第12周GARCH模型的作业

1. For ARCH(1) model $X_t = (a_0 + a_1 X_{t-1}^2)^{1/2} \varepsilon_t$, if $X_t$ is strong stationary and $EX_t^4 < \infty$, show that

$$EX_t^4 = \frac{a_0^2(1 + a_1)Ex_t^4}{(1 - a_1)(1 - a_1^2Ex_t^2)}$$

and the kurtosis

$$\kappa_x = \frac{(1 - a_1^2)\kappa_\varepsilon}{1 - a_1^2Ex_t^2}$$

Therefore, a necessary condition for the existence of the fourth moment is that $a_1 < 1/\sqrt{Ex_t^2}$.

2. Suppose that the daily log-return of General Electric Inc. from January 1995 to December 2012 follows an AR(1)-GARCH(1, 1) model:

$$r_t = \alpha_0 + \alpha_1 r_{t-1} + \eta_t, \eta_t = \sigma_t \varepsilon_t, \sigma_t^2 = a_0 + a_1 \eta_{t-1}^2 + b_1 \sigma_{t-1}^2.$$ 

(a) Write down the conditional likelihood function when $\varepsilon_t$ follows the standard normal distribution.

(b) Write down the conditional likelihood function when $\varepsilon_t$ follows a $t$-distribution of freedom $v$.

(c) Simulate the series of length 2520 with $\alpha_0 = 0.002, \alpha_1 = -0.12, a_0 = 0.000015, a_1 = 0.0414, b_1 = 0.921$, and the Gaussian innovation $\varepsilon_t$.

(d) What is the variance of this AR(1)-GARCH(1, 1) models? Compare it with the sample variance?

3. Suppose that the volatilities of the daily log-return of the Coco-Cola company follow the GARCH(1,1) model:

$$X_t = \sigma_t \varepsilon_t, \sigma_t^2 = a_0 + a_1 X_{t-1}^2 + b_1 \sigma_{t-1}^2$$

with $a_1 + b_1 < 1$ and $\varepsilon_t \sim N(0,1)$.

(a) If $a_0 = 0.006, a_1 = 0.05$ and $b_1 = 0.55$, is the tail of the distribution lighter than that of $t_4$ in terms of kurtosis?

(b) What is the autocorrelation function of the series $\{X_t^2\}$. 

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(c) With the parameters in (a), if $X^2_t = 0.02$, and $\sigma^2_T = 0.03$, give the one-step and two-step forecast of the volatility.

4. Suppose that the log-return $\{r_t\}$ of a portfolio follows an GARCH($p, q$) model:

$$r_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i X^2_{t-i} + \sum_{j=1}^{q} b_j \sigma^2_{t-j},$$

(a) Show that $\text{cov}(r_t^2, \sigma_t^2) > 0$.

(b) What is the unconditional volatility of the $\tau$-period log-returns: $R_{t, \tau} = r_{t+1} + \cdots + r_{t+\tau}$? What is the conditional volatility of $R_{t, \tau}$?

5. Consider the log-returns of a financial data (say S&P from Jan 1990 to Dec 2013).

(a) Does GARCH or ARCH effect exist in this data set?

(b) If GARCH effect exists, fit a GARCH (1,1) model to the return series using Gaussian innovation.

(c) Compute the mean return and long-run volatility (unconditional standard deviation).