$zrv = rzv.$$

Hence the function $v \mapsto zv$ is a $C[G]$-homomorphism from $V$ to $V$. By Schur Lemma, this $C[G]$-homomorphism is equal to $\lambda 1$ for some $\lambda \in C$, and the result follows. \qed

**Definition 1.83.** Let $C_1, \ldots, C_l$ be the distinct conjugacy classes of $G$. For $1 \leq i \leq l$, define

$$\overline{C}_i = \sum_{g \in C_i} g \in C[G].$$

The elements $\overline{C}_1, \ldots, \overline{C}_l$ of $C[G]$ are called *class sums*.

**Proposition 1.84.** The all class sums $\overline{C}_1, \ldots, \overline{C}_l$ form a basis of $Z(C[G])$.

**Proof.** First we show that each $\overline{C}_i$ belongs to $Z(C[G])$. Let $C_i$ consist of the $r$ distinct conjugates $y_1^{-1}gy_1, \ldots, y_r^{-1}gy_r$ of an element $g$, so

$$\overline{C}_i = \sum_{j=1}^r y_j^{-1}gy_j.$$
For all \( h \in G \),
\[
h^{-1}C_i h = \sum_{j=1}^{r} h^{-1}y^{-1}_j gy_j h.
\]

As \( j \) runs from 1 to \( r \), the elements \( h^{-1}y^{-1}_j gy_j h \) run through \( C_i \) since
\[
h^{-1}y^{-1}_j gy_j h = h^{-1}y^{-1}_k gy_k h \iff y^{-1}_j gy_j = y^{-1}_k gy_k.
\]

Hence
\[
\sum_{j=1}^{r} h^{-1}y^{-1}_j gy_j h = C_i,
\]
and so \( h^{-1}C_i h = C_i \). That is,
\[
C_i h = hC_i.
\]

Therefore every \( C_i \) commutes with all \( h \in G \), hence with all \( \sum_{h \in G} \lambda_h h \in C[G] \), and so \( \overline{C}_i \in Z(C[G]) \).

Next, observe that \( \overline{C}_1, \ldots, \overline{C}_l \) are linearly independent: if \( \sum_{i=1}^{l} \lambda_i \overline{C}_i = 0 (\lambda_i \in C) \), then all \( \lambda_i = 0 \) as the classes \( C_1, \ldots, C_l \) are pairwise disjoint.

It remains to show that \( \overline{C}_1, \ldots, \overline{C}_l \) span \( Z(C[G]) \). Let \( r = \sum_{g \in G} \lambda_g g \in Z(C[G]) \). For \( h \in G \), we have \( rh = hr \), so \( h^{-1}rh = r \). That is,
\[
\sum_{g \in G} \lambda_g h^{-1}gh = \sum_{g \in G} \lambda_g g.
\]

So for every \( h \in G \), the coefficient \( \lambda_g \) of \( g \) is equal to the coefficient \( \lambda_{h^{-1}gh} \) of \( h^{-1}gh \). That is to say, the function \( g \mapsto \lambda_g \) is constant on conjugacy classes of \( G \). It follows that \( r = \sum_{i=1}^{l} \lambda_i \overline{C}_i \), where \( \lambda_i \) is the coefficient \( \lambda_{g_i} \) for some \( g_i \in C_i \). This completes the proof.

The above theorem told us that \( Z(C[G]) \) consists of the functions which are constant on conjugacy classes of \( G \). We call these functions to be the class function.

**Proposition 1.85.** Let \( V \) be a \( C[G] \)-module, \( f \in C[G] \), define
\[
\phi_{f,V} = \sum_{g \in G} f(g)g : V \to V.
\]

Then \( \phi_{f,V} \) is a \( C[G] \)-homomorphism for all \( V \) if and only if \( f \) is a class function.
Proof. If \( f \) is a class function, then

\[
\phi_{f,V}(hv) = \left( \sum_{g \in G} f(g)g \right)(hv) = \left( \sum_{g \in G} f(hgh^{-1})hgh^{-1} \right)(hv)
\]

\[
= h \left( \sum_{g \in G} f(hgh^{-1})g v \right) = h \left( \sum_{g \in G} f(g)v \right) = h \phi_{f,V}(v).
\]

If for every \( C[G]\)-module \( V \), \( \phi_{f,V} \) is a \( C[G] \)-homomorphism, taking \( V = R \), then for every \( h \in G \) and \( v \in V \), we must have \( \phi_{f,V}(hv) = \sum_{g \in G} f(g)g(hv) = h(\sum_{g \in G} f(g)v) \).

Let \( v = e \), we get \( (\sum_{g \in G} f(g)g)h = h(\sum_{g \in G} f(g)g) \). Thus, for every \( r \in R \), \( \sum_{g \in G} f(g)g \in Z(C[G]) \). Thus, \( f \) is a class function.

**Theorem 1.86.** The number of irreducible characters of \( G \) is equal to the number of conjugacy classes of \( G \).

**Proof.** Note that the number of conjugacy classes of \( G \) is the dimensional of \( Z(C[G]) \). It follows from above theorem and corollary that the number of irreducible characters of \( G \) is less than the number of conjugacy classes of \( G \). Thus, if \( f \in Z(C[G]) \) and \( (f, \chi) = 0 \) for every irreducible character \( \chi \), then \( f = 0 \).

For any \( C[G]\)-module \( V_0 \), note that the function \( \tilde{f} \) is also a class function. Then consider the \( C[G]\)-homomorphism

\[
\psi = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)}g : V_0 \to V_0.
\]

If \( V_0 \) is irreducible, we have \( \psi = \lambda 1_{V_0} \). Taking character \( \chi_{V_0} \) on two side of \( \psi = \lambda 1_{V_0} \) and note that \( \chi_{V_0}(1_{V_0}) = \dim(V_0) \), we have

\[
\lambda = \frac{1}{\dim(V_0)} \chi_{V_0}(\psi) = \frac{1}{\dim(V_0)} \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} \chi_{V_0}(g) = \frac{1}{\dim(V_0)} (\chi_{V_0}, f) = 0.
\]

Thus, \( \psi = 0 \) on every \( C[G]\)-module \( V \), that is, for every \( v \in V \), \( \psi(v) = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)}gv = 0 \).

In particular, taking \( V = R \) and \( v = e \), we have \( \frac{1}{|G|} \sum_{g \in G} \overline{f(g)}g = 0 \). Thus \( f = 0 \). □

**Corollary 1.87.** The characters \( \chi_{V} \) form an orthonormal basis for \( Z(C[G]) \).
Corollary 1.88. The \( \mathbb{C}[G] \)-module \( V \) decomposes as \( V \cong \bigoplus_a V_a^{m_a} \) if and only if \( \chi_V = \sum_a m_a \chi_a \).

Proposition 1.89. Assume that \( G \) has exactly \( k \) conjugacy classes, with representatives \( g_1, \ldots, g_k \).

Let \( \chi \) and \( \psi \) be characters of \( G \).

(i) \( \langle \chi, \psi \rangle = \langle \psi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \psi(g^{-1}) \), and this is a real number.

(ii) \( \langle \chi, \psi \rangle = \sum_{i=1}^k \frac{\chi(g_i) \psi(g_i)}{|C_G(g_i)|} \).

Theorem 1.90. Let \( \chi_1, \ldots, \chi_k \) be the irreducible characters of \( G \). If \( \psi \) is any character of \( G \), then

\[ \psi = d_1 \chi_1 + \cdots + d_k \chi_k \]

for some non-negative integers \( d_1, \ldots, d_k \). Moreover,

\[ d_i = \langle \psi, \chi_i \rangle \text{ for } 1 \leq i \leq k; \langle \psi, \psi \rangle = \sum_{i=1}^k d_i^2. \]

Theorem 1.91. Let \( V \) be a \( \mathbb{C}[G] \)-module with character \( \psi \). Then \( V \) is irreducible if and only if \( \langle \psi, \psi \rangle = 1 \).

Theorem 1.92. Suppose that \( V \) and \( W \) are \( \mathbb{C}[G] \)-modules with characters \( \chi \) and \( \psi \), respectively. Then \( V \) and \( W \) are isomorphic if and only if \( \chi = \psi \).

Theorem 1.93. Let \( V \) and \( W \) be \( \mathbb{C}[G] \)-modules with characters \( \chi \) and \( \psi \), respectively. Then

\[ \dim(\text{Hom}_{\mathbb{C}[G]}(V, W)) = \langle \chi, \psi \rangle. \]

Definition 1.94. Let \( \chi_1, \ldots, \chi_k \) be the irreducible characters of \( G \) and \( g_1, \ldots, g_k \) be representatives of the conjugacy classes of \( G \). The \( k \times k \) matrix whose \( ij \)-entry is \( \chi_i(g_j) \) is called the character table of \( G \).

Theorem 1.95. Let \( \chi_1, \ldots, \chi_k \) be the irreducible characters of \( G \) and let \( g_1, \ldots, g_k \) be representatives of the conjugacy classes of \( G \). Then the following relations hold for any \( r, s \in 1, \ldots, k \).

(i) The row orthogonality relations:

\[ \sum_{i=1}^k \frac{\chi_r(g_i) \bar{\chi}_s(g_i)}{|C_G(g_i)|} = \delta_{rs}. \]
(ii) The column orthogonality relations:

\[
\sum_{i=1}^{k} \chi_i(g_r)\overline{\chi_i(g_s)} = \delta_{rs} |C_G(g_r)|.
\]

In fact, (i) is clear. Now, we take a new matrix \(M\) by letting the \(ij\)-entry of \(M\) be

\[
\frac{\chi_i(g_j)}{\sqrt{|C_G(g_j)|}}.
\]

Then we have

\[MM^* = 1.\]

So, we have also \(M^*M = 1\). Thus, (ii) is true.

**Proposition 1.96.** Let \(V\) be a \(C[G]\) module, \(\pi : G \to \text{GL}(V)\) be an action of \(G\) on \(V\). For every \(\varphi \in C[G]\), define the Fourier transform \(\tilde{\varphi}(\pi)\) in \(\text{End}(V)\) by the formula

\[
\tilde{\varphi}(\pi) = \sum_{g \in G} \varphi(g)\pi(g).
\]

Then it holds that

(i) \(\varphi \star \psi(\pi) = \tilde{\varphi}(\pi)\tilde{\psi}(\pi)\).

(ii) the Fourier inversion formula:

\[
\varphi(g) = \frac{1}{|G|} \sum_{a=1}^{k} \text{dim}(V_a) \text{Tr} \left( \pi_{V_a}(g^{-1})\tilde{\varphi}(\pi_{V_a}) \right),
\]

where \(\{V_a\}\) is the complete set of all non-isomorphic irreducible \(C[G]\)-modules, and \(\pi_{V_a}\) are the action of \(G\) on the \(V_a\).

(iii) the Plancherel formula for \(\varphi, \psi \in C[G]\):

\[
\sum_{g \in G} \varphi(g^{-1})\psi(g) = \frac{1}{|G|} \sum_{a=1}^{k} \text{dim}(V_a) \text{Tr} \left( \tilde{\varphi}(\pi_{V_a})\tilde{\psi}(\pi_{V_a}) \right).
\]

**Proof.** Note that

\[
(\varphi \star \psi)(g) = \sum_{h \in G} \varphi(h)\psi(h^{-1}g).
\]
By the definition, we have

\[
\tilde{\varphi} \star \tilde{\psi}(\pi) = \sum_{g \in G} (\varphi \star \psi)(g) \pi(g) = \sum_{g \in G} \left( \sum_{h \in G} \varphi(h) \psi(h^{-1}g) \right) \pi(g) \\
= \sum_{h \in G} \varphi(h) \left( \sum_{g \in G} \psi(h^{-1}g) \pi(g) \right) \\
= \sum_{h \in G} \varphi(h) \pi(h) \pi(h^{-1}) \left( \sum_{g \in G} \psi(h^{-1}g) \pi(g) \right) \\
= \sum_{h \in G} \varphi(h) \pi(h) \left( \sum_{g \in G} \psi(h^{-1}g) \pi(h^{-1}g) \right) = \tilde{\varphi}(\pi) \tilde{\psi}(\pi).
\]

We are done for (i).

For (ii), the check is as follows:

\[
\frac{1}{|G|} \sum_{a=1}^{k} \text{dim}(V_a) \text{Tr} \left( \pi_{V_a}(g^{-1}) \tilde{\varphi}(\pi_{V_a}) \right) \\
= \frac{1}{|G|} \sum_{a=1}^{k} \text{dim}(V_a) \text{Tr} \left( \pi_{V_a}(g^{-1}) \left( \sum_{h \in G} \varphi(h) \pi_{V_a}(h) \right) \right) \\
= \frac{1}{|G|} \sum_{h \in G} \varphi(h) \left( \frac{1}{|G|} \sum_{a=1}^{k} \text{dim}(V_a) \chi_{\pi_{V_a}}(g^{-1}h) \right) \\
= \sum_{h \in G} \varphi(h) \left( \frac{1}{|G|} \sum_{a=1}^{k} \chi_{\pi_{V_a}}(e) \chi_{\pi_{V_a}}(g^{-1}h) \right) = \varphi(g).
\]

For (iii), since

\[
\frac{1}{|G|} \sum_{a=1}^{k} \text{dim}(V_a) \text{Tr} \left( \tilde{\varphi}(\pi_{V_a}) \tilde{\psi}(\pi_{V_a}) \right) \\
= \frac{1}{|G|} \sum_{a=1}^{k} \text{dim}(V_a) \text{Tr} \left( \left( \sum_{h \in G} \varphi(h) \pi_{V_a}(h) \right) \left( \sum_{g \in G} \psi(g) \pi_{V_a}(g) \right) \right) \\
= \sum_{h,g \in G} \varphi(h) \psi(g) \left( \frac{1}{|G|} \sum_{a=1}^{k} \text{dim}(V_a) \text{Tr} \left( \pi_{V_a}(h) \pi_{V_a}(g) \right) \right) \\
= \sum_{h,g \in G} \varphi(h) \psi(g) \delta_{h,g^{-1}} = \sum_{g \in G} \varphi(g^{-1}) \psi(g),
\]
the proposition is proved.

Now, by the above proposition, we can establish the following important algebra isomorphism theorem:

**Theorem 1.97.** Let $V_1, \ldots, V_k$ be a complete set of non-isomorphic irreducible $\mathbb{C}[G]$-modules. Then

$$\Phi : \varphi \mapsto \bigoplus_{a=1}^{k} \sum_{g \in G} \varphi(g) \pi V_a(g) = \bigoplus_{a=1}^{k} \tilde{\varphi}(\pi V_a)$$

define an algebra isomorphism between $\mathbb{C}[G]$ and $\bigoplus_{a=1}^{k} \text{End}(V_a)$, where $\pi V_a : G \to V_a$ is the action of $G$ on the irreducible $\mathbb{C}[G]$-modules $V_a$.

**Proof.** In fact, it follows from above Proposition (i) that $\Phi$ is an algebra morphism. Moreover, above Proposition (iii) showed that if $\Phi(\varphi) = 0$, then $\varphi = 0$. On the other hand, note that $\mathbb{C}[G]$ and $\bigoplus_{a=1}^{k} \text{End}(V_a)$ have the same finite dimensional, so $\Phi(\mathbb{C}[G]) = \bigoplus_{a=1}^{k} \text{End}(V_a)$. The theorem is proved.

### 1.10 Projection formulas from $\mathbb{C}[G]$-module onto its submodule

Let $V$ be a $\mathbb{C}[G]$-module,

$$V^G := \{ v \in V : gv = v \ \forall g \in G \}.$$ 

We try to find a way of expressing $V^G$ explicitly. Let

$$\varphi := \frac{1}{|G|} \sum_{g \in G} g \in \text{End}(V).$$

Then $\varphi$ is a class function and so it is a $\mathbb{C}[G]$-homomorphism on $\mathbb{C}[G]$-module $V$.

**Proposition 1.98.** The map $\varphi$ is a projection of $V$ onto $V^G$.

**Proof.** Suppose that $v = \varphi(x)$. Then for any $h \in G$,

$$hv = h\varphi(x) = \frac{1}{|G|} \sum_{g \in G} hgx = \frac{1}{|G|} \sum_{g \in G} gx = v.$$ 

So the image of $\varphi$ is contained in $V^G$, i.e. $\text{im}(\varphi) \subseteq V^G$. Conversely, for $v \in V^G$ it follows from $gv = v$ for all $g \in G$ that

$$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} g v = \frac{1}{|G|} \sum_{g \in G} v = v.$$ 

So $V^G \subseteq \text{im}(\varphi)$ and $\varphi^2 = \varphi$. 

\[37\]
Taking trace on two sides of $\varphi$ and note that $V = V^G \oplus \ker(\varphi)$ and $\varphi|_{V^G} = I_{V^G}$, so we have
\[
dim(V^G) = \Tr(\varphi) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).
\]

Now, we will study how to obtain a more general projection formula. Let $W$ be an irreducible $C[G]$-module. Then for any $C[G]$-module $V_0$,
\[
\psi := \frac{1}{|G|} \sum_{g \in G} \overline{\chi_W(g)g} \in \End(V_0).
\]

It follows from the above section proposition that $\psi$ is a $C[G]$-homomorphism. Hence, if $V_0$ is irreducible, we have $\psi = \lambda \mathbb{1}_{V_0}$. Taking trace on two sides of $\psi = \lambda \mathbb{1}_{V_0}$ we have
\[
\lambda = \frac{1}{\dim(V_0)} \Tr(\psi) = \frac{1}{\dim(V_0)} \frac{1}{|G|} \sum_{g \in G} \overline{\chi_W(g)\chi_{V_0}(g)} = \begin{cases} \frac{1}{\dim(V_0)} = \frac{1}{\dim(W)}, & V_0 = W, \\ 0, & V_0 \neq W. \end{cases}
\]

The above conclusion is:

For arbitrary irreducible $C[G]$-module $V_0$, $\dim(W)\psi = \frac{\dim(W)}{|G|} \sum_{g \in G} \overline{\chi_W(g)g} = \begin{cases} \mathbb{1}_{V_0}, & V_0 = W, \\ 0, & V_0 \neq W. \end{cases}$

Now, we can give the projection formulas from $C[G]$-module $V$ onto its submodule, that is,
\[
\psi_V := \frac{\dim(W)}{|G|} \sum_{g \in G} \overline{\chi_W(g)g} : V \to V,
\]

which is the projection of $V$ onto the factor consisting of the sum of all copies of $W$ appearing in $V$. In other words, if $V = \bigoplus_{\alpha} V^{\oplus \alpha}_\alpha$, where $V_\alpha$ is irreducible $C[G]$-submodule of $V$, then
\[
p_\alpha := \frac{\dim(V_\alpha)}{|G|} \sum_{g \in G} \overline{\chi_{V_\alpha}(g)g}
\]
is the projection of $V$ onto $V^{\oplus \alpha}_\alpha$.

**Definition 1.99.**

(i) An element $p$ in $C[G]$ is said to be a projection if $p^2 = p$.

(ii) A projection $p \neq 0$ is called minimal if it cannot be decomposed into projections $q \neq 0$ and $r \neq 0$ as $p = q + r$.  

38
(iii) Two projections \( p, q \) are called equivalent if there exits an invertible \( u \in \mathbb{C}[G] \) such that \( upu^{-1} = q \).

**Definition 1.100.**  
(i) A central projection \( p \) is an element in \( Z(\mathbb{C}[G]) \) with \( p^2 = p \).

(ii) A central projection \( p \neq 0 \) is called minimal if it cannot be decomposed into central projection \( q \neq 0 \) and \( r \neq 0 \) as \( p = q + r \).

**Theorem 1.101.** Let \( \{V_\alpha\}_{\alpha=1}^k \) be a complete set of non-isomorphic irreducible \( \mathbb{C}[G] \)-modules. Then there is a one-to-one correspondence between equivalence classes of minimal projections and \( \{V_\alpha\}_{\alpha=1}^k \). Moreover, there is a one-to-one correspondence between equivalence classes of minimal central projections and \( \{V_\alpha\}_{\alpha=1}^k \). The minimal central projections are given by

\[
\dim(V_\alpha) \frac{1}{|G|} \sum_{g \in G} \chi_\alpha(g) g,
\]

where \( \chi_\alpha \) is the character of \( V_\alpha \).

**Proof.** We have known that the group algebra \( \mathbb{C}[G] \) is algebra isomorphic to a direct sum of matrix algebras:

\[
\mathbb{C}[G] \cong \bigoplus_{\alpha=1}^k \text{End}(V_\alpha).
\]

Thus, for any \( p \in \mathbb{C}[G] \) corresponds to a sum \( \bigoplus_{\alpha=1}^k p_\alpha \in \bigoplus_{\alpha=1}^k \text{End}(V_\alpha) \), \( p \) is a projection in \( \mathbb{C}[G] \) implies that \( p_\alpha^2 = p_\alpha \). If \( p \) is minimal, then only one \( p_\alpha \neq 0 \) and \( p_\alpha \) is also minimal. If not, then \( p \) will be written into the sum of several projections. Note that every non-zero minimal projection \( p_\alpha \) in \( \text{End}(V_\alpha) \) must be a rank-one projection. Moreover, it is clear that two minimal projections \( \bigoplus_{\alpha \neq \alpha_0} 0_{\alpha} \oplus p_{\alpha_0} \) and \( \bigoplus_{\alpha \neq \beta_0} 0_{\alpha} \oplus q_{\beta_0} \) in \( \bigoplus_{\alpha=1}^k \text{End}(V_\alpha) \) are equivalent if and only if \( \alpha_0 = \beta_0 \). This establishes the one-to-one correspondence between equivalence classes of minimal projections and \( \{V_\alpha\}_{\alpha=1}^k \).

If \( c \in \mathbb{C}[G] \) is a central projection, then as above, we have \( c = \bigoplus_{\alpha=1}^k c_\alpha \), where \( c_\alpha \) is a center projection. If \( c \) is minimal, then only one \( c_\alpha \neq 0 \). On the other hand, it is also in the center of \( \text{End}(V_\alpha) \). So it must be \( \lambda_0 I \) for some \( \lambda_0 \) on \( V_\alpha \). Note that \( p_\alpha^2 = p_\alpha \), so \( \lambda_0 = 1 \). Thus, \( c_\alpha \) and \( \frac{\dim(V_\alpha)}{|G|} \sum_{g \in G} \chi_\alpha(g) g \) have the same properties. Hence,

\[
c_\alpha = \frac{\dim(V_\alpha)}{|G|} \sum_{g \in G} \chi_\alpha(g) g.
\]

\[\Box\]
2 Symmetry classes

In this section, we assume $\mathcal{H}$ is a finite dimensional Hilbert space on field $F$.

2.1 Antisymmetric subspace

The *antisymmetric* tensor product of vectors $x_1, \ldots, x_k$ in $\mathcal{H}$ is defined as

$$x_1 \wedge \cdots \wedge x_k := (k!)^{-\frac{1}{2}} \sum_{\pi \in S_k} \text{sign}(\pi) x_{\pi(1)} \otimes \cdots \otimes x_{\pi(k)},$$

where $\pi$ runs over all permutations of the $k$ indices and \text{sign}(\pi) is $\pm 1$, depending on whether $\pi$ is an even or an odd permutation.

**Proposition 2.1.** (i) If $\{x_1, \ldots, x_k\}$ is an orthonormal set, then $x_1 \wedge \cdots \wedge x_k$ is a unit vector.

(ii) $x_1 \wedge \cdots \wedge x_i \wedge \cdots \wedge x_j \wedge \cdots \wedge x_k = -x_1 \wedge \cdots \wedge x_j \wedge \cdots \wedge x_i \wedge \cdots \wedge x_k$. In particular, $x_1 \wedge \cdots \wedge x_k = 0$ if any two of the factors are equal.

**Proof.** We can see that

$$\langle x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k \rangle = \frac{1}{k!} \sum_{\sigma, \tau} \text{sign}(\sigma \tau) \prod_{i=1}^{k} \langle x_{\sigma(i)}, y_{\tau(i)} \rangle. \quad (2.1)$$

Thus

$$\langle x_1 \wedge \cdots \wedge x_k, x_1 \wedge \cdots \wedge x_k \rangle = \frac{1}{k!} \sum_{\sigma, \tau} \text{sign}(\sigma \tau) \prod_{i=1}^{k} \langle x_{\sigma(i)}, x_{\tau(i)} \rangle. \quad (2.2)$$

If $\sigma \neq \tau$, then there exist some $j \in \{1, \ldots, k\}$ such that $\sigma(j) \neq \tau(j)$. So $\langle x_{\sigma(j)}, x_{\tau(j)} \rangle = 0$. Therefore $\prod_{i=1}^{k} \langle x_{\sigma(i)}, x_{\tau(i)} \rangle = 0$ whenever $\sigma \neq \tau$. It follows from the normalization of the set $\{x_1, \ldots, x_k\}$ that

$$\langle x_1 \wedge \cdots \wedge x_k, x_1 \wedge \cdots \wedge x_k \rangle = \frac{1}{k!} \sum_{\sigma} \text{sign}(\sigma^2) \prod_{i=1}^{k} \langle x_{\sigma(i)}, x_{\sigma(i)} \rangle$$

$$= \frac{1}{k!} \sum_{\sigma} [\text{sign}(\sigma)]^2 = 1,$$

i.e., $x_1 \wedge \cdots \wedge x_k$ is a unit vector.
Now let $\pi = (ij)$. It follows from $\text{sign}(\pi) = -1$ that

\[
x_1 \wedge \cdots \wedge x_j \wedge \cdots \wedge x_i \wedge \cdots \wedge x_k
= (k!)^{-\frac{1}{2}} \sum_\sigma \text{sign}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(j)} \otimes \cdots \otimes x_{\sigma(i)} \otimes \cdots \otimes x_{\sigma(k)}
\]

\[
= (k!)^{-\frac{1}{2}} \sum_\sigma \text{sign}(\sigma)[\text{sign}(\pi)]^2 x_{\sigma\pi(1)} \otimes \cdots \otimes x_{\sigma\pi(i)} \otimes \cdots \otimes x_{\sigma\pi(j)} \otimes \cdots \otimes x_{\sigma\pi(k)}
\]

\[
= (k!)^{-\frac{1}{2}} \sum_\sigma \text{sign}(\sigma) \text{sign}(\pi) x_{\sigma\pi(1)} \otimes \cdots \otimes x_{\sigma\pi(i)} \otimes \cdots \otimes x_{\sigma\pi(j)} \otimes \cdots \otimes x_{\sigma\pi(k)}
\]

\[
= -(k!)^{-\frac{1}{2}} \sum_\sigma \text{sign}(\sigma)\pi x_{\sigma\pi(1)} \otimes \cdots \otimes x_{\sigma\pi(i)} \otimes \cdots \otimes x_{\sigma\pi(j)} \otimes \cdots \otimes x_{\sigma\pi(k)}
\]

\[
= -x_1 \wedge \cdots \wedge x_i \wedge \cdots \wedge x_j \wedge \cdots \wedge x_k.
\]

In particular, set $x_i = x_j = x$. We get that $x_1 \wedge \cdots \wedge x_k = 0$. \hfill \Box

The span of all antisymmetric tensors $x_1 \wedge \cdots \wedge x_k$ in $\otimes^k \mathcal{H}$ is denoted by $\wedge^k \mathcal{H}$, which is called the $k$-th antisymmetric tensor product space of $\mathcal{H}$.

Given an orthonormal basis $\{e_1, \ldots, e_n\}$ in $\mathcal{H}$, there is a standard way of constructing an orthonormal basis in $\wedge^k \mathcal{H}$. Let $Q_{k,n}$ denote the set of all strictly increasing $k$-tuples chosen from $\{1, 2, \ldots, n\}$, i.e., $I \in Q_{k,n}$ if and only if $I = (i_1, i_2, \ldots, i_k)$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. For such an $I$, let

\[
e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}.
\]

Then, $\{e_I : I \in Q_{k,n}\}$ gives an orthonormal basis of $\wedge^k \mathcal{H}$. Such $I$ are sometimes called multi-indices. It is conventional to order them lexicographically. Note that the cardinality of $Q_{k,n}$ is $|Q_{k,n}| = \binom{n}{k}$.

If $k = n$, the space $\wedge^k \mathcal{H}$ is 1-dimensional. This plays a special role later on. When $k > n$, the space $\wedge^k \mathcal{H}$ is $\{0\}$.

**Proposition 2.2.** The inner product $\langle x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k \rangle$ is equal to the determinant of the $k \times k$ matrix $[\langle x_i, y_j \rangle]$.

**Proof.** Let $a_{ij} = \langle x_i, y_j \rangle$. By the definition of determinant of a square matrix $A := [a_{ij}]$, we have

\[
\text{Det}(A) = \sum_\pi \text{sign}(\pi) \prod_{i=1}^k a_{i\pi(i)} = \sum_\pi \text{sign}(\pi) \prod_{i=1}^k \langle x_i, y_{\pi(i)} \rangle.
\]  

(2.3)
Since
\[
\langle x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k \rangle
\]
\[
= \frac{1}{k!} \sum_{\sigma, \tau} \text{sign}(\sigma \tau) \prod_{j=1}^{k} \langle x_{\sigma(j)}, y_{\tau(j)} \rangle
= \frac{1}{k!} \sum_{\sigma, \tau} \text{sign}(\sigma \tau) \prod_{i=1}^{k} \langle x_i, y_{\tau^{-1}(\sigma)(i)} \rangle
\]
\[
= \frac{1}{k!} \sum_{\sigma, \tau} \text{sign}(\tau^{-1} \sigma) \prod_{i=1}^{k} \langle x_i, y_{\tau^{-1}(\sigma)(i)} \rangle
= \frac{1}{k!} \sum \text{sign}(\sigma \tau) \prod_{i=1}^{k} \langle x_i, y_{\tau\pi(i)} \rangle
\]
\[
= \frac{1}{k!} \sum \text{sign}(\sigma \tau) \prod_{i=1}^{k} \langle x_i, y_{\tau\pi(i)} \rangle
= \sum \text{sign}(\sigma \tau) \prod_{i=1}^{k} \langle x_i, y_{\tau\pi(i)} \rangle,
\]
we have
\[
\langle x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k \rangle = \text{Det}(A) = \text{Det}([\langle x_i, y_j \rangle]).
\]
This completes the proof.

From the above result, we see for \(I, J \in Q_{k,n}\),
\[
\langle e_I, e_J \rangle = \langle e_{i_1} \wedge \cdots \wedge e_{i_k}, e_{j_1} \wedge \cdots \wedge e_{j_k} \rangle = \text{Det}([\langle e_{i_\mu}, e_{j_\nu} \rangle]) = \text{Det} [\delta_{i_\mu j_\nu}]. \tag{2.4}
\]
If \(I = J\), then \(\langle e_I, e_I \rangle = 1\); if \(I \neq J\), then \(\langle e_I, e_I \rangle = 0\).

### 2.2 Symmetric subspace

The *symmetric* tensor product of \(x_1, \ldots, x_k\) is defined as
\[
x_1 \vee \cdots \vee x_k := (k!)^{-\frac{1}{2}} \sum_{\pi \in S_k} x_{\pi(1)} \otimes \cdots \otimes x_{\pi(k)}.
\]
The linear span of all these vectors comprises the subspace \(\vee^k \mathcal{H}\) of \(\otimes^k \mathcal{H}\), which is called the \(k\)-th symmetric tensor product space of \(\mathcal{H}\).

Let \(G_{k,n}\) denote the set of all non-decreasing \(k\)-tuples chosen from \(\{1, \ldots, n\}\), i.e., \(I \in G_{k,n}\) if and only if \(I = (i_1, \ldots, i_k)\), where \(1 \leq i_1 \leq \cdots \leq i_k \leq n\). For such an \(I\) consisting of \(l\) distinct indices \(\lambda_1, \ldots, \lambda_l\) with multiplicities \(m_1, \ldots, m_l\), respectively, put \(m(I) = m_1! m_2! \cdots m_l!\). Given an orthonormal basis \((e_1, \ldots, e_n)\) of \(\mathcal{H}\), for every \(I \in G_{k,n}\) define
\[
e_I' := e_{i_1} \vee \cdots \vee e_{i_k}.
\]

42
Proposition 2.3. The set \( \{ m(I)^{-\frac{1}{2}} e'_I : I \in \mathcal{G}_{k,n} \} \) is an orthonormal basis in \( \vee^k \mathcal{H} \).

Proof. Let \( \pi \cdot I = (i_{\pi(1)}, \ldots, i_{\pi(k)}) \) for each \( I \in \mathcal{G}_{k,n} \). The operator \( Q_\pi \) induced by \( \pi \in S_k \) is defined as

\[
Q_\pi(e_{i_1} \otimes \cdots \otimes e_{i_k}) := e_{i_{\pi(1)}} \otimes \cdots \otimes e_{i_{\pi(k)}}.
\]

Thus

\[
e'_I = (k!)^{-\frac{1}{2}} \sum_{\pi \in S_k} Q_\pi(e_{i_1} \otimes \cdots \otimes e_{i_k}).
\]

Moreover \( e'_I = e'_{\pi^{-1}} \) for all \( \pi \in S_k \). For such an \( I \) consisting of \( l \) distinct indices \( \lambda_1, \ldots, \lambda_l \) with multiplicities \( m_1, \ldots, m_l \), respectively, we have that

\[
e'_I = (k!)^{-\frac{1}{2}} \sum_{\pi \in S_k} Q_\pi(e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_l} \otimes e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_l}).
\]

Now, we need to explain what \( Q_\pi(e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_l} \otimes e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_l}) \) means.

In fact, denote \( y_1 = e_{\lambda_1}, y_2 = e_{\lambda_1}, \ldots, y_{m_1} = e_{\lambda_1}, y_{m_1+1} = e_{\lambda_2}, \ldots, y_{m_1+m_2} = e_{\lambda_2}, \ldots, y_{m_1+m_2+\cdots+m_l-1+1} = e_{\lambda_l}, y_{m_1+m_2+\cdots+m_l-1+2} = e_{\lambda_l}, \ldots, y_{m_1+m_2+\cdots+m_l-1+m_l} = e_{\lambda_l} \). Then

\[
Q_\pi(e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_l} \otimes e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_l}) = Q_\pi(y_1 \otimes y_2 \cdots \otimes y_k) = y_{\pi(1)} \otimes y_{\pi(2)} \otimes \cdots \otimes y_{\pi(k)}.
\]

In this form, we have given the explain of \( Q_\pi(e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_l} \otimes e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_l}) \).

Let \( X_1 = \{1, 2, \ldots, m_1\}, X_2 = \{m_1+1, m_1+2, \ldots, m_1+m_2\}, \ldots, X_l = \{m_1+m_2+\cdots+m_{l-1}+1, \ldots, m_1+m_2+\cdots+m_{l-1}+m_l\} \), \( S_X \) denote the symmetric group of \( X \). Then we have

\[
Q_\pi(e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_l} \otimes e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_l}) = e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_l} \otimes e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_l}
\]

if and only if \( \pi \in S_{X_1} \cdots S_{X_l} \).

Let \( I = (i_1, i_2, \ldots, i_k), J = (j_1, j_2, \ldots, j_k) \in \mathcal{G}_{k,n} \), \( I \) consisting of \( l \) distinct indices \( \lambda_1, \ldots, \lambda_l \) with multiplicities \( m_1, \ldots, m_l \), \( J \) consisting of \( q \) distinct indices \( \mu_1, \ldots, \mu_q \) with multiplicities \( n_1, \ldots, n_q \) respectively. If \( I \neq J \), then there are two cases:

1. If \( \{\lambda_1, \ldots, \lambda_l\} \neq \{\mu_1, \ldots, \mu_q\} \), then there is an \( i_{k_0} \) such that \( i_{k_0} \neq j_m, m = 1, 2, \ldots, k \). So for every \( \sigma \in S_k, \langle Q_\sigma(e_{i_1} \otimes \cdots \otimes e_{i_k}), Q_\tau(e_{j_1} \otimes \cdots \otimes e_{j_k}) \rangle \) must be 0 since the inner product have two different terms. Thus, \( \langle e'_I, e'_J \rangle = 0 \).
(2). If \( \{ \lambda_1, \ldots, \lambda_l \} = \{ \mu_1, \ldots, \mu_q \} \), then there are \( i \) and \( j \) such that \( m_i \neq n_j \). So for every \( \sigma \) and \( \tau \in S_k \), \( \langle Q_\sigma(e_{i_1} \otimes \cdots \otimes e_{i_k}), Q_\tau(e_{i_1} \otimes \cdots \otimes e_{i_k}) \rangle \) must be 0 since the inner product have also two different terms. Thus, \( \langle e'_i, e'_j \rangle = 0 \).

If we denote \( y_1, y_2, \ldots, y_k \) and \( X_1, X_2, \ldots, X_i \) as above, then

\[
\langle e'_i, e'_j \rangle = (k!)^{-1} \sum_{\sigma, \tau} \prod_{a=1}^k \langle y_{\sigma(a)}, y_{\tau(a)} \rangle = (k!)^{-1} \sum_{\sigma} \prod_{a=1}^k \langle y_{\sigma(a)}, y_{\tau_{\sigma^{-1}}(a)} \rangle
\]

\[
= \sum_{\sigma} \prod_{a=1}^k \langle y_{\sigma(a)} \rangle = \sum_{\sigma} \prod_{a=1}^k \langle e_{\sigma(a)}, e_{\sigma(a)} \rangle = \sum_{\sigma} \delta_{\sigma, \pi}.
\]

Note that \( I = \pi \cdot I \) if and only if \( \pi \in S_{X_1} \cdots S_{X_i} \). Thus we have

\[
\langle e'_i, e'_j \rangle = \sum_{\sigma} \delta_{\sigma, \pi} = m(I).
\]

Therefore,

\[
\langle m(I)^{-\frac{1}{2}} e'_i, m(I)^{-\frac{1}{2}} e'_j \rangle = \langle e'_i, e'_j \rangle = 1.
\]

This completes the proof.

The cardinality of the set \( G_{k,n} \), and hence the dimensionality of the space \( \bigwedge^k \mathcal{H} \), is \( |G_{k,n}| = \binom{n+k-1}{k} \). Indeed, there is a one-to-one correspondence between \( k \)-tuples \( I = (i_1, \ldots, i_k) \) with \( 1 \leq i_1 \leq \cdots \leq i_k \leq n \) and \( I' = (i'_1, \ldots, i'_k) \) with \( 1 \leq i'_1 < \cdots < i'_k \leq n+k-1 \) via the following transform: \( i'_a := i_a + (a-1) \), where \( a \in \{1, \ldots, k\} \). Therefore \( |G_{k,n}| = |Q_{k,n+k-1}| \).

Let \( V \) be a \( \mathbb{C}[G] \)-module with character \( \chi \). The module \( V \otimes V \) has character \( \chi^2 \). Let \( v_1, \ldots, v_n \) be a basis of \( V \), and define a linear transformation \( T : V \otimes V \to V \otimes V \) by

\[
T(v_i \otimes v_j) = v_j \otimes v_i \quad \text{for all } i, j
\]

and extending it linearly, that is,

\[
T \left( \sum_{i,j} \lambda_{ij} (v_i \otimes v_j) \right) = \sum_{i,j} \lambda_{ij} (v_j \otimes v_i).
\]

For all \( v, w \in V \), we have

\[
T(v \otimes w) = w \otimes v.
\]

Now, define the subsets of \( V \otimes V \) as follows

\[
S(V \otimes V) \overset{\text{def}}{=} \{ x \in V \otimes V : Tx = x \}, A(V \otimes V) \overset{\text{def}}{=} \{ x \in V \otimes V : Tx = -x \}.
\]

It is easy to see that \( S(V \otimes V) \) and \( A(V \otimes V) \) are subspaces of \( V \otimes V \).
Proposition 2.4. \( S(V \otimes V) \) and \( A(V \otimes V) \) are \( C[G] \)-submodules of \( V \otimes V \). In addition,
\[
V \otimes V = S(V \otimes V) \bigoplus A(V \otimes V).
\]

Proof. For all \( \lambda_{ij} \in C \) and \( g \in G \), we have
\[
T \left( \sum_{i,j} \lambda_{ij} v_i \otimes v_j \right) = \sum_{i,j} \lambda_{ij} T(g v_i \otimes v_j) = \sum_{i,j} \lambda_{ij} T(gv_i) \otimes T(gv_j) (2.5)
\]
\[
= T \left( \sum_{i,j} \lambda_{ij} (gv_i) \otimes (gv_j) \right). (2.6)
\]

Therefore \( T \) is a \( C[G] \)-homomorphism from \( V \otimes V \) to itself. Hence, for \( x \in S(V \otimes V), y \in A(V \otimes V) \) and \( g \in G \), we get
\[
T(gx) = g(Tx) = gx, \quad T(gy) = g(Ty) = -gy.
\]

So \( gx \in S(V \otimes V) \) and \( gy \in A(V \otimes V) \). Thus \( S(V \otimes V) \) and \( A(V \otimes V) \) are \( C[G] \)-submodules of \( V \otimes V \).

If \( x \in S(V \otimes V) \cap A(V \otimes V) \), then \( x = Tx = -x \), so \( x = 0 \). Further, for all \( x \in V \), we have
\[
x = \frac{1}{2}(x + Tx) + \frac{1}{2}(x - Tx).
\]
Since \( T^2 \) is the identity, \( \frac{1}{2}(x + Tx) \in S(V \otimes V) \) and \( \frac{1}{2}(x - Tx) \in A(V \otimes V) \). Therefore,
\[
V \otimes V = S(V \otimes V) \bigoplus A(V \otimes V).
\]

Proposition 2.5. Let \( v_1, \ldots, v_n \) be a basis of \( V \).

(i) The vectors \( v_i \otimes v_j + v_j \otimes v_i (1 \leq i \leq j \leq n) \) form a basis of \( S(V \otimes V) \). The dimension of \( S(V \otimes V) \) is \( \frac{n(n+1)}{2} \).

(ii) The vectors \( v_i \otimes v_j - v_j \otimes v_i (1 \leq i < j \leq n) \) form a basis of \( A(V \otimes V) \). The dimension of \( A(V \otimes V) \) is \( \frac{n(n-1)}{2} \).

Proof. Clearly the vectors \( v_i \otimes v_j + v_j \otimes v_i (1 \leq i \leq j \leq n) \) are linearly independent elements of \( S(V \otimes V) \), and the vectors \( v_i \otimes v_j - v_j \otimes v_i (1 \leq i < j \leq n) \) are linearly independent elements of \( A(V \otimes V) \). Hence
\[
\dim(S(V \otimes V)) \geq \frac{n(n+1)}{2}, \quad \dim(A(V \otimes V)) \geq \frac{n(n-1)}{2}.
\]
Note
\[ \dim(S(V \otimes V)) + \dim(A(V \otimes V)) = \dim(V \otimes V) = n^2. \]

Hence the above inequalities are equalities, and the result follows. \hfill \square

Define \( \chi_S \) to be the character of the \( \mathbb{C}[G] \)-module \( S(V \otimes V) \), and \( \chi_A \) to be the character of the \( \mathbb{C}[G] \)-module \( A(V \otimes V) \). Thus
\[ \chi^2 = \chi_S + \chi_A. \]

**Proposition 2.6.** For \( g \in G \), we have
\[ \chi_S(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2)), \quad \chi_A(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2)). \]

**Proof.** We can choose a basis \( e_1, \ldots, e_n \) of \( V \) such that \( ge_i = \lambda_i e_i (1 \leq i \leq n) \) for some complex numbers \( \lambda_i \). Then
\[ g(e_i \otimes e_j - e_j \otimes e_i) = \lambda_i \lambda_j (e_i \otimes e_j - e_j \otimes e_i). \]

Hence
\[ \chi_A(g) = \sum_{i<j} \lambda_i \lambda_j. \]

Since \( g^2 e_i = \lambda_i^2 e_i \), we have \( \chi(g) = \sum_i \lambda_i \) and \( \chi(g^2) = \sum_i \lambda_i^2 \). Therefore
\[ \chi^2(g) = (\chi(g))^2 = \sum_i \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j = \chi(g^2) + 2\chi_A(g). \]

Hence
\[ \chi_A(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2)). \]

In addition, \( \chi^2 = \chi_S + \chi_A \) which implies that
\[ \chi_S(g) = \chi^2(g) - \chi_A(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2)). \]

\hfill \square

**Proposition 2.7.** The elementary tensors \( \otimes^k x \) with all factors equal, are all in the subspace \( \vee^k \mathcal{H} \) and \( \vee^k \mathcal{H} = \text{span}\{ \otimes^k x : x \in \mathcal{H} \} \).
Proof. According to the definition, $\bigvee^k \mathcal{H}$ is spanned by the vectors

$$v_{i_1 \ldots i_k} := \sum_{\pi \in S_k} e_{i_{\pi(1)}} \otimes \cdots \otimes e_{i_{\pi(k)}}.$$ 

Clearly $\text{span}\{x \otimes^k : x \in \mathcal{H}\} \subseteq \bigvee^k \mathcal{H}$. It therefore suffices to show that every $v_{i_1 \ldots i_k}$ can be written in terms of tensor products $x \otimes^k$. To this end, we will study the derivative

$$\frac{d}{dt} \bigg|_{t=0} (x + ty)^{\otimes k}.$$ 

Since

$$\frac{d}{dt} (x + ty)^{\otimes k} = \sum_{i=0}^{k-1} (x + ty)^{\otimes i} \otimes \frac{d}{dt} (x + ty)^{\otimes (k-i-1)}$$

$$= \sum_{i=0}^{k-1} (x + ty)^{\otimes i} \otimes y \otimes (x + ty)^{\otimes (k-i-1)},$$

it follows that

$$\frac{d}{dt} \bigg|_{t=0} (x + ty)^{\otimes k} = \sum_{i=0}^{k-1} x^{\otimes i} \otimes y \otimes x^{\otimes (k-i-1)}.$$ 

Moreover, consider the derivative

$$w_{i_1 \ldots i_k} := \frac{\partial}{\partial t_1 \ldots \partial t_k} \bigg|_{t_1=\cdots=t_k=0} \left( e_{i_1} + \sum_{j=2}^{k} t_j e_{i_j} \right)^{\otimes k},$$

which can be realized by subsequently applying

$$\frac{\partial}{\partial t_j} \bigg|_{t_j=0} (v + t_j e_j)^{\otimes k} = \lim_{t_j \rightarrow 0} \frac{(v + t_j e_j)^{\otimes k} - v^{\otimes k}}{t_j},$$

iteratively going from $j = k$ all the way to $j = 2$. The $w_{i_1 \ldots i_k}$ takes the form of a limit of sums of tensor powers. Since $\text{span}\{x^{\otimes k} : x \in \mathcal{H}\}$ is a finite dimensional vector space this limit is contained in $\text{span}\{x^{\otimes k} : x \in \mathcal{H}\}$. On the other hand, a direct calculation shows that $w_{i_1 \ldots i_k} = v_{i_1 \ldots i_k}$ and hence all vectors $v_{i_1 \ldots i_k}$ are contained in $\text{span}\{x^{\otimes k} : x \in \mathcal{H}\}$. 

\[\square\]

Corollary 2.8. Let $\mathcal{H}_1 = \text{End}(\mathbb{C}^d)$ with the Hilbert-Schmidt inner product $(A, B) = \text{Tr}(A^\dagger B)$, $\Delta = \text{span}\{Z^{\otimes k} : Z \in \text{End}(\mathbb{C}^d)\}$, $X \in \text{End}((\mathbb{C}^d)^{\otimes k})$. If for every $\pi \in S_k, Q_\pi X Q_\pi^{-1} = X$, then $X \in \bigvee^k \mathcal{H}_1 = \Delta$. 

47
Proof. Firstly, we need to understand the meaning of \( X \in \wedge^k \mathcal{H}_1 \). In fact, for every \( X = \sum_{i_1, \ldots, i_k} A_{i_1} \otimes A_{i_2} \otimes \cdots \otimes A_{i_k} \), then \( X \in \wedge^k \mathcal{H}_1 \) asked that for every \( \pi \in S_k, \sum_{i_1, \ldots, i_k} A_{i_{\pi(1)}} \otimes A_{i_{\pi(2)}} \otimes \cdots \otimes A_{i_{\pi(k)}} = X \).

On the other hand, note that for every \( \pi \in S_k, Q_{\pi}XQ_{\pi}^{-1} = X \) can just satisfy this condition. In fact, \( Q_{\pi}XQ_{\pi}^{-1}(x_1 \otimes x_2 \cdots \otimes x_k) = X(x_1 \otimes x_2 \cdots \otimes x_k) \) implies that \( \sum_{i_1, \ldots, i_k} A_{i_{\pi(1)}} \otimes A_{i_{\pi(2)}} \otimes \cdots \otimes A_{i_{\pi(k)}}(x_1 \otimes x_2 \cdots \otimes x_k) = \sum_{i_1, \ldots, i_k} A_{i_1} \otimes A_{i_2} \otimes \cdots \otimes A_{i_k}(x_1 \otimes x_2 \cdots \otimes x_k) \), so \( X \in \wedge^k \mathcal{H}_1 \).

The permanent of a \( k \times k \) matrix \( A = [a_{ij}] \) is defined as

\[
\text{Per}(A) := \sum_{\pi \in S_k} a_{1\pi(1)} \cdots a_{n\pi(n)}.
\]

**Proposition 2.9.** The inner product \( \langle x_1 \vee \cdots \vee x_k, y_1 \vee \cdots \vee y_k \rangle \) is equal to the permanent of the \( k \times k \) matrix \( [\langle x_i, y_j \rangle] \).

*Proof.* Let \( a_{ij} := \langle x_i, y_j \rangle \) and \( A = [a_{ij}] \). Thus

\[
\text{Per}(A) = \sum_{\pi \in S_k} \prod_{i=1}^k a_{i\pi(i)} = \sum_{\pi \in S_k} \prod_{i=1}^k \langle x_i, y_{\pi(i)} \rangle.
\]

Now we have

\[
\langle x_1 \vee \cdots \vee x_k, y_1 \vee \cdots \vee y_k \rangle = \frac{1}{k!} \sum_{\sigma, \tau} \langle x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}, y_{\tau(1)} \otimes \cdots \otimes y_{\tau(k)} \rangle
\]

\[
= \frac{1}{k!} \sum_{\sigma, \tau} \prod_{i=1}^k \langle x_{\sigma(i)}, y_{\tau(i)} \rangle = \frac{1}{k!} \sum_{\sigma, \tau} \prod_{i=1}^k \langle x_i, y_{\tau_{\sigma^{-1}(i)}} \rangle
\]

\[
= \frac{1}{k!} \sum_{\sigma} \sum_{\tau} \prod_{i=1}^k \langle x_i, y_{\tau_{\sigma^{-1}(i)}} \rangle
\]

\[
= \frac{1}{k!} \sum_{\sigma} \sum_{\tau} \prod_{i=1}^k \langle x_i, y_{\tau(i)} \rangle = \sum_{\tau} \prod_{i=1}^k \langle x_i, y_{\tau(i)} \rangle = \text{Per}(A).
\]

This completes the proof. \( \square \)

If \( A \in \text{End}(\mathcal{H}) \), then \( Ax_1 \wedge \cdots \wedge Ax_k \in \wedge^k \mathcal{H} \) for all \( x_1, \ldots, x_k \in \mathcal{H} \). This implies that

\[
(\otimes^k A)(\wedge^k \mathcal{H}) \subseteq \wedge^k \mathcal{H},
\]

that is, the space \( \wedge^k \mathcal{H} \) is invariant under the operator \( \otimes^k A \). The restriction of \( \otimes^k A \) to this invariant subspace is denoted by \( \wedge^k A := \otimes^k A|_{\wedge^k \mathcal{H}} \). Then,

\[
x_1 \wedge \cdots \wedge x_k \mapsto Ax_1 \wedge \cdots \wedge Ax_k
\]
is called the $k$-th antisymmetric tensor power or the $k$-th Grassmann power of $A$.

Similarly,

$$\otimes^k A(\vee^k \mathcal{H}) \subseteq \vee^k \mathcal{H}.$$  

The space $\vee^k \mathcal{H}$ is also invariant under the operator $\otimes^k A$. The restriction of $\otimes^k A$ to this invariant subspace is written as $\vee^k A$ or $A^{\vee k}$, called the $k$-th symmetric tensor power of $A$. Some essential and simple properties of these operators are summarized below:

1. $(\wedge^k A)(\wedge^k B) = \wedge^k (AB)$, $(\vee^k A)(\vee^k B) = \vee^k (AB)$.

2. $(\wedge^k A)^* = \wedge^k A^*$, $(\vee^k A)^* = \vee^k A^*$.

3. $(\wedge^k A)^{-1} = \wedge^k A^{-1}$, $(\vee^k A)^{-1} = \vee^k A^{-1}$.

4. If $A$ is Hermitian, unitary, normal or positive, then so are $\wedge^k A$ and $\vee^k A$.

5. If $\alpha_1, \ldots, \alpha_k$ are eigenvalues of $A$ belonging to eigenvectors $u_1, \ldots, u_k$, respectively, then $\prod_{i=1}^k \alpha_i$ is an eigenvalue of $\wedge^k A$ belonging to eigenvector $u_1 \wedge \cdots \wedge u_k$. In addition, if the vectors $u_1, \ldots, u_k$ are linearly independent, then $\prod_{i=1}^k \alpha_i$ is an eigenvalue of $\wedge^k A$ belonging to eigenvector $u_1 \wedge \cdots \wedge u_k$.

6. If $s_1, \ldots, s_n$ are the singular values of $A$, then the singular values of $\wedge^k A$ are $s_i_1 \cdots s_i_k$, where $(i_1, \ldots, i_k)$ vary over $Q_{k,n}$; the singular values of $\vee^k A$ are $s_i_1 \cdots s_i_k$, where $(i_1, \ldots, i_k)$ vary over $G_{k,n}$.

7. If $A$ is diagonal, then $\text{Tr}(\wedge^k A)$ is the $k$-th elementary symmetric polynomial in the eigenvalues of $A$; $\text{Tr}(\vee^k A)$ is the $k$-th complete symmetric polynomial in the eigenvalues of $A$.

Here, certain polynomials are defined as follows. Given any $n$-tuple $(\alpha_1, \ldots, \alpha_n)$ of numbers, the $k$-th elementary symmetric polynomial in them is the sum of all terms $\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}$ for $(i_1, \ldots, i_k) \in Q_{k,n}$; the $k$-th complete symmetric polynomial is the sum of all terms $\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}$ for $(i_1, \ldots, i_k) \in G_{k,n}$.

We will show the conclusions are correct for antisymmetric power. The same procedure goes for symmetric power. The check is as follows: for the item 1:

$$\wedge^k (AB)(x_1 \wedge \cdots \wedge x_k) = ABx_1 \wedge \cdots \wedge ABx_k$$

$$= \wedge^k A(Bx_1 \wedge \cdots \wedge Bx_k)$$

$$= (\wedge^k A)(\wedge^k B)(x_1 \wedge \cdots \wedge x_k);$$

49
for the item 2:

$$\langle (\wedge^k A)^* (y_1 \wedge \cdots \wedge y_k), x_1 \wedge \cdots \wedge x_k \rangle = \langle y_1 \wedge \cdots \wedge y_k, \wedge^k A (x_1 \wedge \cdots \wedge x_k) \rangle$$

$$= \langle y_1 \wedge \cdots \wedge y_k, Ax_1 \wedge \cdots \wedge Ax_k \rangle = \det(\langle y_i, Ax_j \rangle) = \det(\langle A^*y_i, x_j \rangle)$$

$$= \langle A^*y_1 \wedge \cdots \wedge A^*y_k, x_1 \wedge \cdots \wedge x_k \rangle = \langle (\wedge^k A^*) (y_1 \wedge \cdots \wedge y_k), x_1 \wedge \cdots \wedge x_k \rangle;$$

for the item 3 and 4 are trivially; for the item 5: since $Au_i = \alpha_i u_i$, we have

$$\langle u_1 \wedge \cdots \wedge u_k, u_1 \wedge \cdots \wedge u_k \rangle = \det(\langle u_i, u_j \rangle),$$

implying that the $k \times k$ matrix $A := [\langle u_i, u_j \rangle]$ is full-ranked because $A = M^*M$ for $k$-ranked matrix $M := [u_1, \ldots, u_k]$. Thus $\det(A) \neq 0$. That is, $u_1 \wedge \cdots \wedge u_k \neq 0$. The proof is done.

For the item 6: $s_i$ is a singular value of $A$ if and only if $s_i^2$ is an eigenvalue of $A^*A$. Note $\wedge^k (A^*A) = (\wedge^k A)^* (\wedge^k A)$. Hence $s_i^2 \cdots s_k^2$ is an eigenvalue of $\wedge^k (A^*A)$, and is also an eigenvalue of $(\wedge^k A)^* (\wedge^k A)$. This means that $s_i \cdots s_k$ is a singular value of $\wedge^k A$.

For the item 7, it follows from item 5. Thus, these conclusions are proved.

For $A \in \text{End}(\mathcal{H})$, consider the operator

$$A \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes A \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes A,$$

where there are $k$ summands, each of which is a product of $k$ factors. Both the spaces $\wedge^k \mathcal{H}$ and $\vee^k \mathcal{H}$ are invariant under this operator. One pleasant way to see this is to regard this operator as the $t$-derivative at $t = 0$ of $\otimes^k (1 + tA)$. The restriction of this operator to the space $\wedge^k \mathcal{H}$ will be of particular interest to us, written as $A^{[k]}$. If $u_1, \ldots, u_k$ are linearly independent eigenvectors of $A$ belonging to eigenvalues $\alpha_1, \ldots, \alpha_k$, then $u_1 \wedge \cdots \wedge u_k$ is an eigenvector of $A^{[k]}$ belonging to eigenvalue $\alpha_1 + \cdots + \alpha_k$.

Now, fixing an orthonormal basis $(e_1, \ldots, e_n)$ of $\mathcal{H}$, we identify $A$ with its matrix $[a_{ij}]$. We want to find the matrix representations of $\wedge^k A$ and $\vee^k A$ relative to the standard bases constructed earlier.

The basis of $\wedge^k \mathcal{H}$ that we are using is $\{e_I : I \in \mathcal{Q}_{k,n}\}$. The $(I,J)$-entry of $\wedge^k A$ is $\langle e_I, (\wedge^k A) e_J \rangle$. One may verify that this is equal to a subdeterminant of $A$. Namely, let
$A[I|J]$ denote the $k \times k$ matrix obtained from $A$ by expunging all its entries $a_{ij}$ except those for which $i \in I$ and $j \in J$. Then, the $(I,J)$-entry of $\wedge^k A$ is equal to $\det(A[I|J])$.

The special case $k = n$ leads to the 1-dimensional space $\wedge^n H$. The operator $\wedge^n A$ on this space is just the operator of multiplication by the number $\det(A)$. We can thus think of $\det(A)$ as being equal to $\wedge^n A$.

The basis of $\wedge^k H$ we are using is $\{m(I)^{-1/2}e_i : I \in G_{k,n}\}$. The $(I,J)$-entry of the matrix $\wedge^k A$ can be computed as before, and the result is somewhat similar to that for $\wedge^k A$. For $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$ in $G_{k,n}$, let $A[I|J]$ now denote the $k \times k$ matrix whose $(r,s)$-entry is the $(i_r,j_s)$-entry of $A$. Since repetitions of indices are allowed in $I$ and $J$, this is not a submatrix of $A$ this time. One verifies that the $(I,J)$-entry of $\wedge^k A$ is $(m(I)m(J))^{-1/2}\per(A[I|J])$.

In particular, $\per(A)$ is one of the diagonal entries of $\wedge^n A$: the $(I,I)$-entry for $I = (1,2,\ldots,n)$.

**Proposition 2.10.** For any vectors $u_1, \ldots, u_k, v_1, \ldots, v_k$, we have

\[
|\det(\langle u_i, v_j \rangle)|^2 \leq \det(\langle u_i, u_j \rangle) \det(\langle v_i, v_j \rangle),
\]
\[
|\per(\langle u_i, v_j \rangle)|^2 \leq \per(\langle u_i, u_j \rangle) \per(\langle v_i, v_j \rangle).
\]  

*Proof.* In fact, we have already known that

\[
\det(\langle u_i, v_j \rangle) = \langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle,
\]
\[
\per(\langle u_i, v_j \rangle) = \langle u_1 \vee \cdots \vee u_k, v_1 \vee \cdots \vee v_k \rangle.
\]

By Cauchy-Schwartz’s inequality we obtain the conclusion. \hfill \Box

**Proposition 2.11.** For any two matrices $A, B$ we have $|\per(A^*B)|^2 \leq \per(A^*A) \per(B^*B)$.

*Proof.* Let $A = [u_1, \ldots, u_k]$ and $B = [v_1, \ldots, v_k]$. Then

\[
A^*B = \begin{bmatrix}
    u_1^* \\
    \vdots \\
    u_k^*
\end{bmatrix} [v_1, \ldots, v_k] = [\langle u_i, v_j \rangle], \quad A^*A = [\langle u_i, u_j \rangle], \quad B^*B = [\langle v_i, v_j \rangle].
\]

Thus by Prop. 2.10, we have

\[
|\per(A^*B)|^2 \leq \per(A^*A) \per(B^*B),
\]

implying the proof. \hfill \Box

51
**Proposition 2.12** (Schur’s theorem). If $A$ is positive, then $\text{Per}(A) \geq \text{Det}(A)$.

**Proof.** Let $A = T^*T$ for an upper triangular $T$. Then we have

\[
\text{Det}(A) = \text{Det}(T^*T) = |\text{Det}(T)|^2 \\
= |\text{Per}(T)|^2 = |\text{Per}(\mathbf{1}T)|^2 \\
\leq \text{Per}(\mathbf{1}^*\mathbf{1})\text{Per}(T^*T) = \text{Per}(A).
\]

The proof is done. $\square$

We have observed earlier that for any vectors $x_1, \ldots, x_k$ in $\mathcal{H}$, we have

\[
\text{Det}(\langle x_i, x_j \rangle) = \|x_1 \wedge \cdots \wedge x_k\|^2.
\]

When $\mathcal{H} = \mathbb{R}^n$, this determinant is also the square of the $k$-dimensional volume of the parallelepiped having $x_1, \ldots, x_k$ as its sides. To see this, note that the determinant is not altered if we add to any of these vectors a linear combination of the others. Performing such operations successively, we can reach an orthogonal set of vectors, some of which might be zero. In this case it is obvious that the determinant is equal to the square of the volume; hence that was true initially too.

**Proposition 2.13.** Every $k \times k$ positive matrix $A = [a_{ij}]$ can be realized as a Gram matrix. There is a set $\{x_j : j = 1, \ldots, k\}$ of vectors such that $a_{ij} = \langle x_i, x_j \rangle$ for all $i, j$.

**Proposition 2.14.** Vectors $x_1, \ldots, x_k$ are linearly dependent if and only if $x_1 \wedge \cdots \wedge x_k = 0$. Two sets $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ of linearly independent vectors have the same linear span if and only if $x_1 \wedge \cdots \wedge x_k = cy_1 \wedge \cdots \wedge y_k$ for some constant $c$. Thus there is a one-to-one correspondence between $k$-dimensional subspaces of a vector space $W$ and 1-dimensional subspaces of $\wedge^k W$ generated by elementary tensors $x_1 \wedge \cdots \wedge x_k$.

Every vector $w$ of $W$ induces a linear operator $T_w$ from $\wedge^k W$ to $\wedge^{k+1} W$ as follows. The operator $T_w$ is defined on elementary tensors as

\[
T_w(v_1 \wedge \cdots \wedge v_k) := v_1 \wedge \cdots \wedge v_k \wedge w,
\]

and then it is extended linearly to all of $\wedge^k W$. It is natural to write $T_w(x) = x \wedge w$ for any $x \in \wedge^k W$.

When $W$ is a Hilbert space, the operator $T_w$ are called *creation operators* and their adjoints are called *annihilation operators* in the physics literature.
Proposition 2.15. It holds that $\| \wedge^k A \| \leq \| A \|^k$, $\| \vee^k A \| \leq \| A \|^k$, $\| \text{Det}(A) \| \leq \| A \|^n$, and $|\text{Per}(A)| \leq \| A \|^n$.

Proposition 2.16. (i) Let $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ be two orthonormal bases in $\mathcal{H}$. Show that

$$|\langle e_2 \wedge \cdots \wedge e_n, f_2 \wedge \cdots \wedge f_n \rangle|^2 = |\langle e_1, f_1 \rangle|^2.$$ 

(ii) Let $P$ and $Q$ be orthogonal projections in $\mathcal{H}$, each of rank $n - 1$. Let $x, y$ be unit vectors such that $Px = Qy = 0$. Show that

$$\wedge^{n-1}(PQP) = |\langle x, y \rangle|^2 \wedge^{n-1} P.$$ 

Proof. (i). Denote $A = [a_{ij}]$. Let $\text{Det}(A[i][j])$ be the minor of $a_{ij}$, where $i$ is a string of numbers of length $n - 1$, obtained by deleting $i$ from $1, 2, \ldots, n$. The adjoint matrix of $A$ is

$$\text{Adj}(A) = [(-1)^{i+j} \text{Det}(A[i][j])],$$

and $A^{-1} = \text{Det}(A)^{-1} \text{Adj}(A)$ if $A$ is non-singular.

Now let $U = [u_{ij}]$ be a unitary matrix. Then $U^\dagger = U^{-1}$ and $|\text{Det}(U)| = 1$. It follows that

$$[\hat{u}_{ji}] = \frac{1}{\text{Det}(U)}[(-1)^{i+j} \text{Det}(U[i][j])] \iff \hat{u}_{ij} = \frac{1}{\text{Det}(U)}(-1)^{i+j} \text{Det}(U[i][j]).$$

Hence $|u_{ij}| = |\text{Det}(U[i][j])|$. In particular,

$$|u_{11}| = |\text{Det}(U[1][1])|. \quad (2.9)$$

For this $U$, taking two orthonormal bases $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ such that $u_{ij} = |\langle e_i, f_j \rangle|$. Therefore we have

$$|\langle e_1, f_1 \rangle| = |\text{Det}(U[1][1])| = |\langle e_2 \wedge \cdots \wedge e_n, f_2 \wedge \cdots \wedge f_n \rangle|.$$ 

For (ii). Let $E = 1 - |e_1\rangle\langle e_1|$ and $F = 1 - |f_1\rangle\langle f_1|$. In the following, we show that $\wedge^{n-1}E$ is a projection of rank one, i.e. $\wedge^{n-1}E$ is a projection onto the subspace generated by the vector $e_2 \wedge \cdots \wedge e_n := \hat{e}_1$. Clearly, since $\dim(\wedge^{n-1}\mathcal{H}) = \begin{pmatrix} n \\ n-1 \end{pmatrix} = n$, it follows that

$$\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$$

is an orthonormal basis for $\wedge^{n-1}\mathcal{H}$, where $\hat{e}_i := e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_n$. If $i_1, \ldots, i_{n-1} \in \{1, 2, \ldots, n\}$ with $i_1 < \cdots < i_{n-1}$, then

$$(\wedge^{n-1}E)(e_{i_1} \wedge \cdots \wedge e_{i_{n-1}}) \neq 0 \iff (i_1, \ldots, i_{n-1}) = (2, \ldots, n).$$
Thus we have
\[ \wedge^{n-1} E = |e_2 \wedge \cdots \wedge e_n \rangle \langle e_2 \wedge \cdots \wedge e_n| . \]

Analogously we have
\[ \wedge^{n-1} F = |f_2 \wedge \cdots \wedge f_n \rangle \langle f_2 \wedge \cdots \wedge f_n| . \]

We have
\[ \wedge^{n-1} (EFE) = (\wedge^{n-1} E)(\wedge^{n-1} F)(\wedge^{n-1} E) \]
\[ = |\langle e_2 \wedge \cdots \wedge e_n, f_2 \wedge \cdots \wedge f_n \rangle|^2 |e_2 \wedge \cdots \wedge e_n \rangle \langle e_2 \wedge \cdots \wedge e_n| \]
\[ = |\langle e_1, f_1 \rangle|^2 \wedge^{n-1} E . \]

Finally, we get the desired conclusion. \hfill \Box

**Proposition 2.17.** (i) For any \(A, B \in \text{End}(\mathcal{H})\), we have
\[ \otimes^k A - \otimes^k B = \sum_{j=1}^{k} (\otimes^{k-j} A) \otimes (A - B) \otimes (\otimes^j B) . \]

Hence \( \| \otimes^k A - \otimes^k B \| \leq kM^{k-1} \| A - B \| \), where \(M = \max\{\| A \|, \| B \|\} \).

(ii) The norms of \(\wedge^k A - \wedge^k B\) and \(\vee^k A - \vee^k B\) are therefore also bounded by \(kM^{k-1} \| A - B \| \).

(iii) For \(n \times n\) matrices \(A, B\),
\[ |\text{Det}(A) - \text{Det}(B)| \leq nM^{n-1} \| A - B \|, \quad |\text{Per}(A) - \text{Per}(B)| \leq nM^{n-1} \| A - B \| . \]

**Proposition 2.18.** Let \(A, B\) be positive operators with \(A \geq B\). Then
\[ \otimes^k A \geq \otimes^k B, \quad \wedge^k A \geq \wedge^k B, \quad \vee^k A \geq \vee^k B, \quad \text{Det}(A) \geq \text{Det}(B), \quad \text{Per}(A) \geq \text{Per}(B) . \]

3 Schur-Weyl duality

3.1 Some elementary theorems of matrix algebras

Let \(\mathcal{H}\) be a finite dimensional complex Hilbert space. For any subset \(\mathcal{M} \subseteq \text{End}(\mathcal{H})\), define
\[ \mathcal{M}' \overset{\text{def}}{=} \{ X \in \text{End}(\mathcal{H}) : [X, M] = 0 \quad \forall M \in \mathcal{M} \} \]
and call it the commutant of $\mathcal{M}$.

For any subset $\mathcal{K} \subseteq \mathcal{H}$ and $\mathcal{M} \subseteq \text{End}(\mathcal{H})$ we denote

$$\mathcal{M}\mathcal{K} := \{Tx : x \in \mathcal{K}, T \in \mathcal{M}\}$$

and

$$[\mathcal{M}\mathcal{K}] := \text{span}\{y : y \in \mathcal{M}\mathcal{K}\} = \text{span}\{y : y \in \mathcal{M}\mathcal{K}\}.$$ 

We say that $\mathcal{M} \subseteq \text{End}(\mathcal{H})$ is selfadjoint if for every $T \in \mathcal{M}$, $T^\dagger \in \mathcal{M}$. If $\mathcal{M} \subseteq \text{End}(\mathcal{H})$ is an algebra, and is selfadjoint, then it is said to be a selfadjoint algebra.

An operator $P \in \text{End}(\mathcal{H})$ is said to be an orthogonal projection operator if $P^2 = P$ and $P^\dagger = P$. The orthogonal projection onto $[\mathcal{M}\mathcal{K}]$ is denoted by $[\mathcal{M}\mathcal{K}]$ too.

**Lemma 3.1.** If $\mathcal{M} \subseteq \text{End}(\mathcal{H})$ is a selfadjoint algebra, $P \in \text{End}(\mathcal{H})$ is an orthogonal projection operator, then $P \in \mathcal{M}'$ if and only if for every $T \in \mathcal{M}$, $TP = PTP$ hold. In particular, for every $x \in \mathcal{H}$, $\mathcal{M}x = [\mathcal{M}x] \in \mathcal{M}'$.

If $\mathcal{M}$ is a selfadjoint algebra, $\mathcal{H}_0$ is an invariant subspace of $\mathcal{M}$ if and only if the orthogonal project operator $P_0$ on $\mathcal{H}_0$ satisfies that $P_0 \in \mathcal{M}'$. Thus, if $\mathcal{H}_0$ is an invariant subspace of $\mathcal{M}$, then $\mathcal{H}_0^\perp$ is also an invariant subspace of $\mathcal{M}$.

**Proof.** In fact, if $P \in \mathcal{M}'$, then for every $T \in \mathcal{M}$, we have $TP = PT$. So $TP = PTP$. On the other hand, if for every $T \in \mathcal{M}$, $TP = PTP$, then $T^\dagger P = PT^\dagger P$. Thus, $(PT)^\dagger = (PTP)^\dagger$. Therefore, $PT = PTP = TP$, that is, $P \in \mathcal{M}'$. 

Let $\mathcal{H}^{\oplus n}$ be the direct sum of $\mathcal{H}$. For every $i$ ($1 \leq i \leq n$), we consider the operator $u_i : \mathcal{H} \to \mathcal{H}^{\oplus n}$, where $u_i(x) = (x_1, x_2, \cdots, x_n)$, $x_i = x$ if $j = i$ and $x_j = 0$ otherwise. Then the adjoint operator $u_i^\dagger : \mathcal{H}^{\oplus n} \to \mathcal{H}$ such that $u_i^\dagger(x_1, x_2, \cdots, x_n) = x_i$. It is easy to prove that $u_i^\dagger u_i$ is the identity operator on $\mathcal{H}$, $u_i u_i^\dagger$ is the orthogonal project operator from $\mathcal{H}^{\oplus n}$ onto $\mathcal{H}$.

To any $R \in \text{End}(\mathcal{H}^{\oplus n})$, one can associate a matrix $(R_{ij})$ of operators from $\text{End}(\mathcal{H})$ by the relation

$$R_{ik} = u_i^\dagger Ru_k, \quad 1 \leq i, k \leq n,$$

with its help the operator $R$ that can be recovered by the formula

$$R = \sum_{i,k} u_i R_{ik} u_k^\dagger.$$

For any $A \in \text{End}(\mathcal{H})$, denote $\overline{A} = (\delta_{ik} A)$, then $\overline{A} \in \text{End}(\mathcal{H}^{\oplus n})$. 
Let $\mathcal{X} \subseteq \text{End}(\mathcal{H})$. We will use the notations:

$$M_n(\mathcal{X}) = \{ (R_{ik}) : R_{ik} \in \mathcal{X}, 1 \leq i, k \leq n, \}$$

$$\overline{\mathcal{X}} = \{ \overline{A} : A \in \mathcal{X} \}.$$ 

It is easy to prove that

$$(R_{ik})^\dagger = (R_{ki})^\dagger,$$

$$(R_{ik})(Q_{ik}) = (\sum_j R_{ij}Q_{jk}),$$

$$\overline{A} \overline{B} = \overline{AB}.$$ 

**Theorem 3.2** (von Neumann theorem). Let $\mathcal{H}$ be a finite dimensional complex Hilbert space, $\mathcal{A} \subseteq \text{End}(\mathcal{H})$ be a selfadjoint algebra, $I \in \mathcal{A}$. Then $\mathcal{A} = \mathcal{A}''$.

**Proof.** Note that for every finite dimensional vector space, its all vector topology are same. So for every vector subspace of finite dimensional vector space is a closed vector subspace with respect to any vector topology. Thus, it is sufficient to prove that $\mathcal{A}$ is dense in $\mathcal{A}''$ for strong-operator topology. To this end, let us choose an element $T_0 \in \mathcal{A}''$.

Taking $x_0 \in \mathcal{H}$, then the orthogonal project operator $\mathcal{A}x_0 \in \mathcal{A}' = \mathcal{A}'''$. This implies, in particular, that the subspace $\mathcal{A}x_0$ is invariant for any operator from $\mathcal{A}''$. Since $I \in \mathcal{A}$, it follows that $x_0 \in \mathcal{A}x_0$ and

$$T_0x_0 \in \mathcal{A}x_0.$$ 

Let $x_1, x_2, \cdots, x_n \in \mathcal{H}$, $\overline{\mathcal{A}} = \{ \overline{A} : A \in \mathcal{A} \}$. It is clear that $\overline{\mathcal{A}} \subseteq \text{End}(\mathcal{H}^{\oplus n})$ is a selfadjoint algebra, $I \in \overline{\mathcal{A}}$. Moreover,

$$\overline{\mathcal{A'}} = \{ (T_{ij}) : T_{ij} \in \mathcal{A}', i, j = 1, 2, \cdots, n \} = M_n(\mathcal{A'}).$$ 

Indeed, the relation

$$0 = \overline{A}(T_{ij}) - (T_{ij})\overline{A}$$

in $\text{End}(\mathcal{H}^{\oplus n})$ is satisfied for any $\overline{A} \in \overline{\mathcal{A}}$ if and only if the relations

$$AT_{ij} = T_{ij}A, \quad i, j = 1, 2, \cdots, n$$

is satisfied for any $A \in \mathcal{A}$. Consequently, for any $(T_{ij}) \in \overline{\mathcal{A'}}$, we have

$$\overline{T_0}(T_{ij}) = (T_{ij})\overline{T_0}.$$
that is, \( \overline{T_0} \in A'' \). According to the first part of the proof, it follows that
\[
\overline{T_0}(x_1, x_2, \ldots, x_n) \in \overline{A}(x_1, x_2, \ldots, x_n).
\]
Hence, there is \( A \in A \) such that \((T_0 - A)x_k = 0, k = 1, 2, \ldots, n\). This showed that \( T_0 \) is in the strong-operator topology closure of \( A \), and so in \( A \) because it is a finite dimensional vector space. \( \square \)

**Definition 3.3.** If \( A \subseteq \text{End}(H) \) is a selfadjoint matrix algebra and contains \( I \), then \( A \) is said to be a **matrix von Neumann algebra**.

Thus, for every matrix von Neumann algebra \( A \), we have \( A = A'' \).

**Theorem 3.4.** Let \( V \) and \( W \) be finite dimensional vector spaces. If \( M_1 \subseteq \text{End}(V) \) with \( I \in M_1 \), \( M_2 \subseteq \text{End}(W) \) with \( I \in M_2 \), then
\[
(M_1 \otimes M_2)' = M_1' \otimes M_2'.
\]

**Proof.** It is clear that \((M_1 \otimes M_2)' \supseteq M_1' \otimes M_2'\). Now we prove the converse inclusion.

For every \( R \in (M_1 \otimes M_2)' \), without lose generality, we can assume that \( R = \sum_{i=1}^{k} P_i \otimes Q_i \), where \( P_i \in \text{End}(V), Q_i \in \text{End}(W) \), \( \{Q_i : 1 \leq i \leq k\} \) are linearly independent. Note that for every \( A \in M_1 \), we have \( R(A \otimes I) = (A \otimes I)R \). Thus, it follows from
\[
\sum_{i=1}^{k} P_i A \otimes Q_i = \sum_{i=1}^{k} A P_i \otimes Q_i
\]
and \( \{Q_i : 1 \leq i \leq k\} \) are linearly independent that \( P_i A = A P_i \), that is, \( P_i \in M_1', 1 \leq i \leq k \).

On the other hand, without lose generality again, we assume that \( P_1, P_2 \cdots P_r \) is a maximally linearly independent group of \( P_1, P_2 \cdots P_k \). Then \( \{P_i : t > r\} \) can be expressed linearity by \( P_1, P_2 \cdots P_r \). Rewriting the expression of \( R \), we have \( R = \sum_{i=1}^{r} P_i \otimes T_i \) for some \( T_i \in M_2 \).

Using the above method again, we can show that \( T_i \in M_2', 1 \leq i \leq r \). Thus, the above expression of \( R \) showed that \( R \in M_1' \otimes M_2' \). So, we have proved that \((M_1 \otimes M_2)' \subseteq M_1' \otimes M_2' \). \( \square \)

**Lemma 3.5.** Let \( H \) be a finite dimensional complex Hilbert space, \( A \subseteq \text{End}(H) \) be a matrix von Neumann algebra, \( A \in A \) be a normal operator. Then for each function \( f \) defined on the all eigenvalues of \( A \), the Functional Calculation \( f(A) \in A \).
Proof. In fact, let \( \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \) be the distinct eigenvalues of \( A \) and \( V_1, V_2, \ldots, V_k \) be the corresponding eigensubspaces of \( A \), \( P_i \) be the orthogonal projection operator on \( V_i \), \( B \in \mathcal{A}' \) and \( B \) is a Hermitian operator. Then we have \( A = \sum_i \lambda_i P_i \). If \( x \in V_i \), since \( ABx = BAx = \lambda_i Bx \), so \( Bx \in V_i \), thus we have \( P_i Bx = Bx \). Therefore, \( P_i B P_i = B P_i \), it follows from \( B \) is a Hermitian operator that \( B P_i = P_i B \), \( i = 1, 2, \ldots, k \). Note that \( f(A) = \sum_i f(\lambda_i) P_i \), so \( f(A) B = B f(A) \), by von Neumann theorem, \( f(A) \in \mathcal{A} \). \( \square \)

**Lemma 3.6.** Let \( \mathcal{H} \) be a finite dimensional complex Hilbert space, \( \mathcal{A} \subseteq \text{End}(\mathcal{H}) \) be a matrix von Neumann algebra. Then \( \mathcal{A} = \text{span}\{U : U \in \mathcal{A}, U \text{ is a unitary operator}\} \).

**Proof.** In fact, for every \( A \in \mathcal{A}, A = A^\dagger, ||A|| \leq 1 \), we have \( (I - A^2)^{\frac{1}{2}} \in \mathcal{A} \). So, \( A + i(I - A^2)^{\frac{1}{2}}, A - i(I - A^2)^{\frac{1}{2}} \in \mathcal{A} \) and they are unitary operators. So the lemma holds. \( \square \)

**Definition 3.7.** Let \( \mathcal{L}, \mathcal{H} \) be two finite dimensional complex Hilbert spaces, \( \mathcal{A} \subseteq \text{End}(\mathcal{L}) \) be a matrix von Neumann algebra. If \( \tau : \mathcal{A} \to \text{End}(\mathcal{H}) \) is a unit \(*\)-homomorphism, then \( \tau \) is said to be a representation of \( \mathcal{A} \) on \( \mathcal{H} \). If \( \tau(A) \) has no proper invariant subspaces, then \( \tau \) is said to be an irreducible representation. A representation \( \sigma \) is a subrepresentation of representation \( \tau \) if there is an orthogonal projection operator \( P \in (\tau(A))' \) such that \( \sigma(A) = P \tau(A) P \) for each \( A \in \mathcal{A} \). It is clear that if \( \sigma_1(A) = P \tau(A) P \) is a subrepresentation of \( \tau \), then \( \sigma_2(A) = (I - P) \tau(A) (I - P) \) is also a subrepresentation of \( \tau \), and \( \tau = \sigma_1 \oplus \sigma_2 \).

If \( \mathcal{L} = \mathcal{H} \) and \( \tau = I \), then \( \tau \) is said to be an identity representation. If \( \tau \) is an irreducible identity representation of \( \mathcal{A} \), then \( \mathcal{A} \) is said to be irreducible. Let \( \tau_1 \) and \( \tau_2 \) be two representations of \( \mathcal{A} \) on \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively. If there is a unitary transformation \( U \) from \( \mathcal{H}_2 \) onto \( \mathcal{H}_1 \) such that for every \( A \in \mathcal{A}, U \tau_2(A) U^\dagger = \tau_1(A) \), then we say that \( \tau_1 \) is unitarily equivalent to \( \tau_2 \).

**Lemma 3.8.** Let \( \mathcal{H} \) be a finite dimensional complex Hilbert space, \( \mathcal{A} \subseteq \text{End}(\mathcal{H}) \) be a matrix von Neumann algebra. Then \( \mathcal{A} \) has a minimal orthogonal projection operator \( E \). Moreover, \( EAE = CE \).

If \( \tau \) is a representation of \( \mathcal{A} \), then there is a minimal orthogonal projection operator \( E \) of \( \mathcal{A} \) such that \( \tau(E) \neq 0 \). When \( \tau \) is irreducible, \( \tau(E) \) has rank one.

**Proof.** In fact, by the Functional Calculation of normal operators and induction methods for the dimension numbers of orthogonal project operators in \( \mathcal{A} \), there is a minimal orthogonal projection operator \( E \) of \( \mathcal{A} \). If a Hermitian operator \( T \in \mathcal{A} \) such that \( ETE \) is not a scalar multiple of \( E \), then by the spectral decomposition theorem \( ETE = \sum \lambda_i P_i \),
where \( \{ P_i \} \) is orthogonal project operator and \( \{ P_i \} \subseteq EAE \subseteq A \). It is clear that for each \( P_i \), we have \( P_i \leq E \). So \( P_i = E \). Thus, \( ETE \) is a multiple of \( E \). This leads to a contradiction.

Suppose that \( \tau \) is a representation. Then \( \tau(I) = I \). For each minimal orthogonal projection operator \( E_0 \in A \), if \( I - E_0 \) is a minimal orthogonal projection operator in \( A \), then \( I = E_0 + (I - E_0) \). If \( I - E_0 \) is not a minimal orthogonal projection operator in \( A \), then there are two non-zero orthogonal projection operators \( E_1, E_2 \in A \) such that \( E_1 + E_2 = I - E_0 \). If \( E_1 \) is not a minimal orthogonal projection operator in \( A \), then there are two non-zero orthogonal projection operators \( E_3, E_4 \in A \) such that \( E_3 + E_4 = E_1 \). The process continues and we show that \( I \) is a sum of finite minimal orthogonal projection operators in \( A \). Note that \( \tau(I) = I \), therefore, there is a minimal orthogonal projection operator \( E \) in \( A \) such that \( \tau(E) \) is not zero.

If the rank of \( \tau(F) \) is larger than 1, then we have two different orthogonal unit vectors \( x \) and \( y \) in the range of \( \tau(E) \). For any \( A \in A \), there is a scalar \( \lambda \) such that \( EAE = \lambda E \). Hence
\[
\langle \tau(A)x, y \rangle = \langle \tau(EAE)x, y \rangle = \langle \lambda x, y \rangle = 0.
\]
Thus, \( \tau(A)x \) is a proper invariant subspace of \( \tau(A) \). Therefore \( \tau \) is not irreducible.

\[ \square \]

**Lemma 3.9.** Let \( L, \mathcal{H} \) be two finite dimensional complex Hilbert spaces, \( A \subseteq \text{End}(L) \) be a matrix von Neumann algebra, \( \tau \) be a representation of \( A \) on \( \mathcal{H} \). Then \( \tau \) is an irreducible representation if and only if \( (\tau(A))' = CI \).

**Proof.** If \( (\tau(A))' \neq CI \), then \( (\tau(A))' \) contains a non-scalar positive operator. So, it contains an orthogonal projection operator \( 0 < P < I \). Thus, \( P(\mathcal{H}) \) is a proper invariant subspace of \( \tau(A) \), which leads to a contradiction.

Conversely, suppose that \( \mathcal{H}_1 \) is a proper invariant subspace of \( \tau(A) \). Let \( P \) be the orthogonal projection operator onto \( \mathcal{H}_1 \). Then \( P \in \tau(A)' \) and \( P \) is a non-scalar operator in \( (\tau(A))' \). This is a contradiction. \[ \square \]

**Lemma 3.10.** Let \( \mathcal{H} \) be a finite dimensional complex Hilbert space. Then the only irreducible matrix von Neumann subalgebras of \( \text{End}(\mathcal{H}) \) is itself.

**Proof.** First note that \( \text{End}(\mathcal{H})' = CI \). So \( \text{End}(\mathcal{H}) \) is an irreducible matrix von Neumann algebra. If \( A \subseteq \text{End}(\mathcal{H}) \) is an irreducible matrix von Neumann subalgebra of \( \text{End}(\mathcal{H}) \), then it follows from the above lemma that \( (\tau(A))' = A' = CI \), where \( \tau \) is the identity representation. Since \( A \) is a matrix von Neumann algebra, \( A = A'' \). Thus, \( A = (A')' = (CI)' = \text{End}(\mathcal{H}) \). \[ \square \]
**Lemma 3.11.** Let \( \mathcal{L}, \mathcal{H} \) be two finite dimensional complex Hilbert spaces, \( \mathcal{A} \subseteq \text{End}(\mathcal{L}) \) be a matrix von Neumann algebra, \( \tau \) be a representation of \( \mathcal{A} \) on \( \mathcal{H} \). Then there is an irreducible subrepresentation of \( \tau \) which is unitarily equivalent to the restriction of \( \mathcal{A} \) to its a minimal invariant subspace.

**Proof.** Let \( \tau \) be a representation of \( \mathcal{A} \). Then there is a minimal projection \( E \) such that \( P = \tau(E) \) is non-zero. Let \( f \) be a unit vector in the range of \( P \) and denote \( \mathcal{H}_f^\tau = \tau(\mathcal{A})f \).

In addition, let \( e \) be a unit vector in the range of \( E \) with \( \mathcal{L}_e = \mathcal{A}e \). The subspaces \( \mathcal{L}_e \) and \( \mathcal{H}_f^\tau \) are invariant for \( \mathcal{A} \) and \( \tau(\mathcal{A}) \), respectively. Since \( EAE = CE \), \( \phi \) defined by \( \phi(A) = \langle Ae, e \rangle \) satisfies

\[
EAE = \phi(A)E, A \in \mathcal{A}.
\]

In fact, since \( EAE = CE \), \( EAEe = \lambda e \). Hence \( EAE = \lambda e \). Thus, \( \langle E Ae, e \rangle = \lambda \). Therefore, \( \langle Ae, e \rangle = \lambda \).

Define a linear operator \( U \) from \( \mathcal{L}_e \) to \( \mathcal{H}_f^\tau \) by

\[
U(Ae) := \tau(A)f = \tau(AE)f.
\]

It is easy to compute that for \( A, B \in \mathcal{A} \),

\[
\langle U(Ae), U(Be) \rangle = \langle \tau(AE)f, \tau(BE)f \rangle = \langle \tau(EB^\dagger AE)f, f \rangle
\]

\[
= \phi(B^\dagger A)\langle Pf, f \rangle = \phi(B^\dagger A)\langle f, f \rangle = \langle B^\dagger Ae, e \rangle = \langle Ae, Be \rangle.
\]

Consequently, \( U \) is an isometry from \( \mathcal{L}_e \) onto \( \mathcal{H}_f^\tau \).

For \( A, B \in \mathcal{A} \), we have

\[
\tau(A)(UBe) = \tau(A)f = \tau(AB)f = U(ABe) = UAU^\dagger(UBe).
\]

Thus, \( \tau(A)|_{\mathcal{H}_f^\tau} = UA|_{\mathcal{L}_e}U^\dagger \) for every \( A \in \mathcal{A} \), that is, \( A \rightarrow \tau(A)|_{\mathcal{H}_f^\tau} \) is a representation of \( \mathcal{A} \) on \( \mathcal{H}_f^\tau \) which is a subrepresentation of \( \tau \). Moreover, this representation is unitary equivalent to the restriction of \( \mathcal{A} \) to \( \mathcal{L}_e \).

Let \( \tau_e \) be the restriction map of \( \mathcal{A} \) to \( \mathcal{L}_e \), that is, for every \( A \in \mathcal{A} \), \( \tau_e(A) = A|_{\mathcal{L}_e} \). Then \( \tau_e \) is a representation of \( \mathcal{A} \) on \( \mathcal{L}_e \) and \( \tau_e(E) = |e\rangle \langle e| \). Indeed, it is easy to show that \( \tau_e \) is a representation. So \( \tau_e(E) \) is an orthogonal projection operator in \( \text{End}(\mathcal{L}_e) \). Moreover, since \( \mathcal{L}_e = \{ Ae : A \in \mathcal{A} \} \) and \( EAE = EAEe \) belongs to \( Ce \) for every \( A \in \mathcal{A} \), \( Ce \) is the range of \( \tau_e(E) \). Then \( \tau_e(E) = |e\rangle \langle e| \). Suppose that \( P_0 \) is an orthogonal projection operator in \( \text{End}(\mathcal{L}_e) \) which commutes with \( \tau_e(A) \). Then

\[
P_0e = P_0\tau_e(E)e = \tau_e(E)P_0e.
\]
Note that \( \tau_e(E) = |e\rangle \langle e| \). So \( P_0e \) is a multiple of \( e \). Since \( P_0e = P_0^2e \), this multiple is 0 or 1. In addition, \( P_0^\perp \) commutes with \( \tau_e(A) \), too. Considering \( P_0^\perp \) if necessary, we may suppose that \( P_0e = 0 \). While

\[
P_0 Ae = P_0 \tau_e(A)e = \tau_e(A)P_0e = 0
\]

for all \( A \in \mathcal{A} \). Hence \( P_0 = 0 \). Therefore \( \tau_e(A)' = CI_{L_e} \) and \( \tau_e \) is irreducible. Therefore, the subspace \( L_e \) is an invariant subspace of \( \mathcal{A} \). Moreover, the subspace \( L_e \) is also a minimal invariant subspace of \( \mathcal{A} \). If \( V_1 \subseteq L_e \) is an invariant nonzero subspace of \( \mathcal{A} \), then the orthogonal projection operator \( P_1 \) on \( V_1 \) is an element of \( \mathcal{A}' \), note that \( P_1 = \tau_e(P_1) \in \tau_e(A)' = CI_{L_e} \), thus, we have \( P_1 = I_{L_e} \), therefore, \( V_1 = L_e \). \( L_e \) is a minimal invariant subspace of \( \mathcal{A} \) is proved.

\[\square\]

**Corollary 3.12.** Let \( \mathcal{H} \) be a finite dimensional complex Hilbert space. Then every irreducible representation of \( \text{End}(\mathcal{H}) \) is unitary equivalent to the identity representation of \( \text{End}(\mathcal{H}) \) on \( \mathcal{H} \).

*Proof.* The only non-zero invariant subspace for \( \text{End}(\mathcal{H}) \) is \( \mathcal{H} \).

\[\square\]

**Theorem 3.13.** Let \( \mathcal{K} \) be a finite dimensional complex Hilbert space, \( \mathcal{A} \subseteq \text{End}(\mathcal{K}) \) be a matrix von Neumann algebra. Then every representation \( \tau \) of \( \mathcal{A} \) on the Hilbert space \( \mathcal{H} \) is the direct sum of irreducible representations which are unitarily equivalent to subrepresentations of the identity representation.

*Proof.* According to the above lemma, \( \mathcal{H} \) contains a subspace \( \mathcal{H}_f^\tau \) such that the restriction of \( \tau(\mathcal{A}) \) to \( \mathcal{H}_f^\tau \) is an irreducible representation which is unitarily equivalent to a subrepresentation \( \tau_e \) of the identity representation. By induction, we can get a maximal family \( \{\mathcal{H}_n^\tau\} \) of pairwise orthogonal invariant subspaces with this property. Then \( \mathcal{H} \) is spanned by these \( \{\mathcal{H}_n^\tau\} \). Otherwise, the complement \( (\sum_n \mathcal{H}_n^\tau)^\perp \) would get another such subspace by the above lemma, which contradicts the maximality of the original family. Let \( \tau_n \) be the restriction of \( \tau \) to \( \mathcal{H}_n^\tau \). Then we obtain the decomposition \( \tau = \sum_n \oplus \tau_n \), and every \( \tau_n \) is equivalent to an irreducible subrepresentation of the identity.

\[\square\]

**Theorem 3.14.** Let \( \mathcal{H} \) be a finite dimensional complex Hilbert space, \( \mathcal{A} \subseteq \text{End}(\mathcal{H}) \) be a matrix von Neumann algebra. Then there are Hilbert spaces \( \mathcal{H}_i \) of dimensional \( n_i \) and positive integers \( k_i \) such that

\[
\mathcal{H} \cong \mathcal{H}_0 \bigoplus_i \mathcal{H}_i \otimes \mathbb{C}^{k_i}
\]

61
and

$$\mathcal{A} \cong \bigoplus_i \text{End}(H_i) \otimes I_{\mathbb{C}^{k_i}}.$$  

Proof. Let $\mathcal{H}_0 = \bigcap_{A \in \mathcal{A}} \ker A$. It is clear that $\mathcal{H}_0$ is an invariant subspace of $\mathcal{A}$. So $\mathcal{H}_0^\perp$ is also an invariant subspace of $\mathcal{A}$. Consider the identity representation $I$ of $\mathcal{A}$ on $\mathcal{H}$. For every $A \in \mathcal{A}$, the restriction of $A$ on $\mathcal{H}_0^\perp$ is a representation of $\mathcal{A}$ on $\mathcal{H}_0^\perp$. It follows from the above theorem that the Hilbert space $\mathcal{H}_0^\perp$ may be decomposed as a direct sum of invariant subspaces $\mathcal{L}_j$ of $\mathcal{A}$ such that the restriction $\sigma_j$ of $A$ to $\mathcal{L}_j$ is irreducible.

On the other hand, note that the only irreducible subalgebra of $\text{End}(\mathcal{L}_j)$ is itself. So $\sigma_j(A) = \text{End}(\mathcal{L}_j)$. Now, we define an equivalent relation $\simeq$ on $\{\mathcal{L}_j\}$: $\mathcal{L}_{j_1} \simeq \mathcal{L}_{j_2}$ if and only if $\sigma_{j_1}$ is unitarily equivalent to $\sigma_{j_2}$. For every equivalent class $S_i$ of $\{\mathcal{L}_j\}$, let $\mathcal{H}_i$ be a Hilbert space of dimension $n_i = \dim(\mathcal{L}_j)$, $\mathcal{L}_j \in S_i$, $k_i$ be the cardinality of $S_i$, $\rho_i$ be the identity representation of $\text{End}(\mathcal{H}_i)$ on $\mathcal{H}_i$. Then it is evident that

$$\mathcal{H} = \mathcal{H}_0 \bigoplus \bigoplus_i \mathcal{H}_i \otimes I_{\mathbb{C}^{k_i}}$$

and

$$\mathcal{I} = 0 \bigoplus \bigoplus_i \rho_i \otimes I_{\mathbb{C}^{k_i}}.$$ 

Thus, by $H_i^{\otimes k_i} \cong H_i \otimes \mathbb{C}^{k_i}$ and $\rho_i^{\otimes k_i} \cong \rho_i \otimes I_{\mathbb{C}^{k_i}}$, we have

$$\mathcal{H} \cong \mathcal{H}_0 \bigoplus \bigoplus_i \mathcal{H}_i \otimes \mathbb{C}^{k_i}$$

and

$$\mathcal{A} \cong \bigoplus_i \text{End}(H_i) \otimes I_{\mathbb{C}^{k_i}}.$$ 

$$\boxdot$$

3.2 Schur-Weyl duality

Theorem 3.15 (The dual theorem). Let $G$ be a finite group, $V$ be a $\mathbb{C}[G]$-module and $\pi_V$ be the action of $G$ on $V$, $\{V_a\}_{a=1}^k$ be a complete set of non-isomorphic irreducible $\mathbb{C}[G]$-modules, $\mathcal{A}$ be the matrix algebra generalized by $\{\pi_V(g) : g \in G\}$, $\mathcal{B} = \mathcal{A}'$. If $V = \bigoplus_{a=1}^k V_a^{\otimes n_a} \cong$
\[ \bigoplus_{\alpha=1}^{k} V_\alpha \otimes C^{n_\alpha}, \]  

then

\[
A \cong \bigoplus_{\alpha=1}^{k} \text{End}(V_\alpha) \otimes 1_{C^{n_\alpha}}, \tag{3.1}
\]

\[
B \cong \bigoplus_{\alpha=1}^{k} 1_{V_\alpha} \otimes \text{End}(C^{n_\alpha}). \tag{3.2}
\]

**Proof.** First, note that for every \( x \in V, x = \sum_{\alpha=1}^{k} \sum_{j=1}^{n_\alpha} x_j^{(\alpha)} \), where \( x_j^{(\alpha)} \in V_\alpha \). Thus, for every \( g \in G, gx = \pi_V(g)x = \sum_{\alpha=1}^{k} \sum_{j=1}^{n_\alpha} gx_j^{(\alpha)} = \sum_{\alpha=1}^{k} \sum_{j=1}^{n_\alpha} \pi_V(\alpha)x_j^{(\alpha)} \), we have

\[
\pi_V = \bigoplus_{\alpha=1}^{k} \pi_{V_\alpha} \otimes 1_{C^{n_\alpha}}.
\]

Thus, we get

\[
A = \text{span}\{\pi_V(g) : g \in G\} \subseteq \bigoplus_{\alpha=1}^{k} \text{span}\{\pi_{V_\alpha}(g) \otimes 1_{C^{n_\alpha}} : g \in G\} = \bigoplus_{\alpha=1}^{k} \text{End}(V_\alpha) \otimes 1_{C^{n_\alpha}}.
\]

On the other hand, it is not difficult to see that

\[
\frac{d_\alpha}{|G|} \sum_{g \in G} V_{\alpha,ij}(g)\pi_V(g) \in A.
\]

It follows from

\[
\frac{d_\alpha}{|G|} \sum_{g \in G} V_{\alpha,ij}(g)\pi_V(g) = \frac{d_\alpha}{|G|} \sum_{g \in G} V_{\alpha,ij}(g) \left( \bigoplus_{\beta=1}^{k} \pi_{V_\beta}(g) \otimes 1_{C^{n_\beta}} \right) \tag{3.3}
\]

\[
= \bigoplus_{\beta=1}^{k} \left( \frac{d_\alpha}{|G|} \sum_{g \in G} V_{\alpha,ij}(g)\pi_{V_\beta}(g) \right) \otimes 1_{C^{n_\beta}} \tag{3.4}
\]

\[
= \bigoplus_{\beta=1}^{k} \left[ \frac{d_\alpha}{|G|} \sum_{g \in G} V_{\alpha,ij}(g)V_{\beta,kl}(g)E_{\beta,kl} \right] \otimes 1_{C^{n_\beta}} \tag{3.5}
\]

\[
= \left[ \frac{d_\alpha}{|G|} \sum_{g \in G} V_{\alpha,ij}(g)V_{\alpha,ij}(g) \right] E_{\alpha,ij} \otimes 1_{C^{n_\alpha}} = E_{\alpha,ij} \otimes 1_{C^{n_\alpha}} \tag{3.6}
\]

that \( E_{\alpha,ij} \otimes 1_{C^{n_\alpha}} \in A \). Hence \( \text{End}(V_\alpha) \otimes 1_{C^{n_\alpha}} \subseteq A \). Thus, we get that

\[
A = \text{span}\{\pi_V(g) : g \in G\} \cong \bigoplus_{\alpha=1}^{k} \text{span}\{\pi_{V_\alpha}(g) \otimes 1_{C^{n_\alpha}} : g \in G\} = \bigoplus_{\alpha=1}^{k} \text{End}(V_\alpha) \otimes 1_{C^{n_\alpha}}.
\]
Clearly \( \bigoplus_{a=1}^{k} 1_{V_a} \otimes \text{End}(C^{n_a}) \subseteq A' = B. \) In order to see \( B \subseteq \bigoplus_{a=1}^{k} 1_{V_a} \otimes \text{End}(C^{n_a}) \), consider the projection \( p_a \) onto \( V_a \otimes C^{n_a} \), that is, \( p_a = \frac{\dim(V_a)}{|G|} \sum_{g \in G} \chi_a(g) \pi_V(g) \). Note that \( \frac{\dim(V_a)}{|G|} \sum_{g \in G} \chi_a(g) g \in Z(C[G]) \). So \( p_a \in A' \). On the other hand, it follows from the above proof that \( p_a \in A \), too. The projectors \( \{p_a\}_{a=1}^{k} \) form a resolution of the identity. Since \( A' = B \), any \( B \in B \) must commute with \( p_a \), that is, \( p_a B = B p_a \). This leads to

\[
B = \left( \sum_a p_a \right) B = \sum_a p_a B p_a = \sum_a B p_a.
\]

Moreover, \( B_a = p_a B p_a \), \( p_a \in A' \), \( A' = B \), so \( B_a \in A' \). Note that \( \text{End}(V_a) \otimes 1_{C^{n_a}} \subseteq A \). Therefore, \( B_a \in (\text{End}(V_a) \otimes 1_{C^{n_a}})' = 1_{V_a} \otimes \text{End}(C^{n_a}) \). It must be of the form \( B_a = 1_{V_a} \otimes D_a \), where \( D_a \in \text{End}(C^{n_a}) \). The proof completes.

Let \( U(d) \) be the all unitary operators on the Hilbert space \( C^d, \{ |1\rangle, \ldots, |d\rangle \} \) be the orthonormal basis of \( C^d \). For an ordered \( k \)-tuple \( I = (i_1, \ldots, i_k) \) with \( i_1, \ldots, i_k \in [d] = \{1, \ldots, d\} \), define \( |I\rangle := |i_1 \cdots i_k\rangle \). Then \( \{|I\rangle : I \in [d]^k \} \) is an orthonormal basis of \( (C^d)^\otimes k \).

The group \( S_k \) permutes this basis by \( P(\pi)|I\rangle = |\pi \cdot I\rangle \), where for \( I = (i_1, \ldots, i_k) \) and \( \pi \in S_k \). Define

\[
\pi \cdot (i_1, \ldots, i_k) := (i_{\pi^{-1}(1)}, \ldots, i_{\pi^{-1}(k)}).
\]

We have \((\sigma \pi) \cdot I = \sigma \cdot (\pi \cdot I)\) for \( \sigma, \pi \in S_k \).

Now consider the following representations: \( Q : U(d) \to \text{GL}((C^d)^\otimes k) \) for \( Q(U) = U^\otimes k \), where

\[
Q(U)|i_1 i_2 \cdots i_k\rangle = U^\otimes k|i_1 i_2 \cdots i_k\rangle = (U|i_1\rangle)(U|i_2\rangle) \cdots (U|i_k\rangle) \quad \forall U \in U(d),
\]

and \( P : S_k \to \text{GL}((C^d)^\otimes k) \) for

\[
P(\pi)|i_1 i_2 \cdots i_n\rangle = |i_{\pi^{-1}(1)} \cdots i_{\pi^{-1}(k)}\rangle \quad \forall \pi \in S_k.
\]

It is easy to show that \( Q(U), P(\pi) \in U(d^k) \) and \( Q(U)P(\pi) = P(\pi)Q(U) \).

In order to prove the Schur theorem, first, we need the following result:

**Lemma 3.16.** \( \text{GL}(d) \) is dense in \( \text{End}(C^d) \).

**Proof.** In fact, for any \( A \in \text{End}(C^d) \), by the Singular Valued Decomposition theorem we have \( A = UDV^\dagger \), where \( U, V \in U(d) \) and \( D \) is a nonnegative diagonal matrix. Define \( A_\varepsilon := U(D + \varepsilon \frac{\mathbb{1}}{2})V^\dagger \) for \( \varepsilon > 0 \). Clearly \( A_\varepsilon \in \text{GL}(d) \) and

\[
\| A - A_\varepsilon \| < \varepsilon.
\]
This indicates that \( \text{GL}(d) \) is dense in \( \text{End}(\mathbb{C}^d) \).

\[ \square \]

**Lemma 3.17.** For every \( A \in \text{GL}(n) \), there is an \( X \in \text{End}(\mathbb{C}^n) \) such that \( e^X = A \).

**Proof.** It is easy to show that if \( A \) is a diagonal matrix, then the conclusion is true. For general cases, we prove it in following two steps:

(I). There is a sequence of diagonal matrices \( D_k \) satisfying that

1. \( \lim_k D_k = A \).
2. If \( D_k = e^{X_k} \), then there is a constant \( M \) such that \( \|X_k\| \leq M \) for every \( k \).

(II). When (I) holds, since the exponential function is a continuous function the image of the compact set \( \exp(B(0,M)) \) must be closed, where \( B(0,M) \) is the closed ball with radius \( M \). Thus, the limit of \( e^{X_k} \) is also in \( \exp(B(0,M)) \). This means that there exists an \( X \in B(0,M) \) such that \( e^X = A \).

Now what we need prove is to show the existence of \( D_k \) in (I).

Consider the Jordan decomposition of \( A \) for \( A = P^{-1}JP \). Let \( j_i \) be the diagonal elements of \( J \). Note that \( A \) is an invertible matrix. So \( j_i \neq 0, 1 \leq i \leq n \). Let \( D_k = P^{-1}(J + \Lambda_k)P \), \( \Lambda_k = \text{diag}(\lambda_k^1, \lambda_k^2, \ldots, \lambda_k^n) \). Then \( D_k \) matches the conditions in (I) if

(i). \( \lim_k \lambda_i^k = 0, i = 1, 2, \ldots, n \),

(ii). \( j_i + \lambda_i^k \) are all different when \( i \) runs from 1 to \( n \) for every given \( k \). Thus, \( D_k \) has \( n \) different eigenvalues \( j_i + \lambda_i^k \), and \( D_k \) is diagonalizable.

(iii). There is a constant \( m \) such that \( |\ln(j_i + \lambda_i^k)| \leq m \) for every \( k \) and \( i \).

Note that if (ii) is true, then \( \|X_k\| = \max_i |\ln(j_i + \lambda_i^k)| \). Now, we give the \( \lambda_i^k \) satisfying (i)-(iii) to complete the proof.

For any given \( k \), let \( \lambda_1^k = \frac{j_1}{k} \), and \( \lambda_i^k \) be one of \( \frac{j_i}{k}, \frac{j_i}{k + 1}, \frac{j_i}{k + 2}, \ldots, \frac{j_i}{k + i} \) such that \( j_i + \lambda_i^k \) is different from \( j_t + \lambda_t^k \) for every \( t < i \).

It is easy to see (i) and (ii) hold. Let \( m = \max_i |\ln(j_i)| + \ln 2 \). We have

\[
|\ln(j_i + \lambda_i^k)| = |\ln(j_i) + \ln(1 + \frac{\lambda_i^k}{j_i})| \leq |\ln(j_i)| + |\ln(1 + \frac{\lambda_i^k}{j_i})|.
\]

The proof is completed. \( \square \)

**Theorem 3.18** (Schur). Let \( \mathcal{A} = \text{span} \{P(\pi) : \pi \in S_k\} \), \( \mathcal{B} = \text{span} \{Q(U) : U \in U(d)\} \).

Then

\[ \mathcal{A}' = \mathcal{B} \]
Moreover, $A$ is a matrix von Neumann algebra and $B' = A$.

Proof. Since $B \subseteq A'$, we only need to prove that $A' \subseteq B$. Note that for every $X \in \text{End}(\mathbb{C}^d)$,
\[
\frac{d}{dt} \bigg|_{t=0} (e^{tX})^{\otimes k} = \sum_{j=1}^{k} 1^{\otimes j-1} \otimes X \otimes 1^{\otimes k-j}.
\]

Denote
\[
dU(X) = \frac{d}{dt} \bigg|_{t=0} (e^{tX})^{\otimes k}.
\]

It is clear that $dU(X + iY) = dU(X) + idU(Y)$, that is, $dU$ is a linear operator.

On the other hand, if $X^\dagger = -X$, that is, $X$ is a skew-Hermitian operator, then $e^{tX}$ is a unitary operator. So $(e^{tX})^{\otimes k} \in B$. Thus, if $X^\dagger = -X$ then
\[
\frac{d}{dt} \bigg|_{t=0} (e^{tX})^{\otimes k} \in \text{span}\{U^{\otimes k} : U \in U(d)\} = B.
\]

For arbitrary $M$, we have decomposition $M = X + iY$, $X^\dagger = -X$, $Y^\dagger = -Y$. It follows from $dU(X), dU(Y) \in B$ and $dU(M) = dU(X) + idU(Y)$ that $dU(M) \in B$. For any $T \in \text{GL}(d)$, there exists an $M \in \text{End}(\mathbb{C}^d)$ such that $T = e^M$. Note that
\[
T^{\otimes k} = (e^M)^{\otimes k} = \exp \left( \sum_{j=1}^{k} 1^{\otimes j-1} \otimes M \otimes 1^{\otimes k-j} \right) = \exp(dU(M)) \in B,
\]

which implies that
\[
\{T^{\otimes k} : T \in \text{GL}(d)\} \subseteq B.
\]

So far, we have shown that
\[
\{U^{\otimes k} : U \in U(d)\} \subseteq \{T^{\otimes k} : T \in \text{GL}(d)\} \subseteq B.
\]

Since $\text{GL}(d)$ is dense in $\text{End}(\mathbb{C}^d)$ and $B$ is closed, we have
\[
\{Z^{\otimes k} : Z \in \text{End}(\mathbb{C}^d)\} \subseteq B.
\]

Denote $\Delta := \text{span} \{Z^{\otimes k} : Z \in \text{End}(\mathbb{C}^d)\}$. Then $\Delta \subseteq B$.

Let $B \in A'$. By the definition of $A$, $B$ commutes with all $P(\pi)$. Therefore $B \in \Delta$. In summary $A' \subseteq \Delta \subseteq B$. That is $A' \subseteq B$.

It follows from the proof of the above theorem:

**Corollary 3.19.**
\[
\text{span} \{Q(U) : U \in U(d)\} = \text{span} \{Q(T) : T \in \text{GL}(d)\} = \text{span}\{Q(Z) : Z \in \text{End}(\mathbb{C}^d)\}.
\]

66
4 Schur-Weyl duality of symmetric and unitary groups

4.1 Some elementary notations of the symmetric groups

Definition 4.1. (1) A positive integers set \( \lambda = (\lambda_1, \ldots, \lambda_d) \) with \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_d \) is called a Young diagram, it is usually illustrated by a diagram consisting of empty boxes arranged in rows, which are left-adjusted. The \( i \)-th row, counted from the top, consists of \( \lambda_i \) boxes.

(2) The size of \( \lambda \) is defined by \( |\lambda| = \sum_i \lambda_i = k \), and \( \lambda \) is said to be a partition of \( k \), denoted by \( \lambda \vdash k \).

(3) A Young diagram \( \lambda \) containing integers \( t_{ij} \) is known as \( \lambda \) shape Young tableau \( t = (t_{ij}) \), denoted by \( t^\lambda \), where \( t_{ij} \in \{1, 2, \ldots, k\} \).

Given a Young diagram \( \lambda = (\lambda_1, \ldots, \lambda_d) \), the conjugate Young diagram \( \lambda' = (\lambda'_1, \ldots, \lambda'_r) \) of \( \lambda \) is defined by interchanging rows and columns in the Young diagram \( \lambda \). For example, given \((3, 3, 2, 1, 1)\), whose conjugate is \((5, 3, 2)\).

Definition 4.2. (1) A Standard \( \lambda \) shape Young tableau \( t \) is a \( \lambda \) shape Young tableau with the numbers from 1 to \( k = |\lambda| \) such that the numbers are strictly increasing from left to right and from upwards to downwards.

(2) A Semi-standard \( \lambda \) shape Young tableau \( t \) is a \( \lambda \) shape Young tableau containing numbers, possibly repeatedly, that is increase from left to right and strictly increase from upwards to downwards.

Definition 4.3. Two \( \lambda \) shape Young tableaus \( t_1 \) and \( t_2 \) are row equivalent, denoted by \( t_1 \sim t_2 \), if corresponding rows of the two tableaux contain the same elements. A \( \lambda \)-tabloid is defined by \( \{t\} = \{t_1 : t_1 \sim t\} \).

If \( \lambda = (\lambda_1, \ldots, \lambda_d) \vdash k \), then the number of tableaux in any given equivalence class is \( \lambda! := \lambda_1!\lambda_2! \cdots \lambda_d! \). Thus, the number of \( \lambda \)-tabloids is just \( \frac{k!}{\lambda!} \).

Definition 4.4. Let \( k \) be a positive integer, \( S_k \) is the symmetric group of \( \{1, 2, \ldots, k\} \). If \( \lambda = (\lambda_1, \ldots, \lambda_d) \vdash k \), then for every \( \pi \in S_k \) and \( \lambda \) shape Young tableaux \( t = (t_{ij}) \), we define \( \pi t = (\pi(t_{ij})) \).
Definition 4.5. Let $\lambda = (\lambda_1, \ldots, \lambda_d) \vdash k$. The corresponding Young subgroup of $S_k$ is defined by

$$S_{(\lambda_1, \ldots, \lambda_d)} = S_{\{1, \ldots, \lambda_1\}} \times S_{\{\lambda_1+1, \lambda_1+2, \ldots, \lambda_2\}} \times \cdots \times S_{\{\lambda_{k-\lambda_d+1}, \ldots, \lambda_d\}}.$$ 

It is clear that $S_{(\lambda_1, \ldots, \lambda_d)}$ is isomorphism to group $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_d}$.

For every permutation $\pi \in S_k$, we can display $\pi$ using cycle notation. Given $i \in \{1, \ldots, k\}$, the elements of the sequence $i, \pi(i), \pi^2(i), \ldots$ cannot all be distinct. Taking the first power $p$ such that $\pi^p(i) = i$, we have the cycle

$$(i, \pi(i), \pi^2(i), \ldots, \pi^{p-1}(i)).$$

Equivalently, the cycle $(i, j, \ldots, l)$ means that $\pi$ sends $i$ to $j$, and $l$ back to $i$. Now pick an element not in the cycle containing $i$ and iterate this process until all members of $\{1, \ldots, k\}$ have been used. For example, $\pi \in S_5$, $\pi = (1, 2, 3)(4)(5)$ in cycle notation. Note that cyclically permuting the elements within a cycle or reordering the cycles themselves does not change the permutation. Thus

$$(1, 2, 3)(4)(5) = (2, 3, 1)(4)(5) = (4)(2, 3, 1)(5) = (4)(5)(3, 1, 2).$$

A $p$-cycle, or cycle of length $p$, is a cycle containing $p$ elements. The cycle type, or simply the type, of $\pi$ is an expression of the form

$$(1^{m_1}, 2^{m_2}, \ldots, k^{m_k}),$$

where $m_i$ is the number of cycles of length $i$ in $\pi$. A 1-cycle of $\pi$ is called a fixed-point. Fixed-points are usually dropped from the cycle notation if no confusion will result. It is easy to see that a permutation $\pi$ satisfies that $\pi^2 = 1$ if and only if all cycles of $\pi$ have length 1 or 2.

Now, we show that the cycle type can be corresponded by the Young diagrams.

In fact, if the type of $\pi$ is $(1^{m_1}, 2^{m_2}, \ldots, k^{m_k})$, taking the all nonzero $m_i$, $l = 1, 2, \cdots, d$, then the Young diagram $\lambda = (\lambda_1, \ldots, \lambda_d) = (i_d, i_{d-1}, \cdots, i_1)$. That is, the Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ of $\pi$ can be obtained by lining the length of cycles from bigger to smaller.

For example, $\pi = (1, 2, 3)(4)(5)$ corresponds to a Young diagram $(3, 1, 1)$, and a cycle type $(1^2, 2^0, 3^1, 4^0, 5^0)$. In $S_k$, if $\pi = (i_{11}, i_{12}, \ldots, i_{1n_1}) \cdots (i_{m_1}, i_{m_2}, \ldots, i_{m_{1m}})$ in cycle notation, then for any $g \in S_k$, note that $g\pi g^{-1}(g(i_{11})) = g\pi(i_{11}) = g(i_{12})$, thus, it is not hard to see that

$$g\pi g^{-1} = (g(i_{11}), g(i_{12}), \ldots, g(i_{1n_1}))(g(i_{m_1}), g(i_{m_2}), \ldots, g(i_{m_{1m}})).$$

68
Therefore, two permutations are in the same conjugate class if and only if they have the same cycle type. Thus, there is a natural 1-1 correspondence between partitions of k and conjugate classes of $S_k$.

Let G be a group. Recall that the conjugate class $g^G$ of g and the centralizer $C_G(g)$ have the relationship of

$$|g^G| = \frac{|G|}{|C_G(g)|}.$$ 

Now we compute the size of a conjugate class in $S_k$.

Let $G = S_k$. If $\pi \in S_k$ has the type $(1^{m_1}, 2^{m_2}, \ldots, k^{m_k})$, then

$$|C_{S_k}(\pi)| = 1^{m_1}m_1!2^{m_2}m_2!\cdots k^{m_k}m_k!.$$ 

Indeed, any $h \in C_{S_k}(\pi)$ can either permute the cycles of length i among themselves or perform a cyclic rotation on every of the individual cycles. Since there are $m_i!$ ways to do the former operation and $i^{m_i}$ ways to do the latter, we are done.

Thus,

$$|\pi^{S_k}| = \frac{k!}{1^{m_1}m_1!2^{m_2}m_2!\cdots k^{m_k}m_k!}.$$ 

4.2 The irreducible submodules of $C[S_k]$

Definition 4.6. Suppose that the $\lambda$ shape Young tableaux $t$ has rows $R_1, R_2, \ldots, R_d$ and columns $C_1, C_2, \ldots, C_l$. Then

$$R(t) = S_{R_1} \times S_{R_2} \cdots S_{R_d}$$ 

and

$$C(t) = S_{C_1} \times S_{C_2} \cdots S_{C_l}$$

are the row-stabilizer and column-stabilizer of $t$, respectively, where $S_{R_i}$ is the symmetric group of $R_i$, $S_{C_j}$ is the symmetric group of $C_j$.

Definition 4.7. To every $\lambda$ shape Young tableau $t$ with $|\lambda| = k$, associate two elements of the group algebra $C[S_k]$,

$$r_t := \sum_{\pi \in R(t)} \pi, \quad c_t := \sum_{\pi \in C(t)} \text{sign}(\pi) \pi.$$ 

The Young symmetrizer $e_t$ is given by

$$e_t := r_tc_t.$$ (4.1)
Lemma 4.8. Let $t$ be a $\lambda$ shape Young tableau and $\pi$ be a permutation. Then

$$R(\pi t) = \pi R(t)\pi^{-1}, \quad C(\pi t) = \pi C(t)\pi^{-1}, \quad e_{\pi t} = \pi e_t\pi^{-1}. $$

Proof. In fact, we have the following list of equivalent statements:

$$\sigma \in R(\pi t) \iff \sigma\{\pi t\} = \{\pi t\} \iff \pi^{-1}\sigma\pi \in R(t) \iff \sigma \in \pi R(t)\pi^{-1}. $$

Another two equivalents can be proved similarly.

Lemma 4.9. Let $t$ and $t'$ be two $\lambda$ shape Young tableaus, and $t' = gt$ for some permutation $g \in S_k$. If there are no two integers that are not only in the same row of $t$, but also in the same column of $t'$, then $g = rc$ for some permutations $r \in R(t)$ and $c \in C(t)$. In particular, if $g \neq rc$, then there is a transposition $\sigma \in R(t)$ with $g^{-1}\sigma g \in C(t)$.

Proof. Note that there are no two integers that are not only in the same row of $t$, but also in the same column of $t'$, so, we can get $r_1 \in R(t)$ and $c'_1 \in C(t')$ such that $r_1 t$ and $c'_1 t'$ have the same first row. Now repeat this process continuously until the row $d$, then $r = r_d \cdots r_1 \in R(t), c' = c'_d \cdots c'_1 \in C(t')$ and

$$rt = c't' = c'gt.$$ 

Hence $r = c'g$ and thus $r^{-1} = g^{-1}c'^{-1}$, that is, $r^{-1}g = g^{-1}c'^{-1}g := c \in C(t)$. Therefore $g = rc$. Here $g^{-1}c'^{-1}g \in C(t)$ since $C(gt) = gC(t)g^{-1}$.

In particular, if $g \neq rc$, there are two integers that not only in the same row of $t$, but also in the same column of $t'$. Without loss of generality, we assume that $a, b$ are two such integers. Let $\sigma = (a, b)$. It is clear that $\sigma \in R(t)$, and $\sigma \in C(t')$ if and only if $g^{-1}\sigma g \in C(t)$.

Lemma 4.10. Let $t$ be a $\lambda$ shape Young tableau, $r \in R(t), c \in C(t)$. Denote $c^- = \text{sign}(c)c$. Then

$$rr_t = r_tr = r_t, \quad (4.2)$$
$$c^-c_t = c_tc^- = c_t, \quad (4.3)$$
$$re_tc^- = e_tc, \quad (4.4)$$

and, up to scalar multiplication, $e_t$ is the only such element for which Equations (4.4) holds for all $r \in R(t)$ and $c \in C(t)$. Moreover, $(e_t)^2 = n_te_t$ for some $n_t \in \mathbb{C}$.
Proof. The three equivalents can be proved directly. Now, we show the uniqueness of $e_t$ in Equation (4.4).

Assume that $x = \sum_{g \in S_k} x_g g \in C[S_k]$ satisfies

$$rxc^- = x, \quad \forall r \in R(t), \ c \in C(t).$$

Since $x = \sum_{g \in S_k} x_r g r g$, so $rxc^- = r \left( \sum_{g \in S_k} x_g g \right) c^- = \sum_{g \in S_k} \text{sign}(c) x_g r g$. Put two equations together, we have

$$x_{rg} = \text{sign}(c)x_g, \quad \forall g \in S_k. \tag{4.5}$$

If $g \neq rc$, it follows from above lemma that there is a transposition $\sigma \in R(t)$ such that $\sigma' = g^{-1}\sigma g \in C(t)$ and therefore $g = \sigma g \sigma'$, which by Equation (4.5) implies that

$$x_g = x_{\sigma g \sigma'} = \text{sign}(\sigma')x_g = \text{sign}(g^{-1}\sigma g)x_g = \text{sign}(\sigma)x_g = -x_g.$$ Thus

$$x_g = 0 \quad \text{if} \ g \neq rc.$$ therefore, we have

$$x = \sum_{g \in S_k} x_g g = \sum_{r \in R(t), c \in C(t)} x_{rc} rc.$$ Letting $g = 1$ in Equation (4.5) gives rise to $x_{rc} = \text{sign}(c)x_1$. Finally we have $x = x_1 e_t$.

Moreover, note that for any $r \in R(t), c \in C(t), re'^2 c^- = e'^2$, so $e'^2 = n_t e_t$ for some $n_t \in \mathbb{C}$.

**Definition 4.11.** Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ and $\mu = (\mu_1, \ldots, \mu_p)$ be two Young diagrams. We say that $\lambda > \mu$ with respect to the lexicographical order if $|\lambda| = |\mu|$ and the first difference $\lambda_i - \mu_i > 0$.

**Corollary 4.12.** Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ and $\mu = (\mu_1, \ldots, \mu_p)$ be two Young diagrams, $|\lambda| = |\mu| = k$, $\lambda > \mu$. If $t$ is a $\lambda$ shape Young tableau, $t'$ is a $\mu$ shape Young tableau, then $r_t xc_{t'} = 0$ for all $x \in C[S_k]$, in particular, $e_i e_{i'} = 0$.

**Proof.** For $g \in S_k$, Let $t'' = gt'$. Then $t''$ is also a $\mu$ shape Young tableau. Since $\lambda > \mu$, so there are two integers in the same row of $t$ and the same column of $t''$. Let $\sigma$ be the transposition of those two integers. Then $\sigma \in R(t) \cap C(t'')$, so we have

$$r_t g c_{t'} g^{-1} = r_t c_{t''} = r_t c_{t''} = (r_t \sigma)(\sigma c_{t'}) = -r_t c_{t'}, \tag{4.6}$$
Thus, \( rt_{c_{t'}} = 0 \), that is, \( rt_{g}c_{t'}g^{-1} = 0 \), so

\[
rt_{g}c_{t'} = 0 \quad \text{for all} \ g \in S_k.
\]

Thus, we have

\[
rt_{x}c_{t'} = 0 \quad \text{for all} \ x \in C[S_k].
\]

Letting \( x = c_{t}r_{t'} \), we get that

\[
e_{t}e_{t'} = 0.
\]

**Theorem 4.13.** Let \( t \) be a standard \( \lambda \) shape Young tableau. Then \( f_{\lambda} \) is a minimal projection associated with the irreducible submodule \( V_{\lambda} \) of \( C[S_k] \), and \( \{ V_{\lambda} : \lambda \vdash k \} \) form a complete set of irreducible submodules of \( C[S_k] \), where \( \dim(V_{\lambda}) = f_{\lambda} \).

**Proof.** Since \( e_{t}^2 = n_{t}e_{t} \), so \( e_{t} \) is proportional to a projection \( e \). Now we assume that \( e = e_1 + e_2 \) for two projections \( e_1 \) and \( e_2 \). Note that \( 0 = e^2 - e = e_1e_2 + e_2e_1 \), one has \( 0 = e_1(e_1e_2 + e_2e_1)e_1 = 2e_1e_2e_1 \), so \( e_1e_2e_1 = 0 \). This shows that \( e_1e_2 = e_1^2e_2 = e_1(e_1e_2 + e_2e_1) - e_1e_2e_1 = 0 \) holds. Since \( e_1e_2 + e_2e_1 = 0 \), thus, \( e_1e_2 = 0 \) and \( e_2e_1 = 0 \). Hence \( ee_1e = e_1, ee_2e = e_2 \). Moreover, note that \( re_{t}c = e_{t} \) and \( e \) is proportional to \( e_{t} \), so

\[
re_{1}c = re_{1}ee_{1}c = ee_{1}e = e_{1},
\]

\[
re_{2}c = re_{2}ee_{2}c = ee_{2}e = e_{2},
\]

this implies that \( e_1 \) and \( e_2 \) are both proportional to \( e_{t} \). Since \( e_1e_2 = 0 \), so \( e_1 \) or \( e_2 \) must be 0. Thus, \( e \) is a minimal projection.

Now, we work out the proportionality constant as follows:

Note that for the left action \( R_{g} \) of \( g \in S_k \) on \( C[S_k] \), \( R_{g} : a \mapsto ga \), one has

\[
\text{Tr} \left( R_{g} \right) = \begin{cases} 
\frac{k!}{k!} & \text{if} \ g = 1, \\
0 & \text{if} \ g \neq 1.
\end{cases}
\]

Since

\[
e_{t} = 1 + \sum_{r \in R(t), c \in C(t):rc \neq 1} rc^{-},
\]

it follows that

\[
\text{Tr} \left( R_{e_{t}} \right) = \text{Tr} \left( R_{1} \right) = k!.
\]
But \( e_t z = n_t z \) for \( z \in e_t C[S_k] \) and 0 otherwise. Hence
\[
k! = n_t \dim(e_t C[S_k]).
\]

Note that
\[
\dim(e_t C[S_k]) = f_\lambda,
\]
thus, \( e = \frac{f_\lambda}{k!} e_t \) is a minimal projection.

\( e_t \) and \( e_{t'} \) are equivalent if \( t \) and \( t' \) have the same Young type \( \lambda \), since then \( t' = gt \) for some \( g \in S_k \), thus \( e_{t'} = ge_t g^{-1} \). But, it follows from above corollary that \( e_t \) and \( e_{t'} \) are inequivalent when their Young types are different.

Note that the number of conjugate classes of \( S_k \) equals the number of inequivalent irreducible submodules of \( C[S_k] \). Moreover, to every \( \lambda \) type standard Young tableau \( t \), we have just constructed an irreducible submodule \( e_t C[S_k] \) of \( C[S_k] \). Since there is a natural one to one correspondence between partitions of \( k \) and conjugate classes of \( S_k \), so the theorem is proved.

**Definition 4.14.** Let \( G \) be a topological group, \( M \) be a complex Hilbert space, \( \text{Inv}(M) \) be the all bounded linear inverse operators on \( M \). If \( h : G \to \text{Inv}(M) \) is a group homomorphism, and \( (g, v) \to h(g)v \) is a continuous map of \( G \times M \to M \) with respect to the norm topology of \( \text{Inv}(M) \), then we say that \((h, M)\) is a topological representation of \( G \), and \( M \) is a \( C[G] \) topological module. If every \( h(g) \in \text{Inv}(M) \) is a unitary operator, then we say that \((h, M)\) is a topological unitary representation of \( G \).

If \((h, M)\) is a topological representation of \( G \), then for every \( v \in M \), we denote \( h(g)v = gv \) for conveniently.

**Definition 4.15.** Let \( G \) be a topological group, \((h, M)\) be a topological unitary representation of \( G \). A representative function of \( G \) is a function \( G \to \mathbb{C} \) of the form
\[
f_{M; \xi, \eta}(g) = \langle \eta, g\xi \rangle.
\]
The representative functions of \( G \) form a subalgebra \( C_{alg}(G) \) of the algebra \( C(G) \) of all continuous functions on \( G \) in the following form:
\[
f_{M_1; \xi_1, \eta_1} + f_{M_2; \xi_2, \eta_2} = f_{M_1 \oplus M_2; \xi_1 \oplus \xi_2, \eta_1 \oplus \eta_2},
\]
\[
f_{M_1; \xi_1, \eta_1} \cdot f_{M_2; \xi_2, \eta_2} = f_{M_1 \otimes M_2; \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2},
\]
and
\[
\lambda f_{M; \xi, \eta} = f_{M; \lambda \xi, \eta}.
\]
Theorem 4.16. Let $G = U(d)$ be the $d$-dimensional unitary group. Then $(\mathcal{C}_{alg}(G), \| \cdot \|_\infty)$ is dense in $(\mathcal{C}(G), \| \cdot \|_\infty)$.

Proof. In fact, the conclusion can follow from $G = U(d)$ is a compact group with respect to operator norm topology and the Stone-Weierstrass theorem. □

Theorem 4.17. Let $G = U(d)$ be the $d$-dimensional unitary group. Then the algebra $\mathcal{C}_{alg}(G)$ is the algebra $\mathbb{C}[a_{ij}]$ of rational functions on the group $GL(d, \mathbb{C})$.

Proof. Theorem 4.16 told us that the representative functions on $U(d)$ are the polynomials in $a_{ij}$ and $\bar{a}_{ij}$. But, since $U = (a_{ij}) \in U(d)$ is a unitary matrix, so $(\bar{a}_{ij}) = U^* = U^{-1} = (\frac{p_{ij}}{\lambda^j})$, where $p_{ij}$ is a polynomial of $a_{ij}$. □

Theorem 4.18. Let $G = U(d)$. Then every topological representation $(h, \mathcal{M})$ of $G$ can be extended to $GL(d, \mathbb{C})$ uniquely.

Proof. It follows from Theorem 4.17 that $h$ can be extended to $GL(d, \mathbb{C})$. Note that the extended homomorphism $h : GL(d, \mathbb{C}) \to Inv(\mathcal{M})$ is a holomorphic map and $U(d)$ is a compact subset of $GL(d, \mathbb{C})$, so the conclusion is proved. □

Recall that $U(d)$ is the all unitary operators on the Hilbert space $\mathbb{C}^d$, $\{|1\rangle, \ldots, |d\rangle\}$ is the orthonormal basis of $\mathbb{C}^d$, $P : S_k \to GL((\mathbb{C}^d)^{\otimes k})$ for

$$P(\pi)|i_1i_2\cdots i_n\rangle = |i_{\pi^{-1}(1)}\rangle |i_{\pi^{-1}(2)}\rangle \cdots |i_{\pi^{-1}(k)}\rangle,$$

then $(P, (\mathbb{C}^d)^{\otimes k})$ is a topological representation of $S_k$, $Q : U(d) \to GL((\mathbb{C}^d)^{\otimes k})$ for

$$Q(U)|i_1i_2\cdots i_k\rangle = (U|i_1\rangle)(U|i_2\rangle)\cdots (U|i_k\rangle),$$

then $(Q, (\mathbb{C}^d)^{\otimes k})$ is a topological representation of $U(d)$, moreover, note that $P(\pi), Q(U) \in U(d^k)$ and $P(\pi)Q(U) = Q(U)P(\pi)$, $PQ : S_k \times U(d) \to GL((\mathbb{C}^d)^{\otimes k})$ for

$$PQ(\pi, U) = P(\pi)Q(U),$$

then $(PQ, (\mathbb{C}^d)^{\otimes k})$ is a topological representation of $S_k \times U(d)$.

It follows from Theorem 4.13 that for every $\lambda$, $V_\lambda$ is an irreducible submodule of $(\mathbb{C}^d)^{\otimes k}$, we denote $(P_\lambda, (\mathbb{C}^d)^{\otimes k})$ is the corresponding representation of $S_k$. Then for the finite group $G = S_k$, we have

$$(\mathbb{C}^d)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} V_\lambda \otimes \mathbb{C}^{n_\lambda}.$$

74
If we denote \( A = \text{span}\{P(\pi) : \pi \in S_k\} \), \( B = \text{span}\{Q(U) : U \in U(d)\} \), then it follows from Theorem 3.15 that \( A' = B, B' = A \), and

\[
A \cong \bigoplus_{\lambda \vdash k} \text{End}(V_{\lambda}) \otimes 1_{C^{n_{\lambda}}}.
\]

\[
B \cong \bigoplus_{\lambda \vdash k} 1_{V_{\lambda}} \otimes \text{End}(C^{n_{\lambda}}).
\]

Thus, for every \( \pi \in S_k \) and \( U \in U(d) \), \( P(\pi) \in A, Q(U) \in B \), and note that Theorem 1.27 we have \( C^{n_{\lambda}} \) is a topological module of \( U(d) \), we denote \((Q_{\lambda}, U(d))\) is the topological representation of \( U(d) \), then

\[
P(\pi) \cong \bigoplus_{\lambda \vdash k} P_{\lambda}(\pi) \otimes 1_{C^{n_{\lambda}}}.
\]

\[
Q(U) \cong \bigoplus_{\lambda \vdash k} 1_{V_{\lambda}} \otimes Q_{\lambda}(U),
\]

\[
PQ(\pi, U) \cong \bigoplus_{\lambda \vdash k} P_{\lambda}(\pi) \otimes Q_{\lambda}(U).
\]

Now, we show that \( C^{n_{\lambda}} \) is an irreducible module of \((Q_{\lambda}, U(d))\). It follows from Theorem 4.18 that we only need to show that if \( Q_{\lambda} \) is extended to \( \text{GL}(d) \), then \((Q_{\lambda}, \text{GL}(d))\) is also irreducible. That is, it suffices to show that \( C^{n_{\lambda}} \) is indecomposable under \( \text{GL}(d) \). This is equivalent to showing that \( \text{End}_{\text{GL}(d)}(C^{n_{\lambda}}) \cong \mathbb{C} \). That is, the maps in \( \text{End}(C^{n_{\lambda}}) \) that commute with the action of \( \text{GL}(d) \) are proportional to the identity. Now, we prove the fact.

It follows from Theorem 1.62 that e

\[
\text{End}_{S_k}(C^d)^{\otimes k} \cong \bigoplus_{\lambda} \text{End}(Q_{\lambda}).
\]

Thus

\[
\text{End}_{\text{GL}(d) \times S_k}(C^d)^{\otimes k} \cong \bigoplus_{\lambda} \text{End}_{\text{GL}(d)}(Q_{\lambda}).
\]

By Theorem 3.18 \( \text{GL}(d) \) and \( S_k \) are double commutants,

\[
\text{End}_{S_k}(C^d)^{\otimes k} = \text{span}\left\{T^{\otimes k} : T \in \text{GL}(d)\right\},
\]

and thus \( \text{End}_{\text{GL}(d) \times S_k}(C^d)^{\otimes k} \) is clearly contained in the commutator of \( \text{End}_{S_k}(C^d)^{\otimes k} \). Therefore \( \text{End}_{\text{GL}(d)}(Q_{\lambda}) \) is contained in the commutator of \( \text{End}(Q_{\lambda}) \cong \mathbb{C} \). Finally \( \text{End}_{\text{GL}(d)}(Q_{\lambda}) \cong \mathbb{C} \). We are done.

In summary, we have the following famous theorem:
Theorem 4.19 (Schur-Weyl duality). Let $\mathcal{H} = (\mathbb{C}^d)^{\otimes k}$. Then

$$(\mathbb{C}^d)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} V_\lambda \otimes \mathbb{C}^{n_\lambda},$$

where $(P_\lambda, V_\lambda)$ is an irreducible topological representation of $S_k$, $(Q_\lambda, \mathbb{C}^{n_\lambda})$ is an irreducible topological representation of $U(d)$, and for every $\pi \in S_k$, $U \in U(d)$,

$$P(\pi) \cong \bigoplus_{\lambda \vdash k} P_\lambda(\pi) \otimes \mathbb{1}_{C^{n_\lambda}},$$

$$Q(U) \cong \bigoplus_{\lambda \vdash k} \mathbb{1}_{V_\lambda} \otimes Q_\lambda(U),$$

$$PQ(\pi, U) \cong \bigoplus_{\lambda \vdash k} P_\lambda(\pi) \otimes Q_\lambda(U).$$

4.3 Combinatorics of Young tableaux

Let $f_\lambda = \dim(V_\lambda)$, $t_\lambda(d) = \dim(\mathbb{C}^{n_\lambda})$.

Theorem 4.20. It holds that

$$f_\lambda = \left| \left\{ t_\lambda : t_\lambda \text{ standard Young tableau with frame } \lambda \right\} \right| \quad (4.7)$$

$$t_\lambda(d) = \left| \left\{ t_\lambda : t_\lambda \text{ semi-standard Young tableau with frame } \lambda \text{ and numbers } [d] \right\} \right| \quad (4.8)$$

$$d^k = \sum_{\lambda \vdash (k, d)} t_\lambda(d)f_\lambda. \quad (4.9)$$

Clearly the minimal central projection is proportional to a sum over the corresponding minimal projections. Note also that the minimal projections are linearly independent.

For both $f_\lambda$ and $t_\lambda(d)$, the combinatorial sum can be evaluated and results in so-called hook length formulae. The hook of box $(i, j)$ in a diagram is given by the box itself, the boxes to its right and below. The hook length is the number of boxes in a hook.

Theorem 4.21 (Hook length formulae). It holds that

$$f_\lambda = \frac{k!}{\prod_{(i,j) \in \lambda} h(i,j)}, \quad (4.10)$$

$$t_\lambda(d) = \prod_{(i,j) \in \lambda} \frac{d + j - i}{h(i,j)} = \frac{f_\lambda}{k!} \prod_{(i,j) \in \lambda} (d + j - i). \quad (4.11)$$

This formula is also a reformulation of Weyl's dimension formula, which is best known in the form

$$t_\lambda(d) = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i}. $$
For the purpose of this work, rather than precise formulae, we are concerned with the following inequalities:
\[
\begin{align*}
    f_\lambda & \leq \frac{k!}{\lambda_1! \cdots \lambda_d!} := \binom{k}{\lambda_1 \cdots \lambda_d}, \\
t_\lambda(d) & \leq (k+1)^{\frac{d(d-1)}{2}} = \exp \left( \frac{d(d-1)}{2} \ln(k+1) \right).
\end{align*}
\]

(4.12)

(4.13)

4.4 Applications of Schur-Weyl duality

Example 4.22. Suppose that \( k = 2 \) and \( d \) is greater than one. Then the Schur-Weyl duality is the statement that the space of two-tensors decomposes into symmetric and antisymmetric parts, each of which is an irreducible module for \( \text{GL}_d \):
\[
\mathbb{C}^d \otimes \mathbb{C}^d = S^2 \mathbb{C}^d \oplus \Lambda^2 \mathbb{C}^d.
\]

(4.14)

The symmetric group \( S_2 \) consists of two elements and has two irreducible representations, the trivial representation and the sign representation. The trivial representation of \( S_2 \) gives rise to the symmetric tensors, which are invariant (i.e. do not change) under the permutation of the factors, and the sign representation corresponds to the skew-symmetric tensors, which flip the sign.

In this subsection, we will use Haar measure \( \mu \) over unitary matrix group \( U(d) \).

Proposition 4.23. \( \square \) It holds that
\[
\int_{U(d)} U A U^\dagger d\mu(U) = \frac{\text{Tr}(A)}{d} \mathbb{1}_d,
\]

(4.15)

where \( A \in M(\mathbb{C}^d) \).

Proof. For any \( V \in U(d) \), we have
\[
V \left( \int_{U(d)} U A U^\dagger d\mu(U) \right) V^\dagger = \int_{U(d)} (VU) A (UV)^\dagger d\mu(U)
\]
\[
= \int_{U(d)} (VU) A (UV)^\dagger d\mu(VU)
\]
\[
= \int_{U(d)} WAW^\dagger d\mu(W) = \int_{U(d)} U A U^\dagger d\mu(U),
\]

implying that \( \int_{U(d)} U A U^\dagger d\mu(U) \) commutes with \( U(d) \). Thus \( \int_{U(d)} U A U^\dagger d\mu(U) = \lambda_A \mathbb{1}_d \).

By taking trace over both sides, we get \( \lambda_A = \text{Tr}(A)/d \). Therefore the desired conclusion is obtained. \( \square \)

\(^1\)See Ref. [R. Frank and E. Lieb, arXiv:1306.5358]
Proposition 4.24. It holds that
\[
\int_{U(d)} (U \otimes U) A(U \otimes U)^d \mu(U) = \left( \frac{\text{Tr}(A)}{d^2 - 1} - \frac{\text{Tr}(AF)}{d(d^2 - 1)} \right) \mathbb{1}_{d^2} - \left( \frac{\text{Tr}(A)}{d(d^2 - 1)} - \frac{\text{Tr}(AF)}{d^2 - 1} \right) F,
\]
where \( A \in M(C^d \otimes C^d) \) and the swap operator \( F \) is defined by \( F |ij\rangle = |ji\rangle \) for all \( i, j = 1, \ldots, d \).

Proof. Analogously, we have \( \int_{U(d)} (U \otimes U) A(U \otimes U)^d \mu(U) \) commutes with \( \{ V \otimes V : V \in U(d) \} \). Denote \( P_\wedge := \frac{1}{2}(1_{d^2} - F) \) and \( P_\vee := \frac{1}{2}(1_{d^2} + F) \). It is easy to see that \( \text{Tr}(P_\wedge) = \frac{1}{2}(d^2 - d) \) and \( \text{Tr}(P_\vee) = \frac{1}{2}(d^2 + d) \). Since \( F = \sum_{i,j} |ij\rangle \langle ji| \), it follows that \( F^* = F \) and \( F^2 = 1_{d^2} \). Thus both \( P_\wedge \) and \( P_\vee \) are projectors and \( P_\wedge + P_\vee = 1_{d^2} \).

Because \( C^d \otimes C^d = \wedge^2 C^d \oplus \vee^2 C^d \), we have \( P_\wedge (C^d \otimes C^d) P_\wedge = \wedge^2 C^d \) and \( P_\vee (C^d \otimes C^d) P_\vee = \vee^2 C^d \). Besides, for any \( V \in U(d) \),
\[
V \otimes V \overset{\text{def}}{=} \begin{bmatrix} P_\wedge (V \otimes V) P_\wedge & 0 \\ 0 & P_\vee (V \otimes V) P_\vee \end{bmatrix}
\]

Now write
\[
\int_{U(d)} (U \otimes U) A(U \otimes U)^d \mu(U) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}
\]
is a block matrix, where \( M_{11} \in \text{End}(\wedge^2 C^d) \), \( M_{22} \in \text{End}(\vee^2 C^d) \) and
\[
M_{12} \in \text{Hom}_{U(d)}(\vee^2 C^d, \wedge^2 C^d), M_{21} \in \text{Hom}_{U(d)}(\wedge^2 C^d, \vee^2 C^d).
\]
Thus
\[
\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} P_\wedge (V \otimes V) P_\wedge & 0 \\ 0 & P_\vee (V \otimes V) P_\vee \end{bmatrix} = \begin{bmatrix} P_\wedge (V \otimes V) P_\wedge & 0 \\ 0 & P_\vee (V \otimes V) P_\vee \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.
\]

We get that, for all \( V \in U(d) \),
\[
\begin{align*}
M_{11}(\wedge^2 V) &= (\wedge^2 V)M_{11}, \\
M_{22}(\vee^2 V) &= (\vee^2 V)M_{22}, \\
M_{12}(\vee^2 V) &= (\vee^2 V)M_{12}, \\
M_{21}(\wedge^2 V) &= (\wedge^2 V)M_{21}.
\end{align*}
\]
\text{---}

\text{See Refs. [W. Roga et al., Int. J. Quantum Inform. 09, 1031 (2011).] and [F. Dupuis et al., arXiv: 1012.6044]}

78
Therefore we obtained that

\[ M_{11} = \lambda(A)P_\wedge, \quad M_{22} = \mu(A)P_\vee, \quad M_{12} = 0, \quad M_{21} = 0. \]

That is

\[
\int_{U(d)} (U \otimes U) A(U \otimes U)^\dagger d\mu(U) = \begin{bmatrix} \lambda(A)P_\wedge & 0 \\ 0 & \mu(A)P_\vee \end{bmatrix} = \lambda(A)P_\wedge + \mu(A)P_\vee. \tag{4.17}
\]

If \( A = 1_d \) in Eq. (4.17), then \( 1_d = \lambda(1_d)P_\wedge + \mu(1_d)P_\vee \). Thus \( \lambda(1_d) = \mu(1_d) = 1 \) since \( 1_d = P_\wedge + P_\vee \) and \( P_\wedge \perp P_\vee \).

If \( A = P_\wedge \) in Eq. (4.17), then \( P_\wedge = \lambda(P_\wedge)P_\wedge + \mu(P_\wedge)P_\vee \) since \( U \otimes U \) commutes with \( P_\wedge \). Thus \( \lambda(P_\wedge) = 1 \) and \( \mu(P_\wedge) = 0 \). Note that \( \lambda(A), \mu(A) \) are two linear functional. Thus we have: \( \lambda(F) = -1 \) and \( \mu(F) = 1 \). This indicates that

\[
\int_{U(d)} (U \otimes U) F(U \otimes U)^\dagger d\mu(U) = \lambda(F)P_\wedge + \mu(F)P_\vee = P_\vee - P_\wedge = F.
\]

More simpler approach to this identity can be described as follows: Since \( F(M \otimes N)F = N \otimes M \), it follows that \( F(M \otimes N) = (N \otimes M)F \). Thus

\[
\int_{U(d)} (U \otimes U) F(U \otimes U)^\dagger d\mu(U) = \int_{U(d)} F(U \otimes U)(U \otimes U)^\dagger d\mu(U) = F \int_{U(d)} d\mu(U) = F = P_\vee - P_\wedge.
\]

Apparently

\[
\int_{U(d)} (U \otimes U)^\dagger F(U \otimes U)d\mu(U) = F = P_\vee - P_\wedge.
\]

By taking trace over both sides above, we get

\[ \text{Tr} (A) = \lambda(A) \text{Tr} (P_\wedge) + \mu(A) \text{Tr} (P_\vee). \]

Now by multiplying \( F \) on both sides in Eq. (4.17) and then taking trace again, we get

\[
\int_{U(d)} \text{Tr} \left( (U \otimes U)^\dagger F(U \otimes U)A \right) d\mu(U) = \lambda(A) \text{Tr} (P_\wedge F) + \mu(A) \text{Tr} (P_\vee F) \tag{4.18}
\]

\[ = \mu(A) \text{Tr} (P_\vee) - \lambda(A) \text{Tr} (P_\wedge), \tag{4.19} \]

where we used the fact that \( P_\wedge F = -P_\wedge \) and \( P_\vee F = P_\vee \). Thus we have

\[
\begin{cases}
\frac{d(d-1)}{2} \lambda(A) + \frac{d(d+1)}{2} \mu(A) = \text{Tr} (A), \\
\frac{d(d+1)}{2} \mu(A) - \frac{d(d-1)}{2} \lambda(A) = \text{Tr} (AF).
\end{cases} \tag{4.20}
\]
Solving this group of two binary equations gives rise to
\[
\begin{align*}
\lambda(A) &= \frac{\Tr(A) - \Tr(AF)}{d(d-1)}, \\
\mu(A) &= \frac{\Tr(A) + \Tr(AF)}{d(d+1)}. 
\end{align*}
\] 
(4.21)

Finally we obtained the desired conclusion as follows:

\[
\int_{U(d)} (U \otimes U) A (U \otimes U)^\dagger d\mu(U) = \frac{\Tr(A) - \Tr(AF)}{d(d-1)} p_\lambda + \frac{\Tr(A) + \Tr(AF)}{d(d+1)} p_\nu. 
\]
(4.22)

We are done.

5 Tensor representations of $\text{GL}(d)$

5.1 Representations of $S_k$ on tensors

We defined a representation $P$ of the symmetric group $S_k$ on the $k$-fold tensor product $(\mathbb{C}^d)^\otimes k$ by permutation of tensor positions. Now we will show how an important family of induced representations of $S_k$ naturally occurs on subspaces of $(\mathbb{C}^d)^\otimes k$.

Given an ordered $k$-tuple $I = (i_1, \ldots, i_k)$ of positive integers with $1 \leq i_1, \ldots, i_k \leq d$, we set $|I\rangle = |i_1 \cdots i_k\rangle$, and write $|I| = k$ for the number of entries in $I$. Let $H$ be the diagonal subgroup of $\text{GL}(d)$. It acts on $\mathbb{C}^d$ by

\[ h|i\rangle = h_i|i\rangle \quad \text{for} \quad h = \begin{bmatrix} h_1 \\ \vdots \\ h_d \end{bmatrix}. \]

We parameterize the characters of $H$ by $\mathbb{Z}^d$, where $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d$ gives the character

\[ h \mapsto h^\lambda := h_1^{\lambda_1} \cdots h_d^{\lambda_d}. \]

The action of $H$ on $\mathbb{C}^d$ extends to a representation on $(\mathbb{C}^d)^\otimes k$ by the representation of $\text{GL}(d)$. For $\lambda \in \mathbb{Z}^d$ let

\[ (\mathbb{C}^d)^\otimes k(\lambda) := \left\{ |u\rangle \in (\mathbb{C}^d)^\otimes k : Q(h)|u\rangle = h^\lambda |u\rangle \right\} \]

be the $\lambda$-weight space of $H$. Given a $k$-tuple $I$ as above, define

\[ p_j \overset{\text{def}}{=} |\{ \alpha : i_\alpha = j \}| \]
and set $p_1 = (p_1, \ldots, p_d) \in \mathbb{Z}^d$. Then

$$Q(h)|I\rangle = h^{p_1}|I\rangle \quad \text{for } h \in H,$$

so $|I\rangle$ is a weight vector for $H$ with weight $p_1 \in \mathbb{N}^d$. Furthermore,

$$|p_1| := p_1 + \cdots + p_d = |I| = k.$$

This shows that

$$(\mathbb{C}^d)^\otimes_k(\lambda) = \begin{cases} \text{span} \{ |I\rangle : p_1 = \lambda \} & \text{if } \lambda \in \mathbb{N}^d \text{ and } |\lambda| = k, \\ 0 & \text{otherwise}. \end{cases} \quad (5.1)$$

Since the actions of $H$ and $S_k$ on $(\mathbb{C}^d)^\otimes_k$ mutually commute, the weight space $(\mathbb{C}^d)^\otimes_k(\lambda)$ is a module for $S_k$. We will show that it is equivalent to an induced representation.

Let $\lambda \in \mathbb{N}^d$ with $|\lambda| = k$. Set

$$|u_\lambda\rangle := 1 \otimes \cdots \otimes 1 \otimes 2 \otimes \cdots \otimes 2 \otimes \cdots. \quad (5.2)$$

Then $|u_\lambda\rangle \in (\mathbb{C}^d)^\otimes_k(\lambda)$. Let $S_\lambda$ be the subgroup of $S_k$ fixing $|u_\lambda\rangle$. We have

$$S_\lambda \cong S_{\lambda_1} \times \cdots \times S_{\lambda_d},$$

where the first factor permutes positions $1, \ldots, \lambda_1$, the second factors permutes positions $\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2$, and so on.

**Proposition 5.1.** The restriction of the representation $P$ of $S_k$ to the subspace $(\mathbb{C}^d)^\otimes_k(\lambda)$ is equivalent to the representation $1^\uparrow_{S_\lambda} S_k$ on $\mathbb{C}[S_k/S_\lambda]$. Furthermore, if $\tilde{\lambda} \in \mathbb{N}^d$ is obtained by permuting the entries of $\lambda$, then

$$(\mathbb{C}^d)^\otimes_k(\lambda) \cong (\mathbb{C}^d)^\otimes_k(\tilde{\lambda})$$

as a module for $S_k$.

**Proof.** Let $I$ be a $k$-tuple such that $p_1 = \lambda$. Then there is a permutation $\pi$ such that $P(\pi)|u_\lambda\rangle = |I\rangle$. Then the map

$$\pi \mapsto P(\pi)|u_\lambda\rangle$$

gives a bijection from $S_k/S_\lambda$ to a basis for $(\mathbb{C}^d)^\otimes_k(\lambda)$ and interwines the left multiplication action of $S_k$ with the representation $P$. 

81
To verify the last statement, we observe that $GL(d)$ contains a subgroup isomorphic to $S_d$ (the permutation matrices), which acts on $C^d$ by permuting the basis vectors $|1\rangle, \ldots, |d\rangle$. The action of this group on $(C^d)^{\otimes k}$ commutes with the action of $P(S_k)$ and permutes the weight spaces for $H$ as needed.

**Remark 5.2.** In the above, we described an important construction which produces representations of a group $G$ from representations of $H$—a subgroup of $G$.

Suppose $V$ is a representation of $G$, and $W \subseteq V$ is a subspace which is $H$-invariant. For any $g \in G$, the subspace $g \cdot W = \{g \cdot w : w \in W\}$ depends only on the left coset $gH$ of $g$ modulo $H$, since $gh \cdot W = g \cdot (h \cdot W) = g \cdot W$ for all $h \in H$. For a coset $\sigma \in G/H$, we write $\sigma \cdot W$ for this subspace of $V$. We say that $V$ is *induced* by $W$ if every element in $V$ can be written uniquely as a sum of elements in such translates of $W$, i.e.,

$$V = \bigoplus_{\sigma \in G/H} \sigma \cdot W.$$ 

In this case we write $V = \text{Ind}^G_H W$. For example, the trivial representation on $W$ induced a representation $\text{Ind}^G_H 1 = 1^G \uparrow^G_H$.

### 5.2 Young symmetrizers and Weyl modules

Now we turn to the problem of finding realizations of the representations of $GL(d)$ in subspaces of tensor space determined by symmetry conditions relative to $S_k$.

Recall that to a partition $\lambda$ of $k$ with at most $d$ parts we have associated the following data:

(i) an irreducible representation $Q$ of $GL(d)$ with highest weight $\lambda$;

(ii) an irreducible representation $P$ of $S_k$.

In the Schur-Weyl duality pairing between (i) and (ii), $P$ acts on the space

$$P_\lambda = \left((C^d)^{\otimes k}(\lambda)\right)^{N^+_d}$$

of $N^+_d$-fixed $k$-tensors of weight $\lambda$, where $N^+_d$ stands for the group of upper-triangular unipotent matrices. We now use the symmetry properties of the tensors in $P_\lambda$ relative to certain subgroups of $S_k$ to construct projection operators onto irreducible $GL(d)$ subspaces of type $Q_\lambda$ and the projection onto the full $GL(d) \times S_k$ isotypic subspace in $(C^d)^{\otimes k}$.

If $T$ is a tableau of shape $\lambda \vdash (k,d)$, we write $T_{ij}$ for the integer placed in the $j$-th box of the $i$-th row of $T$ for $i = 1, \ldots, d$ and $j = 1, \ldots, \lambda_i$, and we set $|T| = k$ (we say that $T$ has size $k$).
Given a shape $\lambda$, we denote $T(\lambda)$ be the tableau obtained by numbering the boxes consecutively down the columns of the shape from left to right. For example, if $\lambda = (3,2,1,1)$, then

$$T(\lambda) = \text{young}(157,26,3,4)$$

We denote by $\text{Tab}(\lambda)$ the set of all tableaux of shape $\lambda$. The group $S_k$ operates simply transitively on $\text{Tab}(\lambda)$ by permuting the numbers in the boxes: $(\pi \cdot T)_{ij} = \pi(T_{ij})$.

Given a tableau $T$ of size $|T| = k$, we set $i_\alpha = r$ if $\alpha$ occurs in the $r$-th row of $T$. We define the $k$-tensor

$$|\psi_T\rangle := |i_1\rangle \otimes \cdots \otimes |i_k\rangle = |i_1 \cdots i_k\rangle.$$

Thus the numbers in the first row of $T$ indicate the tensor positions in $|\psi_T\rangle$ containing $|1\rangle$, the numbers in the second row of $T$ indicate the tensor positions in $|\psi_T\rangle$ containing $|2\rangle$, and so on. For example, if $\lambda = (3,2,1,1)$ as in the example above, then

$$|\psi_T(\lambda)\rangle = |1\rangle \otimes |2\rangle \otimes |3\rangle \otimes |4\rangle \otimes |1\rangle \otimes |2\rangle \otimes |1\rangle = |1234121\rangle.$$

From the definition, we see that $|i\rangle$ occurs $\lambda_i$ times in $|\psi_T\rangle$, and as $T$ ranges over $\text{Tab}(\lambda)$ the positions of $|i\rangle$ can be arbitrary. It follows that

$$\text{span}\{ |\psi_T\rangle : T \in \text{Tab}(\lambda) \} = (\mathbb{C}^d)^{\otimes k}(\lambda), \quad (5.3)$$

where $k = |\lambda|$. Thus the tableaux of shape $\lambda$ label a basis for the tensors of weight $\lambda$, and the action of $S_k$ on $k$-tensors is compatible with the action on tableaux:

$$P(\pi)|\psi_T\rangle = |\psi_{\pi \cdot T}\rangle \text{ for } \pi \in S_k.$$  

Obviously $R(T) \cap C(T) = \{1\}$. Since $|\psi_T\rangle$ is formed by putting $|i\rangle$ in the positions given by the integers in the $i$-th row of the tableau $T$, it is clear that $P(\pi)|\psi_T\rangle = |\psi_T\rangle$ if and only if $\pi \in R(T)$. Furthermore, if $S, T \in \text{Tab}(\lambda)$, then $|\psi_S\rangle = |\psi_T\rangle$ if and only if $T = \pi \cdot S$ for some $\pi \in R(S)$.

**Proposition 5.3.** Let $\lambda \vdash (k,d)$. If $T$ has shape $\lambda$, then $c_T|\psi_T\rangle$ is nonzero and $N^+_d$-fixed of weight $\lambda$.

**Proof.** (1). Suppose first that $T = T(\lambda)$. Let $\lambda'$ be the transposed shape (dual partition) to $\lambda$. Then

$$|\psi_T\rangle = |12 \cdots \lambda'_1\rangle \otimes |12 \cdots \lambda'_2\rangle \otimes \cdots \otimes |12 \cdots \lambda'_q\rangle.$$
The group $C(T)$ gives all permutations of positions $1, 2, \ldots, \lambda_1$, all permutations of positions $\lambda_1 + 1, \ldots, \lambda_2$, and so on. Hence

$$c_T|v_T\rangle = \kappa |\omega_{\lambda_1'}\rangle \otimes |\omega_{\lambda_2'}\rangle \otimes \cdots \otimes |\omega_{\lambda_d'}\rangle,$$

where $|\omega_j\rangle = |1\rangle \wedge |2\rangle \wedge \cdots \wedge |j\rangle$ and $\kappa$ is a nonzero constant. In particular, $c_T|v_T\rangle \neq 0$. Since each $|\omega_j\rangle$ is fixed by $N_d^+$, so is $c_T|v_T\rangle$. Indeed, if $X \in N_d^+$, then we have $X = [x_{st}]$ with $x_{st} = 0$ whenever $s > t$ and $x_{ss} = 1$. That is

$$X|j\rangle = x_{1j}|1\rangle + x_{2j}|2\rangle + x_{j-1,j}|j-1\rangle + |j\rangle,$$

which indicates that

$$X^\otimes_j |\omega_j\rangle = X^\otimes_j |1\rangle \wedge \cdots \wedge |j\rangle = (X|1\rangle) \wedge \cdots \wedge (X|j\rangle)$$

$$= |1\rangle \wedge (x_{12}|1\rangle + |2\rangle) \wedge \cdots \wedge (x_{1j}|1\rangle + x_{2j}|2\rangle + x_{j-1,j}|j-1\rangle + |j\rangle)$$

$$= |1\rangle \wedge |2\rangle \wedge \cdots \wedge |j\rangle = |\omega_j\rangle.$$

Thus

$$X^\otimes_k (c_T|v_T\rangle) = X^\otimes_k \kappa |\omega_{\lambda_1'}\rangle \otimes |\omega_{\lambda_2'}\rangle \otimes \cdots \otimes |\omega_{\lambda_d'}\rangle$$

$$= \kappa X^\otimes_{\lambda_1'} |\omega_{\lambda_1'}\rangle \otimes X^\otimes_{\lambda_2'} |\omega_{\lambda_2'}\rangle \otimes \cdots \otimes X^\otimes_{\lambda_d'} |\omega_{\lambda_d'}\rangle$$

$$= \kappa |\omega_{\lambda_1'}\rangle \otimes |\omega_{\lambda_2'}\rangle \otimes \cdots \otimes |\omega_{\lambda_d'}\rangle = c_T|v_T\rangle.$$

(2). Now let $T$ be any tableau of shape $\lambda$. There is some $\pi \in S_k$ such that $T = \pi \cdot T(\lambda)$. Hence

$$|v_T\rangle = |v_{\pi \cdot T(\lambda)}\rangle = P(\pi)|v_{T(\lambda)}\rangle$$

and $c_T = P(\pi)c_{T(\lambda)}P(\pi)^{-1}$. It follows that

$$c_T|v_T\rangle = \left(P(\pi)c_{T(\lambda)}P(\pi)^{-1}\right)\left(P(\pi)|v_{T(\lambda)}\rangle\right)$$

$$= P(\pi)c_{T(\lambda)}|v_{T(\lambda)}\rangle.$$

Now for $X \in N_d^+$, we have

$$X^\otimes_k (c_T|v_T\rangle) = X^\otimes_k P(\pi)c_{T(\lambda)}|v_{T(\lambda)}\rangle = P(\pi)X^\otimes_k c_{T(\lambda)}|v_{T(\lambda)}\rangle$$

$$= P(\pi)c_{T(\lambda)}|v_{T(\lambda)}\rangle = c_T|v_T\rangle.$$

That is $c_T|v_T\rangle$ is $N_d^+$-fixed as claimed. We are done. \qed
Proposition 5.4. Let $\lambda \vdash (k,d)$. It holds that

(i) if $T$ is a tableau of shape $\lambda$, then $c_T(C_d^d)^k(\lambda) = Cc_T|v_T\rangle$.

(ii) $P_\lambda = \sum_{T \in \text{Tab}(\lambda)} Cc_T|v_T\rangle$.

Proposition 5.5. Let $\lambda, \mu \vdash (k,d)$. It holds that

(i) if $\mu > \lambda$, then $c_T(C_d^d)^k(\mu) = 0$ for all $T \in \text{Tab}(\lambda)$.

(ii) if $\mu < \lambda$, then $r_Tc_S = 0$ for all $T \in \text{Tab}(\lambda)$ and $S \in \text{Tab}(\mu)$.

Proposition 5.6. Let $\lambda \vdash (k,d)$ and let $T$ be a tableau of shape $\lambda$. Define $e_T := c_Tr_T$ as an element of the group algebra $C[S_k]$ of $S_k$.

(i) $e_T P_\mu = 0$ for all $\mu \vdash (k,d)$ with $\mu \neq \lambda$.

(ii) $e_T P_\lambda$ is spanned by the $N_\lambda^+$-fixed tensor $c_T|v_T\rangle$ of weight $\lambda$.

References


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References

